ISSN 0974-3235

ProbStat Forum, Volume 01, July 2008, Pages 26–49

A Review of Repairable Systems and Point Process Models

K. Muralidharan

Department of Statistics, Faculty of Science

M. S. University of Baroda, Vadodara 390 002, India.

e-mail: *lmv_murali@yahoo.com*

Abstract. Repairable systems are those systems which in the event of a failure can be repaired, for example, by replacing a component or by repairing a component etc. In some cases, the reliability of a system, after a repair, returns to the same state as before the failure. Thus, models for repairable systems must be able to describe the occurrence of events in time, and are inherently different from models for non-repairable systems. The non-homogeneous Poisson process (NHPP) and the renewal process (RP) are the commonly used models for repairable systems. The other point process models to describe repairable systems are modulated renewal process of Cox (1972) inhomogeneous gamma process (IGP) and modulated gamma process (MGP) introduced by Berman (1981), the modulated power law process (MPLP) by Lakey and Rigdon (1992), trend renewal process (TRP) by Lindqvist et al (2003), exponentiated power law process (EPLP) by Muralidharan and Shah (2006a) etc. We review various point process models exclusively in parametric set up and provide the methods and applications of this models under different repairable policies. Some examples are also discussed.

Mathematics Subject Classification. 82B21, 94A30.

Keywords. Intensity function; Repairable systems; Reliability; Predictive likelihood; Inhomogeneous gamma process; Power law process; Renewal process; Test for trend.

1 Introduction

A system is a collection of two or more parts witch is designed to perform one or more functions. A system can be either non-repairable or repairable. A non-repairable system (electric bulbs, thermometer etc) in the event of failure can not be brought back to working condition as it is discarded for the first time it ceases to perform satisfactorily. Arjas (1981) and Greenwood (1981) consider non-repairable systems with redundant non-repairable subsystems, using point process techniques. A repairable system (machines, industrial plants, software etc.) is a system which, after failing to perform one or more of its functions satisfactorily, can be restored to fully satisfactory performance by any method, other than replacement of the entire system. In some cases, the reliability of a system, after a repair, returns to the same state as before the failure. Thus, models for repairable systems must be able to describe the occurrence of events in time, and are inherently different from models for non-repairable components. Once a system experiences a failure, different repair strategies have different influences on the system reliability, usually defined as the probability of no failures in time intervals.

The models which have been applied to repairable systems are stochastic point processes and differential equations. A stochastic point process is a mathematical model for a physical phenomenon characterized by highly localized events (failure) distributed randomly in continuum (time). The homogeneous Poisson process (HPP), non-homogeneous Poisson process (NHPP) and the renewal process (RP), the superimposed renewal process (SRP) are the commonly used models for repairable systems. For an NHPP, the probability of a failure in a small time interval depends on the system, and not on the previous pattern of failures. For a RP, the times between failures are i.i.d. The other point process models to describe repairable systems are modulated renewal process (MRP) of Cox (1972) inhomogeneous gamma process (IGP) introduced by Berman (1981), the modulated power law process (MPLP) by Lakey and Rigdon (1992), trend renewal process (TRP) by Lindqvist et.al. (2003), exponentiated power law process (EPLP) by Muralidharan and Shah (2006a) etc. For a class of point process models based on differential equations, one may refer to Ruggeri (2006). We stress on stochastic point process models in this review article.

In section 2, we describe some useful characteristics and definitions of repairable systems. Section 3 reviews some existing point process models and their applications in repairable systems. The tests for trend and their statistical significance are discussed in section 4. Some examples and their choices of models are discussed in the last section.

2 Repairable System Characteristics

It is observed that a repairable system might be subject to many repair and adjustments including well directed kicks. If X is a random variable denoted as the time between successive failures of a part and $f(x)$ as its corresponding probability law, then the mean time to failure (MTTF) is $E(x)$. The corresponding mean residual life (MRL) is $E[X - x|X > x]$. Let $N(s,t)$ denote the number of failures in the interval $(s,t]$ and $N(t)$ is the number of failures in $(0,t]$. We now define some important properties of repairable systems.

Definition 2.1 *Independent increments: A counting process* $\{N(t), t \geq 0\}$ *is said to have independent increments if for all* $t_0 < t_1 < \cdots < t_k$, $k =$

$2, 3, \ldots,\ N(t_1) - N(t_0),\ N(t_k) - N(t_{k-1})$ *are independent random variables.*

Definition 2.2 *Stationary increments: A counting process* $\{N(t), t \geq 0\}$ *is said to have stationary increments if for any two points* $0 \leq s \leq t$*, and any* $\delta > 0$*, the random variables* $N(t) - N(s)$ *and* $N(t+\delta) - N(s+\delta)$ *are identically distributed.*

Definition 2.3 *Stationarity of a point process: A point process is said to be stationary if it has stationary increments.*

In general, a stationary point process does not have a stationary sequence of interarrival times. For example, a renewal process is defined as a sequence of independent and identically distributed inter-arrival times.

Definition 2.4 *Rate of occurrence of failures (ROCOF): It is the reciprocal of the mean time between failures. i.e.* $v(t) = \frac{1}{E(X)}$.

Remark: Under some conditions, the ROCOF of a Poisson process is asymptotically proportional to the reciprocal of the mean forward recurrence time w_t.

Definition 2.5 *Intensity functions of a stochastic point process: If* H_t *denote the history of the failure process up to, but not including t and if simultaneous failures are assumed to be not possible, then the intensity function of a point process is defined as*

$$\begin{aligned}\lambda(t|H_t) &= \lim_{\Delta t \to 0} \frac{\textit{Pr[a failure in the interval } (t, t+\Delta t)|H_t\textit{]}}{\Delta t} \\ &\cong \lim_{\Delta t \to 0} \frac{Pr[N(t, t+\Delta t) - N(t) \geq 1]}{\Delta t}\end{aligned}$$

Some post data characteristics of the processes are the estimates of the current intensity function say $\hat{\lambda}(t|H_t)$, the estimate of the mean value function say $\hat{\Lambda}(t|H_t) = \int_0^t \hat{\Lambda}(t|H_t)dt$ and generalized residuals say $\hat{e}_i = \int_{t_i-1}^{t_1} \hat{\Lambda}(t|H_t)dt$ etc.

