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Some characterizations based on bivariate reversed mean residual life

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Abstract. In the present paper we discuss some properties of bivariate reversed mean residual life function and their extensions to the multivariate case. Inversion formula for the distribution function of a bivariate random vector in terms of the reversed mean residual life, some characterization theorems and relationship with bivariate reversed hazard rates are established. Models based on proportional reversed mean residual life and their properties are discussed.

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1 Introduction

The role of the concepts of failure rate and mean residual life in reliability theory and several other disciplines such as actuarial science, survival analysis, economics etc are well known. Recently reliability functions similar to these two, but reversed in time, have been introduced in the form of reversed hazard rate (RHR) defined as

$$r(x) = f(x)/F(x) \tag{1.1}$$

and the reversed mean residual life (RMRL)

$$\begin{aligned} m(x) &= [F(x)]^{-1} \int_0^x F(t)\,dt \\ &= E(x - X \mid X \le x) \end{aligned} \tag{1.2}$$

where X is a continuous random variable defined over $(0, \infty)$, representing the life time of a unit, with distribution function $F(x)$ and density function $f(x)$.

The RHR has been used in characterizing probability distributions (Block et.al (1998)), ordering life distributions (Gupta and Nanda (2001)), analysing survival data (Gupta and Han (2001)), measurement of uncertainty (Di Crescenzo and Longobardi (2002)) and in stochastic processes (Bloch-Mercier (2001)). The definition and properties of RMRL are given in Finkelstein (2002) and Nanda et.al (2003), in which the former interprets $m(x)$ as the mean waiting time since the failure of a unit conditioned on its failure in the time interval $[0, x]$. Since the properties of the usual mean residual life and the RMRL are radically different (for example, there is no life distribution in $(0, \infty)$ with decreasing or constant RMRL) enabling each to be used in different data situations, there is scope for a separate study of RMRL in the univariate and multivariate cases. Although the theoretical discussions or RMRL are similar to those of mean residual life, the models arising from the two under similar functional forms are different. This makes the study of characterizations based on properties of RMRL important. While mean residual life is discussed extensively in the multivariate case, there appears no similar study of RMRL in higher dimensions. The present paper attempts a study of RMRL in the bivariate case.

The contents of the rest of the paper is as follows. In Section 2 the definition and properties of bivariate reversed mean residual life (BRMRL) are discussed. This is followed by an investigation of the relationship between BRMRL and the bivariate reversed hazard rate (BRHR) and a characterization theorem is taken up in Section 3. In Section 4 proportional BRMRL models are studied and some extensions of the important results to the multivariate case are indicated in Section 5.

2 Bivariate reversed mean residual life.

Consider a random vector $\boldsymbol{X} = (X_1, X_2)$ in the two dimensional space $\mathbb{R}_2$ with joint distribution function $F(x_1, x_2)$ and marginal distribution functions $F_i(x_i)$ of X_i, $i = 1, 2$. Let $\boldsymbol{a} = (a_1, a_2)$ and $\boldsymbol{b} = (b_1, b_2)$ be vectors of real numbers satisfying $a_i = \inf(x|F_i(x) > 0)$ and $b_i = \sup(x|F_i(x) < 1)$. Assume further that $E(\boldsymbol{X}) < \infty$.

We define the vector valued BRMRL of $\boldsymbol{X}$ as the B-measurable function

$$\begin{aligned} \boldsymbol{m}(\boldsymbol{x}) &= (E(\boldsymbol{x} - \boldsymbol{X}|\boldsymbol{X} \leq \boldsymbol{x}), \\ &= (m_1(\boldsymbol{x}), m_2(\boldsymbol{x})) \end{aligned} \tag{2.1}$$

for all $\boldsymbol{x} > \boldsymbol{a}$, where the ordering of the vectors is done component wise.

