

AGEING PROPERTIES OF FIRST PASSAGE TIME DISTRIBUTION OF A MULTISTATE RELIABILITY SYSTEM

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Received on 22 July 2008; Revised, 21 January 2009; Accepted, 29 January 2009.

Abstract. A multistate reliability system modeled by a semi-Markov process is discussed. The states of the system consists of two sets - one of acceptable states and other of down states. We derive a necessary and sufficient condition under which the distribution of the first passage time from acceptable states to down state is increasing failure rate (IFR) and increasing failure rate average (IFRA). Some applications are provided.

MSC 2000: Primary 60K10, Secondary 60K15.

Keywords. semi-Markov process, IFR, IFRA, multistate system, monotone paths, first passage time.

1 Introduction

We are concerned with a multistate system (MSS) having $M+1$ states $0, 1, \dots, M$ where '0' is the best state and 'M' is the worst state. At time zero the system begins at its best state and as time passes the system begins to deteriorate. It is assumed that the time spent by the system in each state is random with arbitrary sojourn time distribution. The system stays in some acceptable states for some time and then it moves to unacceptable (down) state. The first time at which the MSS enters the down state after spending a random amount of time in acceptable states is termed as the first passage time (failure time) to the down state of the MSS.

We study the aging properties of the first passage time distribution of the MSS modeled by the semi-Markov process $\{Y_t, t \geq 0\}$. In the MSS with states $\{0, 1, \dots, k-1, k, k+1, \dots, M\}$ where $\{0, 1, \dots, k-1, k\}$ is the acceptable states, the sojourn time between state 'i' to state 'j' is assumed to be distributed with arbitrary distribution F_{ij} . Definitions and properties of a MSS can be seen in El. Neweihi and Proschan (1984) and Barlow and Wu (1978). Markov and semi-Markov modeling of a MSS is given in Lisnianski and Levitin (2003). In this paper, our aim is to derive a necessary and sufficient condition for a MSS failure time distribution to be IFR and IFRA and to highlight some potential applications. Deshpande et al. (1986) and Barlow and Proschan (1975) described various aspects of positive aging in terms of conditional probability distributions of residual lifetimes and failure rates. Bryson and Siddiqui (1969) discussed the concept of 'aging' or progressive shortening of an entity's residual lifetime in terms of survival time distribution.

Let F be the c. d. f. of a continuous random variable T representing lifetime of a unit. Then $R(t) = 1 - F(t) = \bar{F}(t) = P(T > t)$ is called its reliability function (or survival function) and $R_x(t) = \frac{R(t+x)}{R(t)}$ is the survival function of a unit of age t , i.e., conditional probability that a unit of age 't' will survive for an additional 'x' unit of time. Obviously, any study of the phenomenon of aging/ no aging (i.e., age has no effect on the residual life time) has to be based on $R_x(t)$ and functions related to it. Following are the definitions of IFR, IFRA, DFR and DFRA distributions (see Barlow and Proschan (1975)).

Definition 1.1 *Increasing failure rate (IFR) distribution: F is IFR if $R_x(t_1) \geq R_x(t_2)$, $x \geq 0$, $0 \leq t_1 \leq t_2 < \infty$.*

Definition 1.2 *Increasing failure rate average (IFRA) distributions: F is IFRA if $-\frac{1}{t} \log R(t)$ is increasing in t or equivalently F is said to be IFRA if $(R(t))^{1/t}$ is decreasing in t .*

Definition 1.3 *Decreasing failure rate (DFR) distribution: F is DFR if $R_x(t_1) \leq R_x(t_2)$, $x \geq 0$, $0 \leq t_1 \leq t_2 < \infty$.*

Definition 1.4 *Decreasing failure rate average (DFRA) distributions: F is DFRA if $-\frac{1}{t} \log R(t)$ is decreasing in t or equivalently F is said to be DFRA if $(R(t))^{1/t}$ is increasing in t .*

Now for a continuous time Markov process $\{Y_t, t \geq 0\}$ with state space \mathbf{S} , a countable set with a partial ordering, and transition matrix \mathbf{P} , we say the