3 Point Process Models

3.1 Non-homogeneous Poisson Process (NHPP)

A counting process $\{N(t), t \geq 0\}$ is said to be an HPP with intensity $\lambda(t)$ if

(i) $N(0) = 0$;

(ii) $\{N(t), t \geq 0\}$ has independent increments;

(iii) $P[N(t, t+\delta) = 1] = \lambda(t)\delta + o(\delta)$,

(iv) $P[N(t, t+\delta) \geq 2] = o(\delta)$.

The reliability function of the process is $R(s,t) = \exp[-\delta\{t-s\}]$. This reliability could be constant over time. If the intensity function $\lambda(t)$ is a constant then the process reduces to a homogeneous Poisson process (HPP). An HPP is the orderly stochastic point process with stationary, independent increments. In reliability studies, the HPP has enjoyed some success as a model for repairable system failure over portions of system life, but since the arrival rate is constant, it cannot be adequate to model wear out or growth of the system or to set policies for maintenance, overhaul or trade-in. As an application, Cagno et.al. (1998, 2000) have considered failures in a city network of old case gas pipes and modeled the failures using an HPP as the pipes were not subject to ageing.

Note that HPP is a sequence of independent and identically exponentially distributed X_i's, whereas under NHPP, X_i's are neither independent nor identically distributed. An NHPP is characterized through its cumulative intensity function $\Lambda(t) = \int_0^t \lambda(x)dx$, or by its time derivative $\lambda(t)$, which gives the instantaneous rate of failures. The shape of $\lambda(t)$; increasing, decreasing or otherwise, provides information about the time-dependant nature of the reliability of the system. The NHPP models are primarily used for modeling and analysis of failure data for repairable systems, for which repaired units are in exactly the condition as they were just before the failure ("bad-as-old"). A general treatment of repairable system reliability is provided in Ascher and Feingold (1969). Thompson (1988) has studied some point process models with applications to safety and reliability. Systems are subject to reliability decay or growth and can alternate between them at some change points. Sequential detection of bugs in software, without introduction of new ones implies reliability growth all over the testing phase. Conversely, there are systems subject to many early failures, and then a decrease in them is followed by a long period of rare failures and by a final period with an increasing number of failures. In reliability and life testing this phenomenon is often described by bath tub type models.

Consider a NHPP with intensity function $\lambda(t)$ and we observe the system up to time s and let n be the number of failures, occurred at times $t_1 < t_2 < \cdots < t_n$. Then the likelihood function is given by

$$L(\theta; t) = \prod_{i=1}^{n} \lambda(t_i) \exp\{-\int_0^s \lambda(t)dt\}. \tag{3.1}$$

The above set up can be studied in two possible ways: observation of the system up to a given time or until the n-th failure occurs. The former case is called time truncation, whereas the latter is called failure truncation with $s = t_n$. Although they are conceptually different, the two experiments lead to the same estimates in a Bayesian framework, whereas some differences are possible following a frequentist approach (See Ruggeri, 2006, for details). The other forms of NHPP discussed in literature are the log-linear process, the bounded intensity process, the bathtub-type intensity process and so on. One may refer to Attardi and Pulcini (2005) and Guida and Pulcini (2006) for more details.

3.2 Power law process (PLP)

The widely used NHPP is the Power law process (PLP), sometimes called Weibull process whose intensity is given by

$$\lambda(t) = (\beta/\theta)(t/\theta)^{\beta-1},\ \theta > 0,\ \beta > 0,\ t > 0. \tag{3.2}$$

For $\beta = 1$ the process reduces to a homogeneous Poisson Process (HPP). Otherwise, a PLP provides a model for a system whose reliability changes as it ages. If $\beta > 1$, it models a deteriorating system and when $\beta < 1$, it provides a model for reliability growth. Crow (1974, 1982) discuss applications of this model and provide some inference procedures. Finkelstein (1976) discuss the confidence bounds on the parameters of the Weibull process. Lee and Lee (1978), Bain and Engelhardt (1980) have discussed the point estimation and proposed tests for the parameters of (3.2). Rigdon and Basu (1989), Baker (1996), Jani et.al. (1997) and Muralidharan (1999) have proposed various tests for Weibull process. Gaudoin et.al. (2003) have proposed goodness-of-fit test for the Weibull process based on the Duane plot. More recently, Zhao and Wang (2005) and Gaudoin et.al. (2006) have also studied various inferences on the parameters of the above process. The papers which discuss inferences based on Bayesian set up are due to Soland (1969), Calabria and Pulcini (1997), Sen (2002), Pievatolo and Ruggeri (2004) and references contained therein.