From (2.1) we can write

$$m_i(\boldsymbol{x}) = \frac{1}{F(\boldsymbol{x})} \int_{a_i}^{x_i} F(\boldsymbol{x}_i, t)\, dt, \quad i = 1, 2 \tag{2.2}$$

where $(\boldsymbol{x}_i, t)$ stands for the vector $\boldsymbol{x} = (x_1, x_2)$ in which x_i is replaced by t. Equation (2.2) enables the determination of $\boldsymbol{m}(\boldsymbol{x})$ from the distribution of $\boldsymbol{X}$.

We now examine the converse problem of identifying $F(\boldsymbol{x})$ using the form of $\boldsymbol{m}(\boldsymbol{x})$. Differentiating

$$m_i(\boldsymbol{x})F(\boldsymbol{x}) = \int_{a_i}^{x_i} F(\boldsymbol{x}_i, t)$$

with respect to x_i

$$\frac{\partial m_i(\boldsymbol{x})}{\partial x_i} F(\boldsymbol{x}) + m_i(\boldsymbol{x}) \frac{\partial F(\boldsymbol{x})}{\partial x_i} = F(\boldsymbol{x}).$$

For $i = 1$, this gives

$$\frac{1}{F(\boldsymbol{x})} \frac{\partial F(\boldsymbol{x})}{\partial x_1} = \frac{1}{m_1(\boldsymbol{x})} \left(1 - \frac{\partial m_1(\boldsymbol{x})}{\partial x_1}\right)$$

and integrating over (x_1, b_1),

$$F(\boldsymbol{x}) = F_2(x_2) \exp[- \int_{x_1}^{b_1} \frac{1}{m_1(t, x_2)} (1 - \frac{\partial m_1(t, x_2)}{\partial t}) dt]. \tag{2.3a}$$

Similarly for $i = 2$,

$$F(\boldsymbol{x}) = F_1(x_1) \exp[- \int_{x_2}^{b_2} \frac{1}{m_2(x_1, t)} (1 - \frac{\partial m_2(x_1, t)}{\partial t}) dt]. \tag{2.3b}$$

Allowing x_2 to tend to b_2 in (2.3a)

$$F_1(x_1) = \exp[- \int_{x_1}^{b_1} \frac{1}{m_1(t, b_2)} (1 - \frac{\partial m_1(t, b_2)}{\partial t}) dt].$$

Hence the distribution of $\boldsymbol{X}$ is uniquely determined as

$$\begin{aligned} F(\boldsymbol{x}) &= \exp[- \int_{x_1}^{b_1} \frac{1}{m_1(t, b_2)} (1 - \frac{\partial m_1(t, b_2)}{\partial t}) dt \\ &\qquad - \int_{x_2}^{b_2} \frac{1}{m_2(x_1, t)} (1 - \frac{\partial m_2(x_1, t)}{\partial t}) dt] \\ &= \frac{m_1(b_1, b_2)}{m_1(x_1, b_2)} \frac{m_2(x_1, b_2)}{m_2(x_1, x_2)} \\ &\qquad \exp[- \int_{x_1}^{b_1} \frac{dt}{m_1(t, b_2)} - \int_{x_2}^{b_2} \frac{dt}{m_2(x_1, t)}] \end{aligned} \tag{2.4}$$

or equivalently from (2.3b) as

$$
\begin{aligned}
F(\boldsymbol{x}) &= \exp[-\int_{x_1}^{b_1} \frac{1}{m_1(t,x_2)}(1-\frac{\partial m_1(t,x_2)}{\partial t})dt \\
&\quad -\int_{x_2}^{b_2} \frac{1}{m_2(b_1,t)}\,(1-\frac{\partial m_2(b_1,t)}{\partial t})dt] \\
&= \frac{m_1(b_1,x_2)}{m_1(x_1,x_2)}\frac{m_2(b_1,b_2)}{m_2(b_1,x_2)} \\
&\qquad e^{-\int_{x_1}^{b_1}\frac{dt}{m_1(t,x_2)}-\int_{x_2}^{b_2}\frac{dt}{m_2(b_1,t)}} \qquad (2.5)
\end{aligned}
$$

