

ON ASYMPTOTIC BEHAVIOR OF CERTAIN RECURSIONS WITH RANDOM INDICES OF LINEAR GROWTH

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Abstract. Let η_1, η_2, \dots be independent copies of a random variable η with distribution concentrated on $(0, 1)$ and not supported by a geometric sequence. Consider a distributional recursion $X_1 := a$ and $X_n \stackrel{d}{=} X_{[n\eta]+1} + 1$, $n = 2, 3, \dots$, where η is independent of X_2, \dots, X_n , $a \geq 0$ is given, and $[\cdot]$ denotes the integer part. We point out a necessary and sufficient condition which ensures that X_n , properly normalized and centered, possesses a non-degenerate and proper weak limit. Also we provide the complete description of possible limiting laws and normalizations leading to them. A key observation behind the result is that the weak asymptotic behavior of X_n is the same as that of $\inf\{k \geq 1 : n\eta_1 \cdots \eta_k \leq 1\}$. As a consequence, the renewal theory can be brought into play.

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1 Introduction and main results

Assume that the marginal distributions of non-negative random variables $\{X_n : n \in \mathbb{N}\}$ are defined recursively by

$$X_1 := a \quad \text{and} \quad X_n \stackrel{d}{=} X_{J_n} + 1, \quad n = 2, 3, \dots, \quad (1.1)$$

where finite $a \geq 0$ is given, for $n = 2, 3, \dots$ the random index J_n is independent of X_2, \dots, X_n and takes values in the set $\{1, \dots, n\}$, $\mathbb{P}\{J_n = 1\} > 0$, $\mathbb{P}\{J_n = n\} < 1$, and $\stackrel{d}{=}$ denotes the equality of distributions.

Recursions of this type and the like arise in different areas of applied probability, the most prominent examples being related to coalescents with (simultaneous) multiple collisions [10, 11, 12, 13], random trees [3, 12, 16, 17], random regenerative compositions [2, 4, 6, 7, 8, 9], random walks with a barrier [13, 14], and absorption times of non-increasing Markov chains [15, 19].

It seems that the most general description of recursions (1.1), already hinted at by the latter application, is that, for $n = 2, 3, \dots$, $X_n - a$ has the same distribution as the absorption time at point 1 of a non-increasing Markov chain $\{R_k^{(n)} : k \in \mathbb{N}_0\}$ with $R_0^{(n)} = n$ and $R_1^{(n)} \stackrel{d}{=} J_n$.

Let η be a random variable with distribution μ concentrated on $(0, 1)$ and not supported by a geometric sequence (in other words, the distribution of $|\log \eta|$ is non-lattice). The aim of the paper is to provide the complete information about the weak limiting behavior of recursions (1.1) under the assumption

$$J_n \stackrel{d}{=} [n\eta] + 1, \tag{1.2}$$

i.e. $\mathbb{P}\{J_n = k\} = \mu((k-1)/n, k/n)$, $k = 1, 2, \dots, n$. Since the so defined distribution of J_n satisfies

$$\frac{J_n}{n} \xrightarrow{d} \eta \text{ as } n \rightarrow \infty,$$

it is natural to call recursion (1.1) satisfying (1.2) *a recursion with random indices of linear growth*, hence the title of the paper.

We are ready to state our result.

Theorem 1.1. *Let the distribution of X_n satisfy (1.1) and (1.2). The following assertions are equivalent.*

- (i) *There exist sequences of numbers $\{a_n, b_n : n \in \mathbb{N}\}$ with $a_n > 0$ and $b_n \in \mathbb{R}$ such that, as $n \rightarrow \infty$, $(X_n - b_n)/a_n$ converges weakly to a non-degenerate and proper probability law.*
- (ii) *Either the distribution of $(-\log \eta)$ belongs to the domain of attraction of a stable law, or $\mathbb{P}\{-\log \eta > x\}$ slowly varies at ∞ .*

Set $m := \mathbb{E}(-\log \eta)$ and $\sigma^2 := \text{var}(\log \eta)$.