For failure truncated sampling, the maximum likelihood estimators of the parameters are $\hat{\theta} = \frac{t_n}{n^{1/\beta}}$ and $\hat{\beta} = \frac{n}{\sum_{i=1}^{n-1} \ln(t_n/t_i)}$. The corresponding estimates for time truncated sampling at time point T are $\hat{\theta} = \frac{T}{n^{1/\beta}}$ and $\hat{\beta} = \frac{n}{\sum_{i=1}^{n} \ln(T/t_i)}$. The estimate of the mean value function is $\hat{\Lambda}(t) = (\frac{t}{\hat{\theta}})^{\hat{\beta}}$ and the estimate of the current intensity $\hat{\lambda}(t)$ is obtained by substituting the maximum likelihood estimates of the parameters $\hat{\theta}$ and $\hat{\beta}$ in $\lambda(t)$. It is shown that $2n\frac{\beta}{\hat{\beta}} = 2\beta\sum_{i=1}^{n-1} \ln(t_n/t_i)$ has chi-square distribution with $2(n-1)$ degrees of freedom. This important property is used for proposing tests and constructing confidence intervals for the parameter β. The confidence interval for the other parameter can be obtained by various methods as suggested by Gaudoin et.al. (2006). According to the authors, the first order approximation to the estimated variance-covariance matrix is

$$\hat{I} = \begin{bmatrix} \dfrac{1}{n\hat{\theta}^{2\beta}}\left[1+\left\{\ln(n\hat{\theta}^{\hat{\beta}})\right\}^2\right] & -\dfrac{\hat{\beta}}{n\hat{\theta}}\ln(n\hat{\beta}) \\ -\dfrac{\hat{\beta}}{n\hat{\theta}}\ln(n\hat{\beta}) & \dfrac{\hat{\beta}^2}{n} \end{bmatrix}$$

A large sample confidence interval for the parameters of the process can be constructed using the above estimates of variances. For other methods and approximations, one may refer to Gaudoin et al (2006) for more details.

It is observed that the reliability development of a system often takes place by testing a system until it fails, then making repairs and design changes and testing it again. This process continues until a desired level of reliability is achieved. At various times in the development process it becomes important to assess and predict system reliability from an available set of data on system failures. Hence it is necessary to study the reliability inferences of the system with full or partial realizations of failures. Motivated from this, Muralidharan (2002a) has studied the reliability inferences of i-th weibull process. In the reliability growth situation, the failure times $t_1, t_2, \ldots, t_n$ may represent current available data on a system. If testing of the system is planned to end at the $(n+m)$th (m may be called future failures/samples) failures then it may be desired to predict the time t_{n+m} when testing will be complete based on $t_1, t_2, \ldots, t_n$. The reliability estimation in such a situation enables the experimenter to look for future course of action even when the system is in operation. Inspired from this, Muralidharan et.al. (2006) have obtained future reliability estimation of PLP where the system will be tested under additional time of occurrences of failures. The detailed estimates and the predictive intervals based on complete (full likelihood) and predictive likelihood are presented in the above paper.

From applications point of view, Pievatolo et.al. (2003) have considered failures in 40 underground trains observed over a 8-year period and failures of doors were recorded. Since the repairs were almost instantaneous and minimal, a PLP model was chosen to model the failures. Kumar and Klefsjo (1992) have used this model to study the reliability analysis of hydraulic systems of LHD machines and concluded the time between failures (TBFs) can be adequately modeled by the power law assumption. The coal mining disasters data is another example, which has been extensively used by many authors like Jarret (1979), Berman (1981), Rudemo (1982) and Raftery and Akman (1986) for model fitting and other related inferences. The data were recorded from 1851 to 1962 in great Briton with 190 intervals given in days (see Figure 1). The above authors have suggested that there is some trend existing in the model and a change point model may be appropriate for modeling the data. Raftery and Akman (1986) under HPP assumption have concluded the shift is somewhere around 124th event, which we feel that it is slightly an under estimate of the shift. Under PLP assumption, Muralidharan and Shah (2006b) have concluded

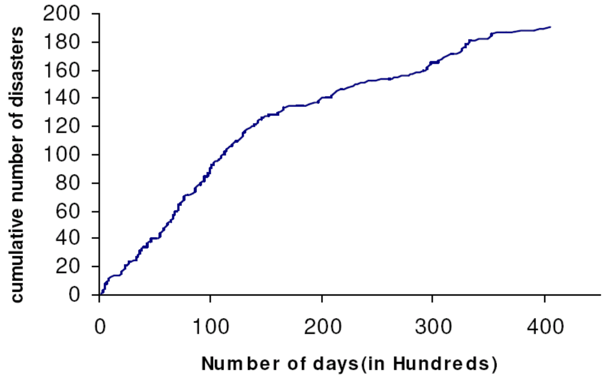

Figure 1: The graph of Coal mining disaster data

that the shift is happening somewhere between 125-th to 128-th event.

Note: The presence of covariates: The NHPP described earlier refer to a unique system with homogeneous characteristics. Sometimes systems can differ for some features, like the location or in scaling the measures etc. A preliminary explorative analysis will then help us in deciding the covariance structure of such prior parameters and their prior distributions. This problem will be more serious, when different classes of failure times are observed over a period of time. In this case, independence among classes could be an unrealistic assumption. A Cox regression model will then be a better approach for such circumstances. The other alternatives may be considering the parameters of NHPP as functions of the covariates $\underline{t}$ (see Ruggeri, 2006). In the subsequent sections at various places we will address this problem.

A general theory of processes with intensity function $\lambda(t) = \rho \exp[\beta' z(t)]$, where $z(t) = (z_1(t), z_2(t), \ldots, z_p(t))'$ is a vector of functions that may depend on both t and the history of failure H_t and $\beta = \beta_1, \beta_2, \ldots, \beta_p)'$ is a vector of unknown parameters was developed by Cox (1972), who called them the modulated Poisson process (MPP). An important feature of this process is that, when they are non-stationary, the intervals between successive events are statistically independent. Thus MPP is a Poisson process with covariates depending on the operating time t and does not depend on the history of failures H_t. The complete intensity function $\lambda(t|H_t)$ of modulated renewal process (MRP) again introduced by Cox (1972) depends both on the time from the last failure and on the operating time t.