We observe that the random variables X_1 and X_2 are independent if and only if

$$m_i(\boldsymbol{x}) = m_i(\boldsymbol{x}_j, b_j),$$

for every $\boldsymbol{x} > \boldsymbol{a}$; that is when the components of BRMRL are respectively equal to the reversed mean residual life of the component variables. Notice that m_1 and m_2 satisfy the relationship

$$\frac{\partial}{\partial x_2}\Big(\frac{1-\frac{\partial m_1}{\partial x_1}}{m_1(\boldsymbol{x})}\Big) = \frac{\partial}{\partial x_1}\Big(\frac{1-\frac{\partial m_2}{\partial x_2}}{m_2(\boldsymbol{x})}\Big).$$

One important application of BRMRL is to identify the distribution based on an assumed functional form of BRMRL. We establish some characterizations based on certain general forms for the BRMRL.

Theorem 2.1 *If* $\boldsymbol{X}$ *is a random vector in the support* $(0,b_1)\times(0,b_2)$, $b_i < \infty$, $i = 1,2$ *with absolutely continuous distribution function* $F(\boldsymbol{x})$ *and* $E(\boldsymbol{X}) < \infty$, *then*

$$m_i(\boldsymbol{x}) = c_i(x_j)x_i \ \ , \ \ i,j = 1,2 \quad i \neq j \qquad (2.6)$$

for some non negative functions $c_i(\cdot)$ *if and only if* $\boldsymbol{X}$ *has bivariate power distribution*

$$F(x_1,x_2) = \Big(\frac{x_1}{b_1}\Big)^{c_1}\Big(\frac{x_2}{b_2}\Big)^{c_2+\theta\log(\frac{x_1}{b_1})}, \theta \leq 0 \qquad (2.7)$$

where $c_i = [c_i(b_j)]^{-1} - 1$.

Proof The distribution function (2.7) verifies that

$$m_i(\boldsymbol{x}) = \frac{x_i}{1+c_i+\theta\log(x_j/b_j)}$$

so that (2.6) holds. Now assuming (2.6), we get from (2.4) and (2.5) that

$$\begin{aligned} F(x_1, x_2) &= \frac{b_1}{x_1}\frac{b_2}{x_2}\left(\frac{x_2}{b_2}\right)^{1/c_2(b_1)}\left(\frac{x_1}{b_1}\right)^{1/c_1(x_2)}, \\ &= \frac{b_1}{x_1}\frac{b_2}{x_2}\left(\frac{x_1}{b_1}\right)^{1/c_1(b_2)}\left(\frac{x_2}{b_2}\right)^{1/c_2(x_1)}. \end{aligned}$$

Hence

$$\frac{c_2(x_1)c_2(b_1)}{c_2(b_1) - c_2(x_1)}\log\left(\frac{x_1}{b_1}\right) = \frac{c_1(x_2)c_1(b_2)}{c_1(b_2) - c_1(x_2)}\log\left(\frac{x_2}{b_2}\right). \tag{2.8}$$

Since (2.8) holds for all x_1, x_2, the expressions on the right and left sides must be a constant, say k, or

$$\frac{c_j(x_i)c_j(b_i)}{c_j(b_i) - c_j(x_i)}\log\left(\frac{x_i}{b_i}\right) = k.$$

Solving

$$[c_j(x_i)]^{-1} = [c_j(b_i)]^{-1} + \theta\log\left(\frac{x_i}{b_i}\right)$$

which leads to (2.7).

Remark: The i^{th} component of BRMRL is linear in x_i, and also that it is proportional to the respective univariate reversed mean residual life derived from the marginals of X_i.

We now illustrate the case when a_i is $-\infty$ and b_i is $+\infty$ with another characterization.

