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Response Surface Methodology in Improving Mean Lifetime

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Abstract: Response surface methodology is a collection of statistical and mathematical techniques useful for developing, improving, and optimizing processes. In this article response surface designs are used for improving the performance of a unit by increasing the reliability for a given mission time or increasing the mean life. A method of estimation of regression parameters of mean lifetime is developed assuming life distribution to follow *Weibull* distribution and experimental errors following independent Normal distribution. In the process, the concept of random effects models is introduced in lifetime distribution. Thus the process parameters can be easily identified and properly set up for improving mean life, and the developed method is illustrated with examples.

Key words: Response Surface, Rotatable Design, Reliability, Mean Life.

AMS 2000 subject classifications. Primary 62K20; Secondary 05B30.

1. INTRODUCTION

In life testing problem, the most important question is how to improve the mean life. Different tools and techniques are used to improve the mean life of a unit or a device or a component. There are two major goals in reliability experiments: *reliability improvement* (i.e., to increase the mean failure time) and *robust reliability* (i.e., to reduce the influence of noise variation on reliability). Applications of design of experiments in improving the quality of a unit or a process is given Taguchi (1986, 1987), Condra (1993), Wu and Hamada

(2000), Myers and Montgomery (2002). We usually assume that a first order model is an adequate approximation to the true surface in a small region of the explanatory variable of the x's. If there is a curvature in the system, then a polynomial of higher degree, such as the second order model must be used. Practitioner is experimenting with a system (perhaps a new system) in which the goal is not to find a *point of optimum response*, but to search for a new *region* in which the process or product is improved.

In reliability theory, the random variable under consideration is the life of a system or a device or a component. A question of fundamental importance is how to improve the mean life or reliability of the system for a given mission time, and both of them mean the same thing in most of the cases such as life distributions are Log-normal, Exponential, Gamma, Extreme value, Weibull etc. Aitkin and Clayton (1980) fitted complex censored survival data to Exponential, Weibull and Extreme value distribution. Of late statistical techniques of design of experiments and concepts of optimality have been used abundantly in reliability theory, in order to improve the mean life of a unit which is expressible as a simple mathematical function of a number of exploratory variables, all of which are non-stochastic and exploratory variables may be purely quantitative type or mixture of qualitative-quantitative type. Much of the reliability literature is concerned with estimating the reliability of an existing product. Response surface methodology (RSM) is a collection of mathematical and statistical techniques useful for analyzing problems where several independent variables influence a dependent variable. See Box and Draper (2007), Myers and Montgomery (2002), Khuri and Cornell (1996). Mukhopadhyay *et al.* (2002) studied lifetime distribution assuming life distribution is Exponential and experimental errors are correlated with associated robust designs (Das; 1997, 2003, 2004), Das and Park (2008).

In this paper we have assumed that the distribution of the performance of a system or life of a component is Weibull. Mean life is conveniently written as a reasonable function (i.e., approximated by a suitable polynomial) of the exploratory variables, taking into consideration the fact that the life as a random variable is non-negative. Very often the form of the function for the mean life can be assumed to be known, with the parameters unknown in the model. These parameters are estimated on the basis of the data realized after conducting an experiment in a planned manner. The planned experimental set up considered in the present article is a response surface design (optimum rotatable first or second order design). In this article, we consider how to improve the reliability i.e., the mean lifetime of a unit or a process through the use of response surface designs. In such experiments, each unit is tested until it fails or is still functioning when the experiment is terminated. If it fails, the experimental response is its *failure time*. If it is still functioning at the termination time t_0, it is said to be *right censored* at t_0. Here we only consider *failure time* as the experimental response. All the earlier authors used response surface designed experiments for improving the quality of a unit without considering the experimental errors and the regression models thus developed are inconsistent. As a result, the estimation procedure is also inconsistent. Some authors (Aalen; 1989, 1994) studied the random effects in frailty model in survival analysis. The use of random effects in the survival analysis models is completely different issues. But it is well known that *experimental error* is the most important part in case of design of experiments. The random effects that used in this article indicates the experimental error which is essential for generating the data set by performing an experiment. Thus the random effects is appropriately used in this lifetime model (in Section 2).

The principal part of this article is the model-building of response i.e., failure time, assuming lifetime follows Weibull distribution, using the technique of RSM. The next part of this article is the analytical techniques, the purpose of which is to optimize the process. The process parameters are suitably set up for increasing mean lifetime. As an illustration of the method developed herein, analysis of two examples (one with simulated data and the other with real experimental data) are given at the end of this article.