3.3 Renewal Process (RP)

This is a generalization of HPP. A RP is defined as a sequence of independent, identically distributed (iid) non-negative random variables $X_1, X_2, \ldots$. In this case, the system after repair will be "good as new" showing a renewal type behavior. If the system is under perfect repair condition, then the failures are according to a RP. The special case where their distribution is exponential corresponds to the HPP. The RP is a plausible first order model for components or parts, since complete replacement of a component after failure implies renewal instead of repair. As described earlier for RP, since the times between failures are iid, a repaired unit is always brought to a like-new condition. For this reason, the RP cannot be used to model a system experiencing deterioration or reliability improvement.

According to Barlow and Proschan (1975), if X_i's are iid with distribution function $F_X(x)$, then the distribution of $T_k = X_1 + X_2 + \cdots + X_k$ is $F^{(k)}(t)$, the k-fold convolution of F. If n renewal processes are working independently of each other, then the process formed by the union of all events is known as super imposed renewal process (SRP). In general, the SRP will not be a renewal process. In fact, Cinlar (1975) shows that if the superposition of two independent renewal process is a renewal process then all three processes must be HPPs.

3.4 Inhomogeneous gamma process (IGP)

The inhomogeneous gamma process (IGP) proposed by Berman (1981) can be seen as follows: Suppose that events or shocks occur according to an NHPP with intensity function $\lambda(t)$ and suppose that a failure occurs not at every shock but at every kth shock, where k is a positive number. If $t_1, t_2, \ldots, t_n$ are the times of the first n events observed after the origin, their joint density is

$$f(t_1, t_2, \ldots, t_n) = \frac{1}{\{\sqrt{\kappa}\}^n} \prod_{i=1}^{n} \lambda(t_i)\{\Lambda(t_i) - \Lambda(t_{i-1})\}^{\kappa-1} e^{-\Lambda(t_n)} \tag{3.3}$$

The above likelihood clearly says that the random variables $\Lambda(t_i) - \Lambda(t_{i-1})$ for $i = 1, 2, \ldots, n$ are independently and identically distributed according to the gamma distribution with unit scale parameter and shape parameter κ. If $\kappa = 1$, (3.3) defines an inhomogeneous Poisson process (IPP), while if $\lambda(t) = \rho$, equation (3.3) defines a renewal process with intervals which have a gamma distribution with scale parameter ρ and shape parameter κ. In this case the process is called an inhomogeneous gamma process (IGP). It also follows that $\Lambda(t_n) = \sum_{i=1}^{n} \Lambda(t_i) - \Lambda(t_{i-1})$ has a gamma distribution with unit scale parameter and shape parameter nk. Thus, t_n has the density

$$g_n(t_n) = \frac{1}{\sqrt{n\kappa}} \lambda(t_n)\{\Lambda(t_n)\}^{n\kappa-1} e^{-\Lambda(t_n)}. \tag{3.4}$$

For more details on distributional properties and inferences, refer to Berman (1981).

3.5 Modulated Gamma process (MGP)

An inhomogeneous gamma process will be called a modulated gamma process if its rate function is of the form $\lambda(t) = \rho e^{\beta z(t)}$, where $\beta = (\beta_1, \beta_2, \ldots, \beta_k)$ and $z(t) = (z_1(t), z_2(t))'$. When $\kappa = 1$, this reduces to Coxs modulated Poisson process. For $\beta = 0$ and $\rho = \kappa = 1$, the process reduces to HPP and for $\beta = 0$ it reduces to GRP. Here this intensity function clearly expresses a possible dependence of the occurrence of events on the vector $z(\cdot)$ of explanatory variables. For instance, the existence of a trend in the data can be examined simply by including a single term $z_1(t) = t$ or $z_2(t) = t^2$. Even cyclic variation can be tested by considering $z_1(t) = \cos(\omega t)$ and $z_2(t) = \sin(\omega t)$. When $\beta = 0$, there is no dependence of the point process on $z(\cdot)$ and then the intervals are independent and identically distributed gamma random variables with scale parameter ρ and shape parameter κ. According to Berman (1981), if $t_1, t_2, \ldots, t_n$ are the times of the first n events, then from (3.3), the likelihood is

$$\begin{aligned} f(t_1, t_2, \ldots, t_n : \beta, \kappa, \rho) = & \frac{1}{\{\sqrt{\kappa}\}^n} \rho^n e^{\beta v} \\ & \exp\Big[-\rho \int_0^{t_n} e^{\beta_z(x)} dx \Big] \prod_{i=1}^{n} \Big[\int_{t_{i-1}}^{t_i} e^{\beta z(x)} dx \Big]^{\kappa - 1}, \end{aligned} \quad (3.5)$$

where $v = (v_1, v_2, \ldots, v_p)'$ and $v_j = \sum_{i=1}^{n} z_j(t_i)$, $j = 1, 2, \ldots, p$. Here β is the parameter of interest and κ and ρ are the nuisance parameters. For given β and κ, t_n is sufficient for ρ. Hence the likelihood of the data conditional on (n, t_n) will be independent of ρ. Therefore, one can study the inferences based on conditional distributions as given in Berman (1981) and Muralidharan (2001).