Markov process is of monotone paths if $P(Y_t > Y_s) = 1$ for $t > s$. Define D a subset of \mathbf{S} to be an increasing set if $i \in D$ and $j \geq i \Rightarrow j \in D$. This Markov process is stochastically monotone if and only if $i \leq j \Rightarrow P(i, D) \leq P(j, D)$ for all increasing sets D . For a state i and a set D define $T_D(i)$ to be first passage time from the state i to D , with $T_D(i) = 0$ if $i \in D$ and $T_D(i) = \infty$ if D is never reached. Brown and Chaganty (1983) proved that, if $\{Y_n, n \geq 0\}$ is a stochastically monotone Markov chain with monotone paths on the partially ordered countable set \mathbf{S} , and D is an increasing set with the complement of D in \mathbf{S} finite, then $T_D(i)$, the first passage time from state i to set D , is IFRA.

Let $E = \{0, 1, 2, \dots, M\}$ be a set representing the state of the MSS and probability space with probability function P , on which we define a bivariate time homogeneous Markov chain $(X, T) = \{X_n, T_n, n \in \{0, 1, 2, \dots\}\}$, X_n takes values of E and T_n on the half real line $R^+ = [0, \infty)$, with $0 \leq T_1 \leq T_2 \leq \dots \leq T_n \leq \dots$. Put $U_n = T_n - T_{n-1}$ for all $n \geq 1$. This Markov process is called a Markov renewal process (MRP) with transition function, the semi-Markov kernel, $\mathbf{Q} = [Q_{ij}]$,

$$\begin{aligned} \text{where } Q_{ij}(t) &= P[X_{n+1} = j, U_n \leq t | X_n = i], i, j \in E, t \geq 0 \\ \text{and } Q_{ii}(t) &= 0, i \in E, t \geq 0. \end{aligned}$$

Now we consider the semi-Markov process (SMP), as defined in Pyke (1961). It is the generalization of Markov process with countable state space. SMP is a stochastic process which moves from one state to another of a countable number of states with successive states visiting form a Markov chain, and that the process stays in a given state a random length of time, the distribution of which may depend on this state as well as on the one to be visited in the next. Define $Z_t = X_{N_t}$, $N_t = \sup\{n, T_n = U_1 + U_2 + \dots + U_n \leq t\}$, it is the semi-Markov process associated with the MRP defined above. In terms of Z , the times T_1, T_2, \dots are successive times of transitions for Z , and X_0, X_1, X_2, \dots are successive states visited. If \mathbf{Q} has the form

$$Q_{ij}(t) = P[X_{n+1} = j | X_n = i][1 - e^{-\lambda(i)t}], i, j \in E, t \geq 0,$$

for some function $\lambda(i), i \in E$, then the process Z_t is a Markov process. That is, in a Markov process, the distributions of the sojourn times are all exponential independent of the next state. The word *semi*-Markov comes from the somewhat limited Markov property which Z enjoys, namely, that the future of Z is independent of its past given the present state provided the "present" is the time of jump.

Let I_{ij} = indicator function of $\{i = j\}$. Define the transition probability that system occupied state $j \in E$ at time $t > 0$, given that it is started at state i at time zero, as, $\forall i, j \in E, t > 0$,

$$p_{ij}(t) = P[Z_t = j | Z_0 = i] = P[X_{N_t} = j | X_0 = i] = h_i(t)I_{ij} + \mathbf{Q} * \mathbf{P}(t)(i, j),$$

where $h_i(t) = 1 - \sum_k Q_{ik}(t)$, $\mathbf{P}(t) = [p_{ij}(t)]$,
and $\mathbf{Q} * \mathbf{P}(t)(i, j) = \sum_{k \in E} \int_0^t Q_{ik}(dx) p_{kj}(t - x)$.

To obtain the reliability function of the semi-Markov system described above, we must define a new process, Y with state space $U \cup \{\nabla\}$, where U denotes set of all up states $\{0, 1, \dots, k\}$ and ∇ is the absorbing state in which all the states $\{k + 1, \dots, M\}$ of the system is united. Let T_D denote the time of first entry to the down states of Z process.