2. WEIBULL LIFE DISTRIBUTION

Suppose T denotes the failure time (or lifetime) of a unit. Here T only takes non-negative values. In general, the failure time (T) distributions are commonly used for reliability studies: (a) Lognormal (b) Exponential and (c) Weibull. The exponential distribution is a special case of the Weibull distribution with shape parameter unity.

In this article, we assume lifetime (T) of a component or a system, measured in some unit follows Weibull distribution. Let x_1, x_2, ...,x_n be n controllable explanatory variables in the system which are highly related with lifetime T. The probability density function (p.d.f.) of lifetime T given the vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)'$, x_i's are so called explanatory variables, is of the form

$$f(t) \;=\; \frac{\delta}{\alpha(\mathbf{x})}\;(\frac{t}{\alpha(\mathbf{x})})^{(\delta-\;1)}\;\; exp.[-\,(\frac{t}{\alpha(\mathbf{x})})^{\delta}]\;;\; t \geq 0,\;\; \alpha(\mathbf{x}) \geq 0,\;\; \delta \geq 0. \quad (2.1)$$

In (2.1) only the scale parameter 'α' depends on $\mathbf{x}$. Some properties of this model are noted in Section 6.1 in Lawless (1982). Here 'δ' is a positive shape parameter constant and in particular, the fact that 'δ' does not depend on $\mathbf{x}$ implies proportional hazards for lifetimes and constant variance for log lifetimes of individuals. This assumption is reasonable in many situations; Pike (1966), Peto and Lee (1973) and Nelson (1972), for example, discuss this in specific contexts.

We shall often work with log lifetime: the p.d.f. of $Y = \ln T$, given $\mathbf{x}$ is

$$f_1(y) = \frac{1}{d}\, exp.[\frac{y-R}{d} \;-\; exp.(\frac{y-R}{d})];\; -\infty \;< y <\; \infty, \qquad (2.2)$$

where $R = g(\mathbf{x},\ \boldsymbol{\beta}) = \ln\,\alpha(\mathbf{x})$ and $d = \frac{1}{\delta}$.

Not that $E(T) =\ e^{g(\mathbf{x},\ \boldsymbol{\beta})}\Gamma(d+1)$, where $\boldsymbol{\beta} = (\beta_0, \beta_1, \beta_2, \ldots, \beta_n)'$, vector of unknown regression coefficients, $\mathbf{x} = (x_1, x_2, \ldots, x_n)'$, x_i's are so called explanatory variables, non-stochastic and x_i's may or may not be mathematically related and 'd' is a positive unknown constant involved in the distribution of T, and $g(\mathbf{x},\ \boldsymbol{\beta})$ is usually taken to represent a polynomial in some well defined factors, the number of which may be denoted by $k(< n)$. We take $g(\mathbf{x},\ \boldsymbol{\beta})$ to be a linear or quadratic function in k factors ($n = k+1$ for a linear function and $n = \binom{k+2}{2}$ for a quadratic function). A well laid out experimental plan which in our case is an optimum rotatable first order or second order design may be utilized for realizing the data on T for, say, N given values of $\mathbf{x}$ which indicate N different operating conditions under the plan. Once the data are obtained, the point is to estimate the parameters $\beta_0, \beta_1, \beta_2, \ldots, \beta_n$ in an appropriate manner.

Now it is assumed that given $\mathbf{x}$, T (lifetime) of the component or system follows a Weibull distribution in an ideal situation (i.e., where lifetime distribution is Weibull for examples vacuum tubes (Kao, 1959), ball bearings (Lieblein and Zelen, 1956), electrical insulation (Nelson, 1972), etc.) with mean $g(\mathbf{x},\ \boldsymbol{\beta})\Gamma(d+1)$. This is same as assuming $y = \ln T = g(\mathbf{x},\ \boldsymbol{\beta}) +\ dh$, where h follows the well known Standard Extreme value distribution so that the density of y is given by (2.2). Assuming that the response surface $g(\mathbf{x},\ \boldsymbol{\beta})$ is of first order, (when the factors are independent) we adopt the model **:**

$$y = \ln\ T = \beta_0\ + \beta_1 x_1 + \beta_2 x_2 + \ldots\ldots + \beta_n x_n + dh,$$

where h follows the well known Standard Extreme value distribution with $E(h) = -\nu$ where $\nu = 0.5772\ldots$, is known as Euler's constant and $\mathrm{Var}(h) = \frac{\pi^2}{6}$.