Since β is the parameter of interest a test of $H_0 : \beta = 0$ makes sense for detecting trend in the process. Berman (1981) have used the efficient score vector and the information matrix to obtain an asymptotic test for the null hypothesis. The efficient score when $\beta = 0$ are

$$U_j(\kappa) = [\frac{\partial \log L}{\partial \beta_j}] = S_j + (\kappa - 1) \sum_{i=1}^{n} A_{i-1,i}(z_j) - (n\kappa - 1) A_{0,n}(z_j), \quad (3.6)$$

where $S_j = \sum_{i=1}^{n-1} z_j(t_i)$, $j = 1, 2, \ldots, p$ and $A_{i,l}(z_j) = \frac{1}{(t_l - t_i)} \int_{t_i}^{t_l} z_j(y) dy$. The elements of the information matrix when $\beta = 0$ are

$$\begin{aligned} I_{j,k}(\kappa) &= -E(\frac{\partial^2 \log L}{\partial \beta_j \partial \beta_k}) \\ &= (n\kappa - 1) B_{0,n}(z_j, z_k) - (\kappa - 1) \sum_{i=1}^{n} E\{B_{i-1,i}(z_j, z_k)\}, \end{aligned} \quad (3.7)$$

where $B_{i,j}(z_j, z_k) = A_{i,l}(z_j, z_k) - A_{i,l}(z_j)A_{i,l}(z_k)$. Then the asymptotic test $\hat{\kappa}_c$, the conditional maximum likelihood of κ under H_0 is found to be the solution of

$$\psi(\kappa) - \psi(n\kappa) = \frac{1}{n}\sum_{i=1}^{n} \log(y_i),$$

where $\psi(x) = \frac{d}{dx}\log\sqrt{x}$ and $y_i = \frac{t_i - t_{i-1}}{t_n}$. For examples on testing for linear trend, quadratic trend and cyclic trend one may refer to Berman (1981). As an application the authors consider the Coal mining disasters data of Jarret (1979) and concluded linear trend in the data.

3.6 Modulated power law process (MPLP)

Lakey and Rigdon (1992) have proposed the modulated power law process as a compromise between the NHPP and the renewal process models. This model, which should be more realistic in nature, is a special case of the inhomogeneous gamma process (IGP) introduced by Berman (1981) and described in the previous sections.

Let $X_1, X_2, \ldots, X_n$ be n iid gamma random variables with shape parameter κ and unit scale parameter. If $Y_j = \sum_{i=1}^{j} X_i$, then Y_j's are the event times of a renewal process in which the times between events are iid gamma. Moreover, if Y_j's were the actual failure times, then a failed and repaired unit would be in better condition than it was just before the failure. Now, if $T_i = \theta Y_i^{1/\beta}$, $i = 1, 2, \ldots, n$, then the process $T_1 < T_2 < \cdots < T_n$ is an MPLP with parameters θ, β and κ. Here the parameter κ has special meaning. If for example $\kappa = 3$, then every third shock cause a failure. A failed and repaired unit would then be better than it was just before the failure, since in order to cause another failure the required number of shocks must accumulate to three again. A failed and repaired unit, however, would not necessarily be as good as a new unit. The larger κ is, the larger the improvement will be. Thus, κ is a measure of the improvement effected by a failure and repair and β is a measure of the system improvement or deterioration over the course of the systems life. When $\kappa = 1$, there is a failure at each shock, and so the MPLP reduces to the power law process. When $\beta = 1$, the times between failure are iid gamma random variables, so the process becomes a GRP. Finally, when $\kappa = \beta = 1$, the process reduces to the HPP. Thus MPLP is a generalization of the PLP and the gamma renewal process. If $\lambda(t)$ is according to (3.2), then the MPLP will have the likelihood as follows:

$$L(\underline{t}|\theta, \beta, \kappa) = \frac{1}{[\Gamma k]^n}(\frac{\beta}{\theta})^n(\prod_{i=l}^{n}\frac{t_i}{\theta})^{\beta-1}\prod_{i=l}^{n}\Big[(\frac{t_i}{\theta})^{\beta} - (\frac{t_{i-1}}{\theta})^{\beta}\Big]^{\kappa-1}\exp\Big[-(\frac{t_n}{\theta})^{\beta}\Big]. \tag{3.8}$$

Black and Rigdon (1996) describe an algorithm for obtaining the ML estimates of the model parameters. Asymptotic results are used to give approximate confidence intervals and hypothesis tests for the parameters. In many cases the asymptotic confidence intervals have coverage rates that are strictly less than the nominal level. This process was further studied by Calabria and Pulcini (1999). Muralidharan (2001) has proposed various tests for the parameters in the presence of nuisance parameters. The reliability inferences correspond to process $\#i$ of MPLP is studied by Muralidharan (2002b). These papers also discuss various examples in respect of modeling. Some more discussions on these are presented in the examples section.

3.7 Trend renewal process (TRP)

A process which incorporates the ordinary renewal process and the HPP is the Trend Renewal Process (TRP) introduced by Lindqvist et.al. (2003). It is the time transformed RP with $T_1, T_2, \ldots$ for which the $\Lambda(T_i) - \Lambda(T_{i-1})$, where $\Lambda(t) = \int_0^t \lambda(u)du$, are iid with any positive distribution F having expected value equal to unity. The TRP class thus includes both the general NHPP, obtained if F is the unit exponential distribution and the general renewal process, obtained if $\lambda(t)$ is constant. If F is gamma then TRP includes Berman (1981) processes. Lindqvist et.al. (2003) considered cases in which several independent processes are observed simultaneously, for example, failures of machines of the same type. Models for such situations may need to include individual effects, for example, caused by the installation of machines in different environments. These differences are often realized in the form of observed covariates. However, there may be individual variation between systems that is not explained by the available covariates and is commonly modeled as a random effect. A TRP is formally defined as follows:

Definition 3.1 *Let $\lambda(t)$ be a non-negative function defined for $t \geq 0$, satisfying $\Lambda(t) = \int_0^t \lambda(u)du\infty$ for each $t \geq 0$ and $\Lambda(\infty) = \infty$. Further, let F be a positive distribution with expected value 1. The process $T_1, T_2, \ldots$ is called TRP $[F, \lambda(\cdot)]$ if the time transformed process $\Lambda(T_1), \Lambda(T_2), \ldots$ is RP (F). i.e. if the $\Lambda(T_i) - \Lambda(T_{i-1})$, $i = 1, 2, \ldots$ are iid with distribution function F.*

In the definition, we call F the renewal distribution and $\lambda(\cdot)$ the trend function of the TRP. If F is a gamma distribution then the process reduces to inhomogeneous gamma process introduced by Berman (1981). The TRP becomes a MPLP if both F is gamma distribution and simultaneously the intensity of the shock process is according to (3.2). Another motivation to the TRP model is that: suppose that failures of a particular system correspond to replacement of a major part, while the rest of the system is not maintained. Then if the rest of the system is not subjected to wear then a renewal process would be a plausible model for the observed failure process. In the presence of wear, however, an increased replacement frequency is to be expected. This is achieved in a TRP model by accelerating the internal time of the renewal process according to a time transformation, $\lambda(t)$, which represents the cumulative wear. Thus in this case it would be natural to call $\lambda(\cdot)$ the wear function rather than the trend function.