(1) *If $\sigma^2 < \infty$, then, with $b_n := m^{-1} \log n$ and $a_n := (m^{-3} \sigma^2 \log n)^{1/2}$, the limiting law is standard normal.*

(2) *If $\sigma^2 = \infty$ and*

$$\int_x^1 \log^2 y \mu(dy) \sim L(-\log x) \text{ as } x \rightarrow 0,$$

for some L slowly varying at ∞ , then, with b_n as in (1) and $a_n := m^{-3/2} c_{[\log n]}$, where c_n is any sequence satisfying $\lim_{n \rightarrow \infty} n c_n^{-2} L(c_n) = 1$, the limiting law is standard normal.

(3) *Assume that the relation*

$$\mu(0, x) \sim (-\log x)^{-\alpha} L(-\log x) \text{ as } x \rightarrow 0, \tag{1.3}$$

holds with L slowly varying at ∞ and $\alpha \in [1, 2)$, and assume that $m < \infty$ if $\alpha = 1$, then, with $b_n := m^{-1} \log n$ and $a_n := m^{-(\alpha+1)/\alpha} c_{[\log n]}$, where c_n is any

sequence satisfying $\lim_{n \rightarrow \infty} n c_n^{-\alpha} L(c_n) = 1$, the limiting law is α -stable with characteristic function

$$t \mapsto \exp\{-|t|^\alpha \Gamma(1 - \alpha)(\cos(\pi\alpha/2) + i \sin(\pi\alpha/2) \operatorname{sgn}(t))\}, \quad t \in \mathbb{R}.$$

(4) Assume that $m = \infty$ and the relation (1.3) holds with $\alpha = 1$. Let c be any positive function satisfying $\lim_{x \rightarrow \infty} xL(c(x))/c(x) = 1$ and set $\psi(x) := x \int_{\exp(-c(x))}^1 \mu(0, y)/y dy$. Let b be any positive function satisfying $b(\psi(x)) \sim \psi(b(x)) \sim x$. Then, with $b_n := b(\log n)$ and $a_n := b(\log n)c(b(\log n))/\log n$, the limiting law is 1-stable with characteristic function

$$t \mapsto \exp\{-|t|(\pi/2 - i \log |t| \operatorname{sgn}(t))\}, \quad t \in \mathbb{R}.$$

(5) If the relation (1.3) holds with $\alpha \in [0, 1)$ then, with $b_n = 0$ and $a_n := \log^\alpha n / L(\log n)$, the limiting law is the scaled Mittag-Leffler distribution θ_α (exponential, if $\alpha = 0$) characterized by moments

$$\int_0^\infty x^n \theta_\alpha(dx) = \frac{n!}{\Gamma^n(1 - \alpha) \Gamma(1 + n\alpha)}, \quad n \in \mathbb{N}.$$

Remark 1.1. Theorem 1.1 is a generalization of the well-known result on asymptotic behavior of recursions (1.1) satisfying (1.2) with μ being the uniform distribution on $[0, 1]$ (this case is covered by part (1) of the theorem). Under this condition on μ , X_{n-1} has the same distribution as the sum of $n - 1$ independent indicators [18]. Therefore, by the Lindeberg-Feller central limit theorem, as $n \rightarrow \infty$, $(X_n - \log n)/\sqrt{\log n}$ weakly converges to the standard normal law. The distribution of X_{n-1} appears in a number of diverse applications. For example, X_{n-1} has the same distribution as (a) the number of upper records in a sample of size n from a continuous distribution, (b) the number of cycles in random permutations of n objects, (c) the number of collision events that take place in the $\beta(3, 1)$ coalescent restricted to the set $\{1, 2, \dots, n\}$ until there is just a single block. Points (a) and (b) along with many other applications and references can be found in [1], point (c) was remarked in [5].

2 Proof of Theorem 1.1

Replacing X_n by $X_n - a$ we can and do assume that $a = 0$.