Theorem 2.2 *If $\boldsymbol{X}$ is a random vector in the support $\mathbb{R}^2$ with absolutely continuous distribution function $F(\boldsymbol{x})$ and $E(\boldsymbol{X}) < \infty$, then X follows bivariate logistic distribution*

$$F(\boldsymbol{x}) = (1 + e^{-x_1} + e^{-x_2})^{-1} \quad -\infty < x_1, x_2 < \infty \tag{2.9}$$

if and only if for all $\boldsymbol{x}$ in $\mathbb{R}^2$

$$m_i(\boldsymbol{x}) = \frac{1 + e^{-x_1} + e^{-x_2}}{1 + e^{-x_j}}\log\left(\frac{1 + e^{-x_1} + e^{-x_2}}{e^{-x_i}}\right), \quad i, j = 1, 2;\ i \neq j. \tag{2.10}$$

Proof Assuming the distribution to be (2.9), we have

$$\int_{-\infty}^{x_i} F(\boldsymbol{x}_i, t)\, dt = (1 + e^{-x_j})^{-1}\log\left(\frac{1 + e^{-x_1} + e^{-x_2}}{e^{-x_i}}\right)$$

from which (2.10) follows. Now to prove the converse, we note that from (2.10)

$$\frac{\partial m_i(\boldsymbol{x})}{\partial x_i} = 1 - \frac{e^{-x_i}}{1+e^{-x_j}} \log \frac{1+e^{-x_1}+e^{-x_2}}{e^{-x_i}} \tag{2.11}$$

so that

$$\frac{1}{m_i}(1-\frac{\partial m_i}{\partial x_i}) = \frac{e^{-x_i}}{1+e^{-x_1}+e^{-x_2}}. \tag{2.12}$$

Using (2.10) and (2.12) in (2.5), we recover (2.9).

Theorem 2.3 *If $\boldsymbol{X}$ is a random vector in the support $(-\infty, b_1) \times (-\infty, b_2)$ with $b_i < \infty$ possessing absolutely continuous distribution function, then*

$$m_i(\boldsymbol{x}) = \alpha_i(x_j), \quad i,j = 1,2,\ i \neq j \tag{2.13}$$

if and only if

$$F(\boldsymbol{x}) = \exp[c_1(x_1-b_1)+c_2(x_2-b_2) + c_3(x_1-b_1)(x_2-b_2)], \quad c_i > 0 \tag{2.14}$$

Proof When the distribution has form (2.14), using the formula (2.2),

$$m_i(\boldsymbol{x}) = [c_i + c_3(x_j - b_j)]^{-1}$$

which is independent of x_i. Thus proves the if part.

Conversely assuming (2.13), substituting it in (2.4) and (2.5) and equating the resulting expressions,

$$\frac{b_1-x_1}{\alpha_1(b_2)} + \frac{b_2-x_2}{\alpha_2(x_1)} = \frac{b_1-x_1}{\alpha_1(x_2)} + \frac{b_2-x_2}{\alpha_2(b_1)}$$

which is equivalent to

$$\frac{b_2-x_2}{(\alpha_1(b_2))^{-1}-(\alpha_1(x_2))^{-1}} = \frac{b_1-x_1}{(\alpha_2(b_1))^{-1}-(\alpha_2(x_1))^{-1}}$$

The last equation holds for all x_1, x_2 iff either expression is a constant, say θ. This solves to

$$[\alpha_i(x_j)]^{-1} = \theta(b_j-x_j)^{-1} - [\alpha_i(b_j)]^{-1}. \tag{2.15}$$

Substituting (2.4)

$$F(x_1,x_2) = \exp[c_1(x_1-b_1)+c_2(x_2-b_2)+c_3(x_1-b_1)(x_2-b_2)]$$

where $c_i = [\alpha_i(b_j)]^{-1} > 0$, $i = 1,2$ and $c_3 = \theta^{-1} > 0$. This completes the proof.