Since the conditional density of T_i given $T_1 = t_1$, $T_2 = t_2, \ldots, T_{i-1} = t_{i-1}$ is $f[\Lambda(t_i) - \Lambda(t_{i-1})]$ and the probability of no failures in the time interval $(T_{n(t)}, t]$ given $T_1, T_2, \ldots, T_{N(t)}$ is $1 - F[\Lambda(t) - \Lambda(T_{N(t)})]$, then the likelihood of a single TRP is written as

$$L = \left\{ \prod_{i=1}^{N(t)} f[\Lambda(t_i) - \Lambda(t_{i-1})]\lambda(T_i) \right\} \{1 - F[\Lambda(t) - \Lambda(T_{N(t)})]\} \tag{3.9}$$

As said earlier, suppose that m systems work independently of each other then the failure mechanisms may differ due to environmental or operational conditions. The difference between system performances can then be attributed to an observable covariate vector $\mathbf{x}$. To incorporate the unobserved heterogeneity between systems a common way to modify the intensity function for the j-th system is $\lambda_j(t) = a_j g(x_j)\lambda(t)$, where a_j is called the failure intensity level (Anderson et.al. 1993, Chap IX) and g is a function of the covariate vector x_j of system j. Thus the presence of a_j makes the TRP heterogeneous and hence it is called heterogeneous trend renewal process (HTRP). For different choices of F and $\lambda(\cdot)$ one can obtain the other heterogeneous models of HPP, RP, NHPP etc.

As an application of these models the authors have considered the air-conditioner failure data of Proschan (1963), Tractor Engine data of Barlow and Davis (1977) and Gas compression data of Erlingsen (1989) and concluded TRP models with suitable choices of F and $\lambda(t)$. For more details one may refer to Lindqvist et al (2003).

3.8 Exponentiated power law process (EPLP)

The intensity function of exponentiated power law process (EPLP), introduced by Muralidharan and Shah (2006 a) is

$$u(t) = [\lambda(t)]^c, \quad c > 0 \tag{3.10}$$

where $\lambda(t)$ as defined in (3.2). The process defined with the intensity in (3.10) has three parameters where the parameter β is used as a measure of the system improvement and c as a measure of improvement affected by failure and repair. Larger the value of c, larger the improvement will be. Some special cases of interest are: (i). for $c = 1$, the model reduces to PLP, (ii) for $\beta = 1$, it reduces to a homogeneous Poisson process (HPP) with constant intensity $\lambda = \theta^{-c}$ and (iii) for $\beta = 1$ and $c = 1$, it reduces to a HPP. For $\lambda(t) = \exp(\beta t)$, the process may be called exponentiated Cox process (ECP).

Consider a Non-Homogeneous Poisson process with rate function $\lambda(t)$ defined as above. If $t_1, t_2, \ldots, t_n$ are the observed failure times of the system and $U(t) = \int_0^t u(x)dx$, then likelihood function of the system is given by

$$\begin{aligned} L &= \prod_{i=1}^{n} u(t_i) e^{-U(t_n)} \\ &= (\frac{\beta}{\theta})^{nc} \prod_{i=1}^{n} (\frac{t_i}{\theta})^{(\beta-1)c} \exp\Big\{ - \Big[(\frac{\beta}{\theta^{\beta}})^c \frac{t_n^{c(\beta-1)+1}}{\{c(\beta-1)+1\}} \Big] \Big\} \end{aligned} \tag{3.11}$$

Theorem 3.1 *H_t denotes the history of the process through time t, then the complete intensity is given according to* (3.10).

Proof. As in the case of PLP, the distribution of the nth failure depends only on the time of the $(n-1)$st failure, so the complete intensity is

$$\begin{aligned} u(t/t_{n-1}) &= \lim_{\Delta t \to 0} \frac{Pr[t < T_n \leq t\Delta t/T_{n-1} = t_{n-1}]}{\Delta t}, t > t_{n-1} \\ &= \frac{f_n(t/t_{n-1})}{1 - F_n(t/t_{n-1})} \\ &= \Big[\frac{\beta}{\theta} (\frac{t}{\theta})^{\beta-1} \Big]^c. \quad \text{Hence the proof.} \end{aligned}$$

Thus $u(t|H_t)\Delta t$ is approximately the probability of failure in the time interval $[t, t+\Delta t]$, conditional on the experienced failure history before time t. The corresponding unconditional intensity is called the rate of occurrence of failure (ROCOF). For more details on estimation and testing procedures we refer to Muralidharan and Shah (2006 a). A detailed study on the current and future reliability estimation are also provided in the paper along with few examples.

3.9 Models of systems subject to imperfect repairs

The stochastic point processes able to describe the failure pattern of repairable systems subject to imperfect repairs are the Proportional Age Reduction models and the Proportional Intensity Variation models In the age reduction model by Kijima (1989), each repair modifies the intensity function for a systems virtual age to some extent at each corrective maintenance action. For these repairs, the virtual age at any given time is determined by a variety of additive age reduction factors. Doyen and Gaudoin (2004) extended and analysed this model. Jack (1998) used the concept of age reduction successfully to model event data for systems subject to periodic maintenance and corrective maintenance. See also Malik (1979) and Shin et.al. (1996) for more details.