That is,

$$Y_t = Z_t(\omega) \text{ if } t < T_D(\omega) \text{ and } Y_t = \nabla \text{ if } t \geq T_D(\omega).$$

Let $\mathbf{1} = (1, 1, \dots, 1)^T$, a unit row vector with appropriate dimension. The process Y_t is a semi-Markov process with semi-Markov kernel

$$\begin{pmatrix} \overbrace{Q_{11}(t)}^{Up} & \overbrace{Q_{12}(t)}^{Down} \\ 0 & 0 \end{pmatrix}$$

We denote $\alpha = (\overbrace{\alpha(0), \dots, \alpha(k)}^{up, \alpha_1}, \overbrace{\alpha(k+1), \dots, \alpha(M)}^{down, \alpha_2})$ where $\alpha(i) = P(Y_0 = i)$.

The reliability function is

$$\begin{aligned} R(t) &= P[\forall u \in [0, t], Z_u \in U] = P[Y_t \in U] = \sum_{j \in U} P[Y_t = j] \\ &= \sum_{i \in U} \sum_{j \in U} P[Y_t = j, Y_0 = i] = \sum_{i \in U} \sum_{j \in U} P[Y_t = j | Y_0 = i] P[Y_0 = i] \\ &= \sum_{i \in U} \sum_{j \in U} p_{ij}(t) \alpha(i). \end{aligned}$$

2 Aging properties

The sojourn time of the MSS in each state or from one state to another in a semi-Markov setup is a random variable. Consider the random lifetime of the MSS, T_D , the first passage time to the down state from upstate U , with distribution F . In the following we assume that $\forall i, j \in U, p_{ij}(t)$ is either monotone increasing or decreasing in t .

Now we obtain a necessary and sufficient condition for the distribution of semi-Markov system to be IFR.

Theorem 2.1 *For a semi-Markov system with monotone decreasing transition probability functions, and first passage time distribution F , the following statements are equivalent:*

- (a) F is IFR
- (b) $\sum_{i,j \in U} p'_{ij}(t+x)\alpha(i) \leq \sum_{i,j \in U} p'_{ij}(t)\alpha(i), t \geq 0.$

Proof: In the semi-Markov setup described above, if $R_x(t) = \frac{R(t+x)}{R(t)} = \frac{\sum_{i,j \in U} p_{ij}(t+x)\alpha(i)}{R(t)}$ is decreasing in t then the rate of decrease of $\sum_{i,j \in U} p_{ij}(t+x)\alpha(i)$ is larger than the rate of decrease of $R(t)$. Therefore if $R_x(t)$ is decreasing, $\sum_{i,j \in U} p'_{ij}(t+x)\alpha(i) \leq \sum_{i,j \in U} p'_{ij}(t)\alpha(i), t \geq 0.$

Conversely suppose that (b) holds, then the rate of decrease of $\sum_{i,j \in U} p_{ij}(t+x)\alpha(i)$ is larger than rate of decrease of $R(t)$. Then obviously we have $R_x(t)$ is decreasing in t , which implies that F is IFR. \square

However for a DFR distribution F the 'rate of increase' of $\sum_{i,j \in U} p_{ij}(t+x)\alpha(i)$ does not affect that of $R_x(t)$. It is easy to prove $R_x(t)$ is increasing if and only if $\forall i, j \in U, p_{ij}(t+x)$ is increasing in t , because $1/R(t)$ is an increasing function of t and product of two increasing functions, $\sum_{i,j \in U} p_{ij}(t+x)\alpha(i)$ and $1/R(t)$, is again an increasing function. Hence we have the following theorem.

Theorem 2.2 *For a semi-Markov system with monotone increasing transition probability functions, and first passage time distribution F , F is DFR if and only if $\forall i, j \in U, p_{ij}(t+x)$ increasing in t .*

Now we prove a necessary and sufficient condition for the IFRA property of first passage time distribution of the semi-Markov system.