If there is a curvature in the system, then a polynomial of higher degree, such as the second order model must be used as given below (standard response surface designs i.e. all factors appear at quantitative levels):

$$y = \ln\ T = \beta_0\ + \sum_{i=1}^{n} \beta_i x_i + \sum \sum_{i \le j=1}^{n} \beta_{ij} x_i x_j + dh.$$

If in the system there are 'p' qualitative factors (characters) or treatments viz. t_1, t_2, ..., t_p and n quantitative factors viz. x_1, x_2, ..., x_n the mean life of T is given by $e^{g_1(\mathbf{t},\ \mathbf{x},\ \mathbf{a},\ \boldsymbol{\beta})}\Gamma(d+1)$. This is same as assuming $y =$ ln $T = g_1(\mathbf{t},\ \mathbf{x},\ \mathbf{a},\ \boldsymbol{\beta})+\ dh$. The model of a Quality-Quantity type second order response surface model $g_1(\mathbf{t},\ \mathbf{x},\ \mathbf{a},\ \boldsymbol{\beta})$ (Adhikary and Panda (1992)) is given below:

$$y = \ln\ T = \beta_0\ +\ \sum_{j=1}^{p} a_j t_j\ +\ \sum_{i=1}^{n} \beta_i x_i + \sum \sum_{i \le j=1}^{n} \beta_{ij} x_i x_j\ +\ dh.$$

For u-th observation, the model is

$$y_u = \ln\ T_u = \beta_0\ +\ \sum_{j=1}^{p} a_{ju} t_j\ +\ \sum_{i=1}^{n} \beta_i x_{iu} + \sum \sum_{i \le j=1}^{n} \beta_{ij} x_{iu} x_{ju}\ +\ dh_u.$$

where $\beta_0 =$ fixed but unknown constant, $t_j =$ effect due to the j-th qualitative character, say the j-th treatment A_j, j =1,2,..., p; $a_{ju} = 0$ or 1, according as A_j is absent or not on the u-th observation; only one of a_{1u}, a_{2u}, ... , a_{pu} is unity and others are zero for the u-th observation, u = 1,2, ..., N (Adhikary and Panda, 1992), and x_i's, β_i's are same as standard response model.

Below we have considered only for standard response designs but similar treatment can be done for Quality-Quantity type designs. Let us take y as our response variable, when the experiment is conducted to collect the data on the basis of which the parameters $\beta_0, \beta_1, \beta_2, \ldots, \beta_n$ are to be estimated.

As soon as an experiment is conducted, the ideal situation mentioned in connection with the distribution of T (Weibull distribution) or y = ln T (Extreme value distribution) is violated. The experiment introduces some noise factors which may be numerous, some of which may be even undefinable or unidentifiable. The total impact on y of all these noise factors represented by the experimental condition is denoted by 'e'. We write (assuming first order model)

$$y\ =\ \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_n x_n + dh + e,$$

$$\text{or,}\quad y_u\ =\ \beta_0 + \beta_1 x_{1u} + \beta_2 x_{2u} + \ldots + \beta_n x_{nu} + dh_u + e_u;\ 1 \le u \le N, \quad (2.3)$$

where β_i's are unknown regression parameters and x_{iu}'s are experimental levels which are non-stochastic, h_u follows Standard Extreme value distribution, and e_u's represent experimental error given rise to by the noise factors. It is assumed that h_u's are all uncorrelated and h_u and e_u are uncorrelated. But e_u's representing the experimental error with $E(\mathbf{e}) = \mathbf{0}$, $D(\mathbf{e}) = \sigma^2 I_N$ and e_u's are independent and uncorrelated with variance σ^2 may be known or unknown.

In this connection the choice of the experimental levels x_{iu}'s, i.e., $X = ((x_{iu});\ \ 1 \leq i \leq n,\ \ 1 \leq u \leq N)$, the design matrix is the most important part for getting maximum information regarding the unknown regression parameters. The prescription of the proper design matrix is a problem of regression design of experiments (Box and Draper, 2007; Khuri and Cornell, 1996, etc.). Construction and the properties of such designs are under the problem of regression design of experiments. We have developed some designs under these models. In this article, we are not considering the problem of construction of design matrix 'X'. Construction and properties of such designs will be presented in our subsequent articles. Herein we have considered the problem of estimation of regression parameters and model building for mean lifetime 'T', when a design matrix 'X' is given. No mention has so far been made about the distributional assumption of 'e'. The most reasonable assumption of 'e' seems to be normal. In this case, the likelihood or the joint distribution takes a complicated shape and finding maximum likelihood (ML) estimate of the parameters appears to be a formidable task.