The proportional intensity variation models assumes that a system needs more frequent maintenance with increased age. Hence the introduction of improvement factors in a maintenance scheduling problem becomes important.

Malik (1979), Chan and Shaw (1993) and Calabria and Pulcini (1996) have considered such models. Doyen and Gaudoin (2004) used an improvement factor after each repair in their proposed arithmetic reduction of intensity and geometric reduction of intensity models for imperfect maintenance, which allow maintenance actions to lie between good-as-new and bad-asold. Percy et.al. (1998) considered an extension to the NHPP based on the proportional intensity model of Cox (1972b), wherein the baseline intensity function is amended at each corrective maintenance to avoid the assumption of minimal repair. This is further simplified by Percy and Alkali (2006), where the model involves a multiplicative scaling of the intensity function upon each failure and repair. The intensity function of this model is $\lambda(t) = \lambda_0(t)\prod_{i=1}^{N(t)} s_i$, where $s_i > 0$ are constants representing the intensity scaling factors and $\lambda_0(t)$ is the baseline intensity function. This model is suitable for systems that are deteriorating with time and provide a perfect description of the physical situation.

3.10 Models of systems subject to Complex maintenance policy

The other approaches to analysis of recurrent events are observed under complex maintenance policy are the minimal repairs interspersed with perfect preventive maintenance, imperfect repairs interspersed with perfect preventive maintenance, minimal repairs interspersed with imperfect preventive maintenance etc. Some such models have been reviewed by Lawless (1995), Lawless and Nadeau (1995), Baxter et.al. (1996), Dorado et.al. (1997), Bhattacharjee et.al. (2004) etc and the references contained therein. Lawless and Thiagarajah (1996) have studied models for recurrent events that incorporate both time trends and effects of past events, such as renewal-type behavior with the above intensity function with $\rho = 1$. Many common models are special cases of this including Poisson process with intensity functions $\exp(\alpha+\beta t)$ and αt^{β}. A recent paper by Syamsundar and Naikan (2007) discusses segmented point process models for maintained systems. We now provide an algorithm to generate samples from various NHPP models discussed above.

Algorithm.

Input the value of n, θ, β, κ and c
Generate a uniform sample from $(0, 1)$
Generate n gamma sample with shape parameter κ, say $x_1, x_2, \ldots, x_n$
Evaluate $t_i = \left[\{c(\beta-1)+1\}(\frac{\theta^{\beta}}{\beta})^{c}\sum_{j=1}^{i} x_j\right]^{\frac{1}{\{c(\beta-1)+1\}}}$, $i = 1, 2, \ldots, n$

Note: For suitable choice of θ, β, κ and c, one can get samples from corresponding NHPP. The above algorithm may be modified for exponential and logarithmic intensities suitably.

4 Test for Trend

It is necessary to verify system improvement or deterioration at some stages during the process to know the state of the art of repairable system. Under stationarity, the successive X_i's has identical marginal distributions but is not necessarily independent. It is necessary to deal with transient, rather than stationary processes, when synchronous sampling is done. Therefore, it is important to test the successive interarrival times for independence. If neither a trend nor statistical dependence is disclosed there is no evidence that the X_i's is not iid and cannot be modeled by a renewal process. The next step is to see the order statistics of these X_i's and to determine whether an HPP is the appropriate model. If this assumption is rejected then one should test NHPP models with different intensities. Ascher and Feingold (1978b) and Jewell (1978a, 1978b) have presented a survey and the need of trend testing. Jani et.al. (1997) considered a broad class of intensities such as logarithmic and exponential intensities to test trend in the model.

There are a number of procedures used to help determine whether a system is improving or deteriorating. Such techniques are particularly useful for inferring the salient features of the data set. The simplest method is to plot cumulative failures versus cumulative time graphically. This method will help to judge whether the inter-arrival times of an improving (deteriorating) system tend to become larger (smaller), if the cumulative number of failures on the graph will tend to be concave down (up). Duane (1964) introduced the technique of plotting $\frac{t}{N(t)}$ against t on log-log paper. Although plotting on log-log scale will tend to linearize any function, doe not necessarily indicate that an NHPP with any intensity function is appropriate. A corrective action will then be needed to conclude the appropriate model. A method which is relatively simple is by estimating average ROCOF in successive time periods. The average ROCOF in the ith subinterval can be estimated as $\hat{v}_i(t) = \frac{N_i(t)-N_{i-1}(t)}{\delta t}$, $(i-1)\delta t < t < i\delta t$, where $N_j(t)$ is the total number of failures from time zero to the end of the jth interval and δt is the length of each subinterval. If a system is improving (deteriorating) there will be a tendency for successive estimates, $\hat{v}_i(t)$, $i = 1, 2, \ldots$ to decrease (increase). The number of intervals to consider is a function of the number of observed failures.

For recurrent events data, the possible presence of time trends in residuals within the individual system j can be checked using total time test (TTT) plots introduced by Barlow and Davis (1977). It is expected that under HPP assumption the plot will be near the diagonal of the unit square. Recently, Gaudoin et.al. (2003) and Garcia et.al. (2005) have done some goodness-of-fit tests to assess the trends in Duane plots.