Theorem 2.3 *For a semi-Markov system with monotone decreasing transition probability functions, and first passage time distribution F , the following two statements are equivalent:*

- (a) F is IFRA
- (b) $-t^2 \sum_{i,j \in U} p'_{ij}(t)\alpha(i) \geq 1, t \geq 0.$

Proof: Suppose that, F is IFRA. Then $(R(t))^{1/t}$ is decreasing in t . But $(R(t))^{1/t} = (\sum_{i,j \in U} p_{ij}(t)\alpha(i))^{1/t}$ is decreasing in t only when rate of decrease of $\sum_{i,j \in U} p_{ij}(t)\alpha(i)$ is larger than rate of decrease of $1/t$. That is, $\sum_{i,j \in U} p'_{ij}(t)\alpha(i) \leq -\frac{1}{t^2}, t \geq 0$, i.e., $-t^2 \sum_{i,j \in U} p'_{ij}(t)\alpha(i) \geq 1, t \geq 0.$

Conversely suppose that (b) holds, then $\sum_{i,j \in U} p_{ij}(t)\alpha(i)$ is decreasing at a greater rate than $1/t$, so that $(R(t))^{1/t}$ is a decreasing function of t . Hence,

first passage time distribution F of the semi-Markov system is IFRA. \square

On a similar lines we prove a necessary and sufficient condition for the DFRA property of first passage time distribution of the semi-Markov system.

Theorem 2.4 *For a semi-Markov system with monotone increasing transition probability functions, and first passage time distribution F , the following two statements are equivalent:*

- (a) F is DFRA
- (b) $\sum_{i,j \in U} p'_{ij}(t)\alpha(i) \geq \sum_{i,j \in U} p_{ij}(t)\alpha(i), t \geq 0$.

Proof: Suppose that, F is DFRA. Then $(R(t))^{1/t}$ is increasing in t . Now consider the logarithmic transformation of $(R(t))^{1/t}$.

$\log(R(t))^{1/t} = \frac{\log(\sum_{i,j \in U} p_{ij}(t)\alpha(i))}{t}$ is increasing in t only when rate of increase of $\log(\sum_{i,j \in U} p_{ij}(t)\alpha(i))$ is larger than rate of increase of t . That is, $\frac{\sum_{i,j \in U} p'_{ij}(t)\alpha(i)}{\sum_{i,j \in U} p_{ij}(t)\alpha(i)} \geq 1, t \geq 0$, i.e., $\sum_{i,j \in U} p'_{ij}(t)\alpha(i) \geq \sum_{i,j \in U} p_{ij}(t)\alpha(i), t \geq 0$.

Conversely suppose that (b) holds, then $\log(\sum_{i,j \in U} p_{ij}(t)\alpha(i))$ is increasing at a greater rate than increase of t , so that $(R(t))^{1/t}$ is a increasing function of t . Hence, first passage time distribution F of the semi-Markov system is DFRA. \square

3 Applications

Major application of the above results is in maintenance policies such as age and block replacement policies. A variety of applications of IFR, DFR, IFRA distributions in maintenance policies of a binary system can be seen in Barlow and Proschan (1996). Under the IFR property the expected number of failures will be less under block replacement than under age replacement. When we identify the distribution of semi-Markov system is IFR or DFR, it will be easy to employ suitable maintenance policies according to the above theorems. We consider some examples that arise in practical applications such as power generation system with multistate performance levels.

Example 3.1 *Consider a Markov process in continuous time and discrete state space $\{1, 2, \dots, M\}$, given in Doob (1953), p.241. The system start in state '1' at time zero and as it enters 'M', it remains there. Consider the intensity matrix, $\mathbf{Q} = [Q_{ij}]$, with entries $q_{ij} = 0, i \in \{1, 2, \dots, M-1\}, j \neq i+1, q_{i+1} = q$, and $q_M = 0$. The Kolmogorov's system of differential equation becomes, for $p_{ij}(t-u) = P(Y_t = j | Y_u = i), 0 \leq u < t$ and we take $u = 0$,*

$$p'_{ik}(t) = -qp_{ik}(t) + qp_{i+1k}(t), i < M$$

$$p'_{Mk}(t) = 0$$

with initial conditions, $p_{ik}(0) = \delta_{ik}$, the indicator of $\{i = k\}$. Then, $p_{Mk}(t) = 0$, $k \neq M$, $p_{MM}(t) = 1$ and it is easily verified that the solution is