Commercial software such as the RELIABILITY procedure in SAS (SAS Institute, 1989) and the Splus functions due to Meekar and Escobar (1998) methods of estimation will not give the estimates of regression parameters in the mean response function in our developed regression model. Hence, for estimating the parameters $\beta_0, \beta_1, \beta_2, \ldots, \beta_n$ the method applied is the Least Squares or some modification of it. As the distributional assumption is not of much importance in applying the method of least squares, we do not bother very much about the distribution of 'e' in further developments of the computational method. The computational method described in the following section for the mentioned model is illustrated with the help of a simulated example.

3. ESTIMATION OF REGRESSION PARAMETERS

The model as explained in (2.3) is

$$y_u \ = \ \beta_1 x_{1u} + \beta_2 x_{2u} + \ldots + \beta_n x_{nu} + dh_u + e_u;\ 1 \leq u \leq N, \tag{3.1}$$

we have $E(\mathbf{e}) = \mathbf{0}$ and $D(\mathbf{e}) = \sigma^2 I_N$ and the other assumptions are given in (2.2). Let $\epsilon_u = dh_u + e_u$. Then $E(\epsilon_u) = -d\nu$, $Var(\epsilon_u) = d^2\frac{\pi^2}{6} + \sigma^2$, and corr. $(\epsilon_u, \epsilon_{u'}) = 0$, for $u \neq u'$. (3.2)

For constant term (intercept), in a design problem, it is normally assumed that $(x_{11}, x_{12}, \ldots, x_{1N}) = (1, 1, \ldots, 1)$. Writing $\mathbf{Y} = X\boldsymbol{\beta} + \boldsymbol{\epsilon}$, where $\mathbf{Y} = (y_1, y_2, \ldots, y_N)'$, $X = ((x_{iu});\ 1 \leq i \leq n,\ 1 \leq u \leq N)$, $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_n)'$, $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, \ldots, \epsilon_N)'$. Now $E(\mathbf{Y}) = X\xi$, where $\boldsymbol{\xi} = (\beta_1 - d\nu, \beta_2, \ldots, \beta_n)'$, $E(\boldsymbol{\epsilon}) = -d\nu\mathbf{J}_N$, $\mathrm{Var}(\boldsymbol{\epsilon}) = (\frac{d^2\pi^2}{6} + \sigma^2)I_N$, where $\mathbf{J}_N$ is the vector of unities of order $N \times 1$.

For N observations, let us define $z_u = y_u - y_N;\ \ 1 \leq u \leq (N-1)$,

$$\begin{aligned} \text{or,}\ \ z_u &= \beta_2(x_{2u} - x_{2N}) + \beta_3(x_{3u} - x_{3N}) + \ldots + \beta_n(x_{nu} - x_{nN}) \\ &+ (dh_u - dh_N) + (e_u - e_N), \quad \text{as} \quad x_{1u} = x_{1N} = 1 \end{aligned}$$

$$\text{or,}\ \ z_u = \beta_2 s_{2u} + \beta_3 s_{3u} + \ldots + \beta_n s_{nu} + \epsilon_u^*, \quad \text{say}, \quad 1 \leq u \leq (N-1), \tag{3.3}$$

where $s_{iu} = (x_{iu} - x_{iN})$; $2 \leq i \leq n$; $\epsilon_u^* = \epsilon_u - \epsilon_N = (dh_u - dh_N) + (e_u - e_N), 1 \leq u \leq (N-1)$,

$$\text{or,}\ \ \mathbf{Z} = S\boldsymbol{\eta} + \boldsymbol{\epsilon}^*, \tag{3.4}$$

where $\mathbf{Z} = (z_1, \ldots, z_{N-1})'$, $S = ((s_{iu});\ 2 \leq i \leq n,\ 1 \leq u \leq (N-1))$, $\boldsymbol{\eta} = (\beta_2, \ldots, \beta_n)'$, $\boldsymbol{\epsilon}^* = (\epsilon_1^*, \ldots, \epsilon_{(N-1)}^*)'$, $E(\epsilon_u^*) = 0, 1 \leq u \leq (N-1);\mathrm{Var}\ (\epsilon_u^*) = 2[\frac{d^2\pi^2}{6} + \sigma^2]; 1 \leq u \leq (N-1)$,

$$\mathrm{Cov.}(\epsilon_u^*,\ \epsilon_{u'}^*) = [\frac{d^2\pi^2}{6} + \sigma^2] \ \text{ and } \ \mathrm{Corr.}(\epsilon_u^*,\ \epsilon_{u'}^*) = \frac{1}{2}; 1 \leq u \neq u' \leq (N-1), \tag{3.5}$$

$$E(\boldsymbol{\epsilon}^*) = \mathbf{0},\ D(\boldsymbol{\epsilon}^*) = 2[d^2\frac{\pi^2}{6} + \sigma^2]W,\ \text{say, with}\ W = [\frac{1}{2}I_{(N-1)} + \frac{1}{2}E_{(N-1)\times(N-1)}].$$

Applying generalized least squares method to the linear model (3.4), we have the following results.