3 Bivariate reversed hazard rates.

Let $\boldsymbol{X} = (X_1, X_2)$ be a random vector representing the life times of a two-component system so that $\boldsymbol{X}$ is defined on $\mathbb{R}_2^+ = \{(x,y)|x,y > 0\}$. We write $H(\boldsymbol{x}) = \log F(\boldsymbol{x})$ for all $\boldsymbol{x}$ for which $F(\boldsymbol{x}) > 0$. If the gradient of $H(\boldsymbol{x})$ is

$$\boldsymbol{h}(\boldsymbol{x}) = \nabla H(\boldsymbol{x}), \tag{3.1}$$

then $\boldsymbol{h}(\boldsymbol{x}) = (h_1(\boldsymbol{x}), h_2(\boldsymbol{x}))$ is called the vector valued reversed hazard rate of $\boldsymbol{X}$ with components $h_i(\boldsymbol{x})$, where

$$h_i(\boldsymbol{x}) = \frac{\partial}{\partial x_i} H(\boldsymbol{x}) = \frac{\partial \log F(\boldsymbol{x})}{\partial x_i}. \tag{3.2}$$

From (3.2) we find

$$h_i(\boldsymbol{x}) = [m_i(\boldsymbol{x})]^{-1}[1 - \frac{\partial}{\partial x_i} m_i(\boldsymbol{x})], \quad i = 1,2 \tag{3.3}$$

as a fundamental relationship between BRHR and BRMRL. In this connection, we show that milder conditions on the product of components of BRHR and BRMRL can characterize lifetime models.

Theorem 3.1 *The only absolutely continuous distribution of the random vector* $\boldsymbol{X}$ *defined on* $\mathbb{R}_2^+$ *satisfying*

$$h_i(\boldsymbol{x}) m_i(\boldsymbol{x}) = A_i(x_j), i,j = 1,2 \quad i \neq j \tag{3.4}$$

is the bivariate power distribution defined in (2.7)*, where,* $A_i(x_j)$ *in* (3.4) *is a positive function independent of* x_i*.*

Proof The bivariate power distribution (2.7) verifies

$$h_i(\boldsymbol{x}) = (c_i + \theta \log(x_j/b_j)) x_i^{-1}$$

so that from the expression for $m_i(\boldsymbol{x})$ derived in Theorem 2.1, the property (3.4) holds.

Conversely if (3.4) is assumed, using (3.3) we get,

$$1 - \frac{\partial}{\partial x_i} m_i(\boldsymbol{x}) = A_i(x_j)$$

which on integration gives,

$$m_i(\boldsymbol{x}) = [1 - A_i(x_j)] x_i + k_i(x_j)$$

where $k_i(x_j)$ is a constant of integration. As $x_i \to 0$, $m_i(\boldsymbol{x}) \to 0$ and hence $k_i(x_j) = 0$. The proof is completed by appealing to Theorem 2.1.

4 Proportional BRMRL models

Proportional hazard models are well known for their applications in reliability, epidemiology and survival analysis. As a more flexible alternative to this model, the notion of proportional mean residual life model in the univariate case has been introduced by Oakes and Dasu (1990), which was later extended to higher dimensions. Gupta and Han (2001) considered analysis of survival data using proportional reversed hazard models (see also Gupta et. al (1998)). In this section we define bivariate proportional reversed mean residual life model and study some of its properties.

If $\boldsymbol{X} = (X_1, X_2)$ represents the random lifetimes of a two - component system with distribution function $F(\boldsymbol{x})$, the bivariate proportional mean residual life model is expressed in terms of a random vector $\boldsymbol{Y}$ whose BRMRL is proportional to the BRMRL of $\boldsymbol{X}$. Denoting by $\boldsymbol{m}(\boldsymbol{x}) = (m_1(\boldsymbol{x}), m_2(\boldsymbol{x}))$, and $\boldsymbol{r}(\boldsymbol{x}) = (r_1(\boldsymbol{x}), r_2(\boldsymbol{x}))$ the BRMRL's of $\boldsymbol{X}$ and $\boldsymbol{Y}$ respectively at the time point $\boldsymbol{x} = (x_1, x_2)$, our requirement for proportional model is that

$$\boldsymbol{r}(\boldsymbol{x}) = (c_1 m_1(\boldsymbol{x}), c_2 m_2(\boldsymbol{x})) \tag{4.1}$$

for some positive real numbers, c_1 and c_2. Component-wise (4.1) means that

$$r_i(\boldsymbol{x}) = c_i m_i(\boldsymbol{x}), i = 1, 2. \tag{4.2}$$

Our primary interest is to express the distribution function $K(\boldsymbol{x})$ of $\boldsymbol{Y}$ in terms of the baseline distribution function $F(\boldsymbol{x})$ and the corresponding BRMRL.