Let η_1, η_2, \dots be independent copies of η and let $\{S_n : n \in \mathbb{N}_0\}$ be a zero-delayed random walk with a step distributed like $(-\log \eta)$. Set

$$M_n := \inf\{k \geq 0 : n\eta_1\eta_2 \cdots \eta_k \leq 1\} = \inf\{k \geq 0 : S_k \geq \log n\}, \quad n \in \mathbb{N}.$$

For fixed $k, i \in \mathbb{N}$ define the sequence

$$R_{0,k}(i) := k, \quad R_{j,k}(i) := [R_{j-1,k}(i)\eta_{i+j-1}] + 1, \quad j = 1, 2, \dots,$$

and set

$$U_k(i) := \inf\{j \geq 0 : R_{j,k}(i) = 1\}.$$

When $i = 1$, we will write just $R_{j,k}$ and U_k . Notice that, for fixed $i \in \mathbb{N}$, the sequences $\{U_k(i) : k \in \mathbb{N}\}$ and $\{U_k : k \in \mathbb{N}\}$ have the same distribution and moreover we have

$$U_1 = 0, \quad U_n = U_{[n\eta_1]+1}(2) + 1 \stackrel{d}{=} U_{[n\eta]+1} + 1, \quad n = 2, 3, \dots,$$

where on the right-hand side η is independent of $\{U_n : n \in \mathbb{N}\}$. Thus, we have managed to provide a pathwise construction of a random sequence the n th element of which has the same marginal distribution as X_n .

By Proposition 8.1 in [6], the claim of the theorem holds with X_n replaced by M_n . The idea of the subsequent proof is to show that the weak asymptotic behavior of U_n (equivalently of X_n) coincides with that of M_n . To this end, notice the following a.s. equality

$$U_n = M_n + U_{R_{M_n,n}}(M_n + 1), \quad n \in \mathbb{N}.$$

Our idea would be realized if we could show that the sequence $\{\mathbb{E}R_{M_n,n} : n \in \mathbb{N}\}$ was bounded. Indeed, the latter would imply that

$$\lim_{n \rightarrow \infty} \frac{U_n - M_n}{y_n} = 0 \quad \text{in probability}$$

for any sequence $\{y_n : n \in \mathbb{N}\}$ such that $\lim_{n \rightarrow \infty} y_n = +\infty$.

With $k \in \mathbb{N}$ fixed we have $R_{1,k} \leq k\eta_1 + 1$ a.s. and

$$R_{i,k} \leq k\eta_1 \cdots \eta_i + 1 + \sum_{j=2}^i \eta_j \eta_{j+1} \cdots \eta_i, \quad i = 2, 3, \dots \quad \text{a.s.}$$

Therefore, almost surely on the event $\{M_n \geq 2\}$

$$R_{M_n,n} \leq 2 + \sum_{j=2}^{M_n} \eta_j \eta_{j+1} \cdots \eta_{M_n} =: 2 + \Theta_n, \quad n \in \mathbb{N}.$$

Furthermore, for $k = 2, 3, \dots$

$$\Theta_n 1_{\{M_n=k\}} \leq \frac{1}{n} (e^{S_1} + e^{S_2} + \dots + e^{S_{k-1}}) 1_{\{M_n=k\}} \quad \text{a.s.},$$

which implies that

$$\Theta_n 1_{\{M_n \geq 2\}} \leq \frac{1}{n} \sum_{k=1}^{\infty} e^{S_k} 1_{\{M_n > k\}} = \frac{1}{n} \sum_{k=1}^{\infty} e^{S_k} 1_{\{S_k < \log n\}} \quad \text{a.s.}$$

By the key renewal theorem,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E} \sum_{k=0}^{\infty} e^{S_k} 1_{\{S_k < \log n\}} &= \lim_{n \rightarrow \infty} \int_0^{\log n} e^{-(\log n - x)} \left(\sum_{k=0}^{\infty} \mathbb{P}\{S_k \in dx\} \right) \\ &= \frac{1}{\mathbb{E}(-\log \eta)}, \end{aligned}$$

no matters finite or infinite the value of $\mathbb{E}(-\log \eta)$ is. Therefore, as $n \rightarrow \infty$, $\mathbb{E}R_{M_n, n} 1_{\{M_n \geq 2\}} = O(1)$. Also, $\lim_{n \rightarrow \infty} \mathbb{E}R_{M_n, n} 1_{\{M_n = 1\}} = 0$. The last two relations prove the boundedness of $\{\mathbb{E}R_{M_n, n} : n \in \mathbb{N}\}$. The proof is complete.

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