Result 1. From (3.4), the best linear unbiased estimator (BLUE) of $\boldsymbol{\eta}$ is

$$\hat{\boldsymbol{\eta}} = (S'W^{-1}S)^{-1} \ \ (S'W^{-1}\mathbf{Z}). \tag{3.6}$$

Result 2. An unbiased estimator (UE) of

$$2[d^2\frac{\pi^2}{6} + \sigma^2] \ \text{ is } \ \frac{(\mathbf{Z}'W^{-1}\mathbf{Z}) - \hat{\boldsymbol{\eta}}'(S'W^{-1}S)\hat{\boldsymbol{\eta}}}{(N-1) - (n-1)}. \tag{3.7}$$

Case I: When σ^2 is known

Result 3. Using (3.6) in (3.7), for known σ^2 the estimate of 'd' is given by

$$\hat{d^2} = [\frac{1}{2}\ \frac{(Z'W^{-1}Z) - \hat{\eta}'(S'W^{-1}S)\hat{\eta}}{(N-1) - (n-1)} - \sigma^2]/\frac{\pi^2}{6}. \tag{3.8}$$

Result 4. From (3.1), the best linear unbiased estimator (BLUE) of $\boldsymbol{\xi}$ is given by

$$\hat{\boldsymbol{\xi}} = (X'X)^{-1} \ (X'\mathbf{Y}), \tag{3.9}$$

Result 5. An unbiased estimate of

$$[d^2\frac{\pi^2}{6} + \sigma^2] \text{ is } \frac{\mathbf{Y}'\mathbf{Y} - \hat{\boldsymbol{\xi}}' X'\mathbf{Y}}{(N-n)}. \tag{3.10}$$

Using the estimate of "d" as in (3.8), we get $\hat{\xi}_1 = \hat{\beta}_1 - \hat{d}\nu$ or $\hat{\beta}_1 = \hat{\xi}_1 + \hat{d}\nu$, $\hat{\beta}_2 = \hat{\xi}_2, \ldots, \hat{\beta}_n = \hat{\xi}_n$.

Case II: When σ^2 is unknown

Assuming error component is absent in the model (2.3) and using the original distribution (Extreme value distribution) of $y = \ln T$, the maximum likelihood function is

$$L(\boldsymbol{\beta},\ d) = \prod_{u=1}^{N} \frac{1}{d}\ exp.[\frac{y_u - R_u}{d} \ - \ exp.(\frac{y_u - R_u}{d})],$$

where $R_u = \ \sum_{i=1}^{n} \beta_i x_i$ for first order model and for second order model R_u $= \sum_{i=1}^{n} \beta_i x_i \ + \ \sum\sum_{i \le j=1}^{n} \beta_{ij} x_i x_j$.

Maximum likelihood estimates for $d,\ \beta_1,\ \ \beta_2, \ldots\ldots,\ \beta_n$ are obtained from the following maximum likelihood equations (considering first order model), by Newton Raphson method:

$$\sum_{u=1}^{N} e^{p_u} x_{iu} = \sum_{u=1}^{N} x_{iu}; \quad i = 1,\ 2,\ \ldots..,\ n,$$

$$\text{and} \ \sum_{u=1}^{N} p_u\ e^{p_u} \ = \ N \ + \ \sum_{u=1}^{N} p_u, \tag{3.11}$$

where $\ p_u = \ \frac{1}{d}\ (y_u - \ \sum_{i=1}^{n} \beta_i x_{iu})$.

Suppose the estimates are $\hat{\hat{d}},\ \ \hat{\hat{\beta}}_1,\ \ \hat{\hat{\beta}}_2, \ldots\ldots,\ \ \hat{\hat{\beta}}_n$. Hence, the scheme for calculations which can be followed is given hereunder.

(i) Considering the observational equations in (3.1), we get the BLUE of $\boldsymbol{\xi}$ as

$$\hat{\boldsymbol{\xi}} = (X'X)^{-1} \ (X'\mathbf{Y}).$$

Using the estimate of "*d*" as in (3.11), we get $\hat{\xi}_1 = \hat{\beta}_1 - \hat{\hat{d}}\nu$, or $\hat{\beta}_1 = \hat{\xi}_1 + \hat{\hat{d}}\nu$, $\hat{\beta}_2 = \hat{\xi}_2, \ldots, \hat{\beta}_n = \hat{\xi}_n$.