Starting from (4.2), we use the inversion formula (2.4)
for the functions $r_i(\boldsymbol{x})$ to write $K(\boldsymbol{x})$

$$K(\boldsymbol{x}) = \frac{m_1(\boldsymbol{b}) m_2(x_1, b_2)}{m_1(x_1, b_2) m_2(\boldsymbol{x})} \Big(\exp[-\int_{x_1}^{b_1} \frac{dt}{m_1(t, b_2)}] \Big)^{k_1} \Big(\exp[-\int_{x_2}^{b_2} \frac{dt}{m_2(x_1, t)}] \Big)^{k_2}, \tag{4.3}$$

with $k_i = c_i^{-1}$, $i = 1, 2$. Letting $x_2 \to b_2$ in (2.4),

$$\exp[-\int_{x_1}^{b_1} \frac{dt}{m_1(t, b_2)}] = \frac{m_1(x_1, b_2)}{m_1(\boldsymbol{b})} F_1(x_1) \tag{4.4}$$

and using this relationship again in (2.4),

$$\exp[-\int_{x_2}^{b_2} \frac{dt}{m_2(x_1, t)}] = \frac{m_2(\boldsymbol{x})}{m_2(x_1, b_2)} \frac{F(x)}{F_1(x_1)}. \tag{4.5}$$

Inserting (4.4) and (4.5) in (4.3), we have the distribution function of the proportional model

$$K(\boldsymbol{x}) = \Big(\frac{m_1(x_1, b_2)}{m_1(\boldsymbol{b})}\Big)^{k_1-1}\Big(\frac{m_2(\boldsymbol{x})}{m_2(x_1, b_2)}\Big)^{k_2-1}\frac{F^{k_2}(\boldsymbol{x})F_1^{k_1}(x_1)}{F_1^{k_2}(x_1)}. \tag{4.6}$$

Another equivalent expression is

$$K(\boldsymbol{x}) = \Big(\frac{m_2(b_1, x_2)}{m_2(\boldsymbol{b})}\Big)^{k_2-1}\Big(\frac{m_1(\boldsymbol{x})}{m_1(b_1, x_2)}\Big)^{k_1-1}\frac{F^{k_1}(\boldsymbol{x})F_2^{k_2}(x_2)}{F_2^{k_1}(x_2)}. \tag{4.7}$$

Example 4.1: Let $\boldsymbol{X}$ be distributed as bivariate uniform with

$$F(\boldsymbol{x}) = x_1^{1+\theta \log x_2} x_2, 0 < x_1, x_2 < 1,$$

then $\boldsymbol{Y}$ has bivariate power distribution function,

$$K(\boldsymbol{x}) = x_2^{2k_2-1} x_1^{2k_1-1+\theta \log x_2}, 0 < x_1, x_2 < 1.$$

Since (4.6) is proposed as alternative to the proportional reversed hazard model it is informative to examine the relationship between the two models corresponding to the same baseline distribution function $F(\boldsymbol{x})$ of $\boldsymbol{X}$. A random vector $\boldsymbol{Z} = (Z_1, Z_2)$ with distribution function $G(\boldsymbol{x})$ and reversed hazard rate

$$\boldsymbol{a}(\boldsymbol{x}) = (a_1(\boldsymbol{x}), a_2(\boldsymbol{x}))$$

is said to be the proportional reversed hazard rate model with respect to $\boldsymbol{X}$ if there holds the relation

$$a_i(\boldsymbol{x}) = \theta_i h_i(\boldsymbol{x}), \theta_i > 0, i = 1, 2$$

or

$$\frac{\partial \log G(\boldsymbol{x})}{\partial x_i} = \theta_i \quad \frac{\partial \log F(\boldsymbol{x})}{\partial x_i}.$$