Specifically, the tests of $H_0 : \lambda(x)$ is constant against $H_1 : \lambda(x)$ is monotonic increasing or decreasing are of interest. The results of such tests would indicate whether the simple HPP may be adequate or whether a more general NHPP is required. Bartholomew (1956), Boswell (1966), Ascher and Feingold (1978b), Bain and Engelhardt (1980), Bain et.al. (1984), and Jani et.al. (1997) have discussed the tests for H_0 verses H_1 based on time truncated data. One such important test is the Laplaces test which is discussed in Cox and Lewis (1966). Under H_0, if $T_1, T_2, \ldots, T_n$ are the first n arrival times, then the test statistics is of the form

$$U = \frac{\frac{1}{n-1}\sum_{i=1}^{n-1} T_i - \frac{T_n}{2}}{T_n\sqrt{\frac{1}{12(n-1)}}} \tag{4.1}$$

has a standard normal variate. Cox and Lewis (1966) showed that Laplaces test is optimum against the NHPP, with intensity function $\lambda(t) = \exp(\alpha_0 + \alpha_1 t)$. Bates (1955) showed that the approximation is adequate for $n \geq 4$ and is also optimum against a generalized version of the Jelinski-Moranda (1972) software reliability growth model, which is not an NHPP model. For testing null hypothesis of time trends with renewal process Lewis-Robinson (1974) has modified the Laplace statistics as

$$LR = U\left\{\frac{\overline{x}^2}{\sum_{i=1}^{n}(x_i - \overline{x})^2/(n-1)}\right\} \tag{4.2}$$

where the x_i's are the times between events. For large class of NHPP models Zhao and Wang (2005) have also proposed tests based on the Laplace statistics.

Another test which is widely used for checking the presence of trend is the MILHDBK- 189 (Military Handbook) (1981) test and is based on the test statistic

$$M = 2\sum_{i=1}^{n-1} \ln\,[\frac{T_n}{T_i}] \tag{4.3}$$

Under the null hypothesis of an HPP, M is distributed as χ^2 with $2(n-1)$ degrees of freedom. Bain et.al. (1984) point out that the above test is optimum against the NHPP model with intensity $\lambda(t) = \theta\beta t^{\beta-1}$, and recommend its use for testing the HPP against the broad class of NHPPs with monotonically increasing failure rates which tends to infinity as $t \to \infty$, but which are of unknown functional form.

The test statistics of Jani et.al. (1997) has the form

$$Q_1 = 2n(n+1)\sum_{i=1}^{n-1}\frac{T_i}{T_n}\left[\frac{T_i}{T_n} - \frac{T_{i+1}}{T_n}\right] + (n^2-1). \tag{4.4}$$

Under H_0, the mean and variance of Q_1 is respectively obtained as $E(Q_1) = n-1$ and $V(Q_1) = \frac{4n^2(n-1)}{(n+2)(n+3)}$. The authors have proposed a similar kind of test with transformed variables as $\ln(T_1)$, $\ln(T_2)$, $\ldots, \ln(T_n)$ instead of $T_1, T_2 \ldots, T_n$ and has the form

$$Q_2 = \sum_{i=1}^{n-1} [1 - (n-i)\{\ln(T_{n-i+1}) - \ln(T_{n-i})\}]^2, \tag{4.5}$$

which has $E_{H_0}(Q_1) = n - 1$ and $V_{H_0}(Q_1) = 24(n-1)$. Further, if $V_i = \ln(T_n) - \ln(T_{n-i})$, $i = 1, 2, \ldots, n$ then $U_i = (n-i)(V_i - V_{i-1})$ are iid standard exponential random variables with $E(U_i) = 1$, then the modified form of the test statistic Q_2 is

$$Q_3 = \sum_{i=1}^{n-1} [U_i - \overline{U}]^2, \tag{4.6}$$

where $\overline{U} = \frac{1}{n-1}\sum_{i=1}^{n-1} U_i$. The mean and variance of this statistic under HPP assumption is $n-2$ and $\frac{2(4n^2-15n+14)}{n-1}$ respectively. The above tests are useful for testing a broad range of intensities including exponential and logarithmic intensities.

For testing HPP against non-monotonic trend, we refer to Lewis (1972), Hollander and Proschan (1974) and Barndorff-Nielson and Cox (1979). Tests which distinguish between a renewal process and a monotonic trend have been developed by Mann (1945), Lehman (1975), Srinivasan (1978) and Lawless and Thiagarajah (1996). Specifically, if the intensity is of the form $\lambda(t) = \exp\{\alpha + \beta t + \gamma g(t)\}$ then the time trend can be tested by considering $H_0 : \beta = 0$ and the renewal process can be tested by considering $H_0 : \gamma = 0$ If the test for renewal is accepted then the process may have inter-arrival distribution exponential and can be checked by computing the generalized residuals $\hat{e}_i$.

5 Examples

In addition to the specific examples discussed above, here, we discuss some more examples and their corresponding choice of models. The first is the failure times of an aircraft air generator data ($n = 14$) read from a plot in Duanes paper by Black and Rigdon (1996), the second example consists of 21 failure times from the second aircraft air generating unit and the third example consists of the set of 30 failure times of airplane air-conditioning equipment both from Proschan (1963) and the data consists of the ages at successive failures ($n = 92$) of a photocopier in use considered by Baker (1996). It is found that a PLP model is not adequate to model Duans data, whereas Proschan (1963) aircraft air generating unit data can be modeled by using a PLP. As far as the other two sets of data are concerned, a MPLP model is found to be a suitable model for representing the failure times. The same conclusions are drawn by Black and Rigdon (1996), Calabria and Pulcini (1997) and Muralidharan

(2001) in different contexts. Lawless and Thiagarajah (1996) have used the Proschan (1963) data in order to illustrate both time trends and renewal type behavior. On the basis of ML estimation they concluded that there is no evidence of renewal type behavior and there is a strong evidence of a time trend existing in the data. A best example for EPLP model is the failure times of a vertical boring machine due to Majumdar (1993). This conclusion is based on first 27 observations. Another choice for EPLP is the Duans data as the value of β is very small and c is very large (see Muralidharan and shah, 2006a).

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Paper received on 21 March 2008; revised, 13 June 2008; accepted, 26 June 2008.