(ii) Calculate $S(\hat{\hat{d}}, \hat{\boldsymbol{\beta}}) = \sum_{u=1}^{N}(y_u + \hat{\hat{d}}\,\nu - \hat{\beta}_1 x_{1u} - \hat{\beta}_2 x_{2u} - \ldots\ldots\ldots - \hat{\beta}_n x_{nu})^2$, and $S(\hat{\hat{d}}, \hat{\hat{\boldsymbol{\beta}}}) = \sum_{u=1}^{N}(y_u + \hat{\hat{d}}\,\nu - \hat{\hat{\beta}}_1 x_{1u} - \hat{\hat{\beta}}_2 x_{2u} - \ldots\ldots\ldots - \hat{\hat{\beta}}_n x_{nu})^2$.

It is seen from numerical computation (with simulated data) that $S(\hat{\hat{d}}, \hat{\boldsymbol{\beta}}) < S(\hat{\hat{d}}, \hat{\hat{\boldsymbol{\beta}}})$.

(iii) Unbiased estimate of σ^2 can be obtained from (3.7) as

$$\hat{\sigma}^2 = [\frac{1}{2}\,\frac{(\mathbf{Z}'W^{-1}\mathbf{Z}) - \hat{\boldsymbol{\eta}}'(S'W^{-1}S)\hat{\boldsymbol{\eta}}}{(N-1)-(n-1)} - \hat{\hat{d}}^2\,\frac{\pi^2}{6}] = \hat{\sigma}_1^2, \text{ say.} \tag{3.13}$$

Also an unbiased estimate of σ^2 can be obtained from (3.10) as

$$\hat{\sigma}^2 = [\frac{\mathbf{Y}'\mathbf{Y} - \hat{\boldsymbol{\xi}}'X'\mathbf{Y}}{(N-n)} - \hat{\hat{d}}^2\,\frac{\pi^2}{6}] = \hat{\sigma}_2^2, \text{ say.} \tag{3.14}$$

It is seen from numerical computation (with simulated data) that $\hat{\sigma}_1^2 \cong \hat{\sigma}_2^2$.

(iv) Better estimate of '*d*', say, '$\hat{d}_0$' can be obtained from

$$\sum_{u=1}^{N} e^{\frac{1}{d}\,(y_u - \hat{\xi}_1 - d\nu - \sum_{i=2}^{n}\hat{\beta}_i x_{iu})} = N,$$

by computer programming, where $\hat{\xi}_1,\ \hat{\beta}_i;\ i = 2, 3, ..., n$ are as in the scheme (i).

Final estimates are obtained by searching through computer programme. Final estimate of "*d*" can be obtained from scheme (iv) by iterative method and the final estimates of $\beta_1,\ \beta_2,\ .....,\beta_n$, can be obtained from the scheme (i), and also the estimate of σ^2 can be obtained from (3.14), using the estimate of d, say '$\hat{d}_0$' as in the scheme (iv). Here the estimates of $\beta_2,\ \beta_3,\ .....,\beta_n$ are the BLUEs. Also it is seen from numerical computation (with simulated data) that $S(\hat{d}_0, \hat{\boldsymbol{\beta}}) < S(\hat{\hat{d}}, \hat{\boldsymbol{\beta}}) < S(\hat{\hat{d}}, \hat{\hat{\boldsymbol{\beta}}})$, where $S(\hat{d}_0, \hat{\boldsymbol{\beta}}) = \sum_{u=1}^{N}(y_u + \hat{d}_0\,\nu - \hat{\beta}_1 x_{1u} - \hat{\beta}_2 x_{2u} - \ldots\ldots\ldots - \hat{\beta}_n x_{nu})^2$. This method of estimation as developed herein gives better estimates of unknown regression parameters and error variance than the classical likelihood method of estimation which is not relevant in this situation.

4. ANALYSIS OF TWO EXAMPLES

Example 4.1. The example (with simulated data) considered is the performance or quality of metalized glass plates which involves various factors. Main factors (i.e. explanatory variables) taken may be designated as $x_1 =$ Argon sputter pressure, $x_2 =$ Target current, $x_3 =$ Back ground pressure, $x_4 =$ Plate cool-down time, $x_5 =$ Glow discharge time. Let 'T' be the performance or quality or thickness of metalized glass plates in some convenient unit. The study conducted is to estimate the unknown parameters involved in the mean response function of 'T' and error variance and to locate the levels of the parameters so as to reduce the unevenness and imperfections in the metalized glass plates. Five factors as above are considered for this experiment.