Integrating from x_i to b_i and proceeding as above,

$$G(\boldsymbol{x}) = \frac{F^{\theta_i}(\boldsymbol{x}) F_j^{\theta_j}(x_j)}{F_j^{\theta_i}(x_j)}, \quad i, j = 1, 2, \ i \neq j. \tag{4.8}$$

A relationship between the reliability characteristics of $\boldsymbol{X}$ and those of the proportional RMRL model is sometimes useful in characterization problems. Taking logarithms and differentiating (4.7) leads to

$$\begin{aligned} H_1(\boldsymbol{x}) &= (k-1)\frac{m_1'}{m_1} + k_1 h_1(x_1, x_2) \\ &= \frac{k - m_1'}{m_1} \end{aligned}$$

and similarly

$$H_2(\boldsymbol{x}) = \frac{k - m_2'}{m_2},$$

where H_1 and H_2 are the reversed hazard rate components of proportional RMRL model. For the uniform distribution cited earlier,

$$G(\boldsymbol{x}) = x_2^{\theta_2} \quad x_1^{\theta_1 + \theta\ \theta_1 \log x_2}$$

which is different from the corresponding proportional BRMRL model. In general, the two models give different distributions and therefore it is of interest to investigate whether there are bivariate distributions for which $G(\boldsymbol{x})$ and $K(\boldsymbol{x})$ are identical. Our next Theorem provides an answer to this question for a random variable defined on $\mathbb{R}_2^+$.

Theorem 4.1 *A necessary and sufficient condition for the proportional reversed mean residual and reversed hazard rate models to be identical is that the baseline distribution has independent marginal power distributions with shape parameters* $\left(\frac{c_1-1}{1-c_1\theta_1}, \frac{c_2-1}{1-c_2\theta_2}\right)$.

Proof The distributions of $\boldsymbol{Y}$ and $\boldsymbol{Z}$ are identical if and only if they have the same reversed hazard rates and this in turn means that

$$\theta_i \frac{1 - \frac{\partial m_i}{\partial x_i}}{m_i} = \frac{1 - c_i \frac{\partial m_i}{\partial x_i}}{c_i m_i}, i = 1, 2$$

which reduces to

$$\frac{\partial m_i(\boldsymbol{x})}{\partial x_i} = \frac{1 - c_i\theta_i}{c_i(1-\theta_i)}$$

or

$$m_i(\boldsymbol{x}) = \frac{1 - c_i\theta_i}{c_i(1-\theta_i)} x_i + A_i(x_j), i = 1, 2 \quad i \neq j.$$

Using the condition $\lim_{x_i \to 0} m_i(\boldsymbol{x}) = 0, A_i(x_j) = 0$.
From the above expressions for $m_i(\boldsymbol{x})$,

$$F(\boldsymbol{x}) = \Big(\frac{x_1}{b_1}\Big)^{\frac{c_1-1}{1-c_1\theta_1}} \Big(\frac{x_2}{b_2}\Big)^{\frac{c_2-1}{1-c_2\theta_2}}, \frac{c_i - 1}{1 - c_i\theta_i} > 0, 0 < x_i < b_i, \tag{4.9}$$

where $b_i = \sup(x|F_i(x) < 1)$, $i = 1, 2$. The converse is easily established by noting that for the distribution (4.9),

$$m_i(\boldsymbol{x}) = \frac{1 - c_i\theta_i}{c_i(1 - \theta_i)} x_i \quad \text{and} \quad h_i(\boldsymbol{x}) = \frac{c_i - 1}{(i - c_i\theta_i)} x_i.$$