$$\begin{aligned} p_{ik}(t) &= 0 && k < i \\ &= \frac{(qt)^{k-i} e^{-qt}}{(k-1)!}, && i \leq k < M \\ &= e^{-qt} [e^{qt} - 1 - qt - \dots - \frac{(qt)^{M-i-1}}{(M-i-1)!}], && k = M \end{aligned}$$

which is increasing initially (i. e., system is DFR) for $t < t_0$, where t_0 is the time at which $p'_{ik}(t) = 0$, (i. e., the time at which $p_{ik}(t) = p_{i+1k}(t)$) and then decreasing in $t > t_0$ (i. e., system is not DFR) for $k \in \{1, 2, \dots, M-1\}$, the set of acceptable states. Here the process is of monotone paths.

Example 3.2 Consider a continuous time Markov process $\{Y_t, t \geq 0\}$ with state space $\{0, 1, \dots, M\}$ and $Y_0 = 0$, such that the process stays in state i for a random length of time whose distribution is exponential with mean $1/\lambda_i$ then moves to state $(i+1)$, this continues until down state M is reached. Consider the intensity matrix

$$\begin{pmatrix} -\lambda_0 & \lambda_0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & -\lambda_{M-1} & \lambda_{M-1} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The last row $(0, 0, \dots, 0)$ means that state ' M ' is absorbing. Bhat (2000), p.197, obtained the forward Kolmogorov's differential equation, with initial conditions $p_{00}(0) = 1$, as

$$p'_{0k}(t) = -\lambda_k p_{0k}(t) + \lambda_{k-1} p_{0k-1}(t), 0 \leq k \leq M-1.$$

Then,

$$p_{0k}(t) = \lambda_{k-1} e^{-\lambda_k t} \int_0^t e^{\lambda_k x} p_{0k-1}(x) dx, k = 1, 2, \dots, M-1.$$

The nature of the above transition probability functions shows the first passage time is IFRA or IFR or DFR or no aging.

For a numerical realization, Lisnianski and Levitin (2003), p.145, considered an electric generator that has four possible performance levels 100MW (state 0), 70MW (state 1), 50MW (state 2) and 0MW (state 3). The constant demand is 60MW. The best state with performance rate 100MW is the initial state. Times to failures are distributed exponentially with parameters,

$\lambda_{0,1} = 10^{-3}$ (hours(-1)), $\lambda_{1,2} = 5.10^{-3}$ (hours(-1)), and $\lambda_{2,3} = 2.10^{-3}$ (hours(-1)). Hence, times to failures $T_{0,1}$, $T_{1,2}$ and $T_{2,3}$ are random variables distributed according to the c. d. f., $F_{0,1} = 1 - e^{-\lambda_{0,1}t}$, $F_{1,2} = 1 - e^{-\lambda_{1,2}t}$ and $F_{2,3} = 1 - e^{-\lambda_{2,3}t}$, for $t > 0$. The system is having monotone paths. Thus,

$$p_{0,1}(t) = \lambda_{0,1}e^{-\lambda_{1,2}t} \int_0^t e^{\lambda_{1,2}u} p_{0,0}(u) du = \frac{\lambda_{0,1}}{\lambda_{1,2}} = 0.2,$$

$$p_{0,2}(t) = \lambda_{1,2}e^{-\lambda_{2,3}t} \int_0^t e^{\lambda_{2,3}x} \lambda_{0,1}e^{-\lambda_{1,2}x} \int_0^x e^{\lambda_{1,2}u} p_{0,0}(u) dudx = \frac{\lambda_{0,1}}{\lambda_{2,3}} = 0.5,$$

This means that with exponentially distributed sojourn times, the system failure time distribution is IFR as well as DFR, no ageing property of MSS.