Appropriate change of origin and scale is used for each exploratory variable so that it lies between -1 and $+1$ (the range within which the experimentation is conducted). We assume

$$y \;=\; \ln\, T \;=\; \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 x_5 + dh + e, \qquad (4.1)$$

Regression coefficients $\boldsymbol{\beta} = (\beta_0,\, \beta_1,\, \beta_2,\, \beta_3,\, \beta_4,\, \beta_5)$, $\boldsymbol{\xi} = (\xi_0,\, \xi_1,\, \xi_2,\, \xi_3,\, \xi_4,\, \xi_5)$ and $\boldsymbol{\eta} = (\eta_1,\, \eta_2,\, \eta_3,\, \eta_4,\, \eta_5)'$. In the absence of real life data we generate observations according to the formula (4.1) with $\beta_0 = 65.0$, $\beta_1 = 4.2$, $\beta_2 = 3.8$, $\beta_3 = -2.2$, $\beta_4 = -5.2$, $\beta_5 = 4.6$, $d = 2$ and $\sigma^2 = \mathrm{Var}(\text{ e }) = 2.25$, i.e. $\sigma = 1.5$, using the design matrix 'X' as given below. The observations so obtained are given as the column vector '$\mathbf{Y}$', where $\mathbf{Y} =$ (73.952779, 57.775263, 68.831522, 43.436004, 63.606459, 78.075309, 69.508753, 69.784374, 69.704526, 51.007922, 65.249739, 45.417056, 71.548210, 77.622893, 69.810310, 61.420423)$'$.

$$X = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & 1 & 1 & -1 \\
1 & 1 & 1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 & -1 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 \\
1 & -1 & -1 & 1 & -1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & 1 & 1 & -1 \\
1 & 1 & 1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 & -1 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 \\
1 & -1 & -1 & 1 & -1 & 1
\end{pmatrix}$$

With the observation vector '**Y**' and the corresponding design matrix 'X' assumed to be given and the model assumed as (4.1), the method of estimation explained in Section 3 is applied to the simulated data, the final estimates obtained are as follows:

For Case I:

$\sigma = 1.5$ (known) , $\hat{d} = 2.016580$, $\hat{\beta}_0 = 65.960939$, $\hat{\beta}_1 = 4.229565$, $\hat{\beta}_2 = 3.114698$,
$\hat{\beta}_3 = -2.438243$, $\hat{\beta}_4 = -5.375120$, $\hat{\beta}_5 = 5.783224$, $S(\hat{d}, \hat{\boldsymbol{\beta}}) = 125.325965$.

For Case II:

$\hat{\sigma} = 1.924180$, $\hat{d}_0 = 2.416580$, $\hat{\beta}_0 = 66.211111$, $\hat{\beta}_1 = 4.229565$, $\hat{\beta}_2 = 3.114698$, $\hat{\beta}_3 = -2.438243$, $\hat{\beta}_4 = -5.375120$, $\hat{\beta}_5 = 5.783224$, $S(\hat{d}, \hat{\boldsymbol{\beta}}) = 133.163453$, $\hat{\hat{\sigma}} = 2.301571$, $\hat{\hat{d}} = 2.658362$, $\hat{\hat{\beta}}_0 = 66.528643$, $\hat{\hat{\beta}}_1 = 4.636682$, $\hat{\hat{\beta}}_2 = 3.541864$, $\hat{\hat{\beta}}_3 = -1.970335$, $\hat{\hat{\beta}}_4 = -5.662045$, $\hat{\hat{\beta}}_5 = 5.931915$, $S(\hat{\hat{d}}, \hat{\hat{\boldsymbol{\beta}}}) = 149.064935$.

In this case we have to select the values of x_1, x_2, and x_5 as large as possible, and the values of x_2 and x_3 as small as possible to maximize the mean life.

Thus the estimated regression coefficients (under the method developed herein) are very close to the true values with the help of which the data have been simulated. The estimates of σ^2 and 'd' although reasonable are not found to be so good. However, if we can guess the right value of $\sigma^2 = 2.25$, i.e., $\sigma = 1.5$, the estimate of $d = 2.016580$ moves closer to the true value 2.00. However with only 16 observations we do not hope to estimate both 'd' and σ^2 very accurately.

A real life data are given below.

Example 4.2. The data in Table 4.1 [from Myers and Montgomery(2002), p. 68] show the number of cycles to failure of Worsted gain (t) and three factors defined as follows:

Length of test specimen (mn): $x_1 = \frac{Length\ -\ 300}{50}$, Amplitude of load cycle (MM):
$x_2 = (Length - 9)$, Load (grams): $x_3 = \frac{Length - 300}{50}$.

Table 4.1. Worsted Yarn Data

$RunNumbers:$	1	2	3	4	5	6	7	8	9	10
$Length, x_1$	-1	0	1	-1	0	1	-1	0	1	-1
$Amplitude, x_2$	-1	-1	-1	0	0	0	1	1	1	-1
$Load, x_3$	-1	-1	-1	-1	-1	-1	-1	-1	-1	0
$Cycles\ to\ Failure, t$	674	1414	3636	338	1022	1368	170	442	1140	370

$RunNumbers:$	11	12	13	14	15	16	17	18	19	20
$Length, x_1$	0	1	-1	0	1	-1	0	1	-1	0
$Amplitude, x_2$	-1	-1	0	0	0	1	1	1	-1	-1
$Load, x_3$	0	0	0	0	0	0	0	0	1	1
$Cycles\ to\ Failure, t$	1198	3184	266	620	1070	118	332	884	292	634

$RunNumbers:$	21	22	23	24	25	26	27
$Length, x_1$	1	-1	0	1	-1	0	1
$Amplitude, x_2$	-1	0	0	0	1	1	1
$Load, x_3$	1	1	1	1	1	1	1
$Cycles\ to\ Failure, t$	2000	210	438	566	90	220	360

These factors form a 3^3-factorial experiment. This experiment will support a complete second order polynomial. Model:

$$y = \beta_0+\beta_1x_1+\beta_2x_2+\beta_3x_3+\beta_{11}x_1^2+\beta_{22}x_2^2+\beta_{33}x_3^2+\beta_{12}x_1x_2+\beta_{13}x_1x_3+\beta_{23}x_2x_3+dh+e$$

$\hat{\sigma}$ =0.183477, $\hat{d}_0$ =0.143652, $\hat{\beta}_0$=6.415563, $\hat{\beta}_1$ = 0.824804, $\hat{\beta}_2$ =$-$0.630992, $\hat{\beta}_3$ =$-$0.384913, $\hat{\beta}_{11}$ =$-$0.093281, $\hat{\beta}_{22}$ =0.039378, $\hat{\beta}_{33}$=$-$0.075034, $\hat{\beta}_{12}$=$-$0.038240, $\hat{\beta}_{13}$=$-$0.057046, $\hat{\beta}_{23}$=$-$0.020832, $S(\hat{d}, \hat{\boldsymbol{\beta}})$ =1.890458, $\hat{\hat{\sigma}}$ = 4.0430468, $\hat{\hat{d}}$ =0.770329, $\hat{\hat{\beta}}_0$=7.092225, $\hat{\hat{\beta}}_1$ = 2.840451, $\hat{\hat{\beta}}_2$ =0.205108, $\hat{\hat{\beta}}_3$ =$-$0.714751, $\hat{\hat{\beta}}_{11}$ =$-$2.420932, $\hat{\hat{\beta}}_{22}$ =$-$1.047342, $\hat{\hat{\beta}}_{33}$=$-$1.607445, $\hat{\hat{\beta}}_{12}$=1.240965, $\hat{\hat{\beta}}_{13}$=0.532387, $\hat{\hat{\beta}}_{23}$=$-$0.265243, $S(\hat{\hat{d}}, \hat{\boldsymbol{\beta}})$ = 294.493125.

In this case we have to select the value of x_1 as large as possible, and the values of x_2 and x_3 as small as possible to maximize the mean life. From these two numerical examples, it is clear that the estimates obtained by following the new method as developed herein are better than the estimates of likelihood method.

5. CONCLUDING REMARKS

In this article, we have considered the life distribution of a component or a system is Weibull distribution. In general, the mean life time of a component or a system is a reasonable function of the explanatory variables. The functional form may be known but the parameters involved in it are unknown. Using the RSM, the mean life time can be improved due to its optimizing property. Whenever the RSM is used for improving the mean life, the original distribution of the lifetime will be changed. As a result some random effects models are introduced. Under this situation ML method for estimating the unknown parameters is inconsistent. In this article we have developed a new method of estimation which gives BLUE of *all* the regression parameters except the intercept. Mean life time can be efficiently estimated using the method which is developed herein and as a result the process parameters can be exactly setup so that the mean lifetime of the component can be improved. In the process random effects models is introduced in lifetime distribution.

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