On the other hand, when we consider the Weibull distribution for the sojourn times, we can expect specific IFR or DFR property of first passage time distribution. Let $F_{0,1} = (1 - e^{-\lambda_{0,1}t})^{\theta_0}$, $F_{1,2} = (1 - e^{-\lambda_{1,2}t})^{\theta_1}$ and $F_{2,3} = (1 - e^{-\lambda_{2,3}t})^{\theta_2}$, for $t > 0$. Let state 0 and 1 be the up states and 3 and 4 be the down states.

$$p_{01}(t) = \theta_0 \lambda_{0,1} \int_0^t (1 - e^{-\lambda_{0,1}x})^{\theta_0-1} (1 - (1 - e^{-\lambda_{1,2}x})^{\theta_1}) dx$$

$$p_{02}(t) = \theta_0 \lambda_{0,1} \int_0^t (1 - e^{-\lambda_{0,1}x})^{\theta_0-1} \int_0^x \theta_1 \lambda_{1,2} (1 - e^{-\lambda_{1,2}u})^{\theta_1-1} (1 - (1 - e^{-\lambda_{2,3}u})^{\theta_2}) dudx$$

For $\lambda_{0,1} = 2$, $\lambda_{1,2} = 3$, $\theta_0 = 3$, $\theta_1 = 3$ and $\theta_2 = 3$, the functions $p_{01}(t)$ and $p_{02}(t)$, where

$$p_{01}(t) = k_1 \left((-e^{10t} + \frac{3}{10}e^{3t} + \frac{1}{2}e^{7t} - \frac{1}{13} - \frac{1}{9}e^{4t} + \frac{6}{5}e^{8t} - \frac{3}{4}e^{5t} + \frac{2}{11}e^{2t} - \frac{3}{7}e^{6t})e^{-13t} + 0.18 \right)$$

$$p_{02}(t) = k_2 \left(\left(\frac{1}{225}e^{4t} - \frac{2}{5}e^{14t} + \frac{4}{84}e^{5t} + \frac{1}{3}e^{16t} - \frac{5}{144}e^{7t} + \frac{47}{180}e^{17t} - \frac{20}{99}e^{8t} + \frac{47}{180}te^{19t} + \frac{10}{81}e^{10t} + \frac{1}{285} + \frac{3}{8}e^{11t} - \frac{2}{225}e^{2t} - \frac{1}{4}e^{13t} - \frac{5}{192}e^{3t} + \frac{10}{117}e^{6t} - \frac{3}{20}e^{9t} + \frac{1}{7}e^{12t} - \frac{47}{720}e^{15t} \right) e^{-19t} - 0.25 \right)$$

increases in t for constants k_1 and k_2 . This implies that, the first passage time distribution is DFR.

Acknowledgement

The authors wish to thank the financial support from CSIR¹, India and UGC², India. Also they are grateful to the anonymous referee for his/her comments on the earlier version of this paper which improved its presentation.

References

- [1] Barlow, R. E., Proschan, F. (1975). *Statistical Theory of Reliability and Life Testing*, Holt, Rinehart and Winston, New York.
- [2] Barlow, R. E., Proschan, F.(1996). *Mathematical Theory of Reliability*, SIAM, New York.
- [3] Barlow, R. E.,Wu, A. S. (1978). Coherent system with multistate components, *Math. Oper. Res.*, 3, 275-281.
- [4] Bhat, B. R. (2000). *Stochastic Models-analysis and applications*, New Age Int., New Delhi.
- [5] Brown, M. and Chaganty, N. R. (1983). On the first passage time distribution for a class of Markov chains, *Ann. Probab.*, 11, 1000-1008.
- [6] Bryson, M. C. and Siddiqui, M. M. (1969). Some criteria for aging, *J. Amer. Stat. Asso.*, 64, 1472-1483.
- [7] Deshpande, J. V., Kochar, S. C. and Singh, H. (1986). Aspects of positive aging, *J. Appl. Probab.*, 23, 748-758.
- [8] Doob, J. L. (1953). *Stochastic Process*, John Wiley and sons, New York.
- [9] El.Neweihi, Proschan, F. (1984). Degradable systems: a survey of multistate reliability theory, *Comm. Stat. Theory and Methods*, 13, 403-432.
- [10] Limnios, N. (1997). Dependability analysis of semi-Markov systems, *Reliab. Eng. Sys. Saf.*, 55, 203-207.
- [11] Lisnianski, A., Levitin, G. (2003). *Multistate System Reliability-Assessment, Optimization and Applications*, World Scientific, Singapore.
- [12] Pyke, R. (1961). Markov renewal processes: definitions and preliminary properties, *Ann. Math. Statist.*, 32, 1231-1242.