5 Multivariate extensions

The definition of BRMRL can be extended to the multivariate case, by replacing $\boldsymbol{X}$ in (2.1) by the vector $\boldsymbol{X} = (X_1, X_2, \ldots, X_p)$ in the p-dimensional Euclidean space $\mathbb{R}_p$. We define the multivariate reversed mean residual life as the vector

$$\begin{aligned} \boldsymbol{m}(\boldsymbol{x}) &= E(\boldsymbol{x} - \boldsymbol{X}|\boldsymbol{X} \leq \boldsymbol{x}) \\ &= (m_1(\boldsymbol{x}), m_2(\boldsymbol{x}), \ldots, m_p(\boldsymbol{x})) \end{aligned}$$

for all $\boldsymbol{x} > \boldsymbol{a}$, where $\boldsymbol{a} = (a_1, a_2, \ldots, a_p)$ and $\boldsymbol{b} = (b_1, b_2, \ldots, b_p)$ are defined by $-\infty \leq a_i < X_i < b_i \leq \infty$, $i = 1, 2, \ldots p$. In this case

$$m_i(\boldsymbol{x}) = [F(\boldsymbol{x})]^{-1} \int_{a_i}^{x_i} F(\boldsymbol{x}_i, t) dt$$

and the inversion formula (2.4) generalises to

$$\begin{aligned} F(\boldsymbol{x}) = \exp[&- \int_{x_1}^{b_1} \frac{1}{m_1(t, x_{2,p})} (1 - \frac{\partial m_1(t, x_{2,p})}{\partial t}) dt \\ &- \int_{x_2}^{b_2} \frac{1}{m_2(b_1, t, x_{3,p})} (1 - \frac{\partial m_2(b_1, t, x_{3,p})}{\partial t}) dt \\ &- \int_{x_p}^{b_p} \frac{1}{m_p(b_{1,p-1}, t)} (1 - \frac{\partial m_p(b_{1,p-1}, t)}{\partial t}) dt] \end{aligned} \tag{5.1}$$

or to $(p-1)$ equivalent expressions for $F(\boldsymbol{x})$ like (5.1) depending on the order in which the arguments x_i's in $F(\boldsymbol{x})$ are allowed to tend to b_i in the p basic equations for $F(\boldsymbol{x})$ derived from definition as in the bivariate case. (A single expression covering all p cases of $F(\boldsymbol{x})$ like (2.4) in the bivariate case seems notationally complicated to write.) In (5.1), we have used the notation,

$$s_{i,j} = (s_i, s_{i+1}, \dots s_j), i, j = 1, 2, \dots p \text{ and } j > i.$$

Defining the multivariate reversed hazard rate for $\boldsymbol{X} = (X_1, X_2, \dots, X_p)$ as

$$\boldsymbol{h}(\boldsymbol{x}) = \nabla H(\boldsymbol{x}),$$

where $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \dots, \frac{\partial}{\partial x_p}\right)$ is the p-dimensional gradient and $H(\boldsymbol{x}) = \log F(\boldsymbol{x})$ the multivariate counterpart of (3.3) is

$$h_i(\boldsymbol{x}) = [m_i(\boldsymbol{x})]^{-1}[1 - \frac{\partial}{\partial x_i} m_i(\boldsymbol{x})], \ i = 1, 2, \dots p$$

and the characteristic property

$$h_i(\boldsymbol{x}) m_i(\boldsymbol{x}) = A_i(x_{1,i-1}, x_{i+1,p}), \ i = 1, 2, \dots p$$

holds for all $x_i > 0$ if and only if the distribution function $F(\boldsymbol{x})$ is given by

$$\log F(\boldsymbol{x}) = \sum_{i=1}^{p} \theta_i \log\left(\frac{x_i}{b_i}\right) + \sum_{\substack{i,j=1 \\ j>i}}^{p} \theta_{ij} \log\left(\frac{x_i}{b_i}\right) \log\left(\frac{x_j}{b_j}\right) + \cdots + \theta_{12\dots p} \log\left(\frac{x_1}{b_1}\right) \cdots \log\left(\frac{x_p}{b_p}\right). \quad (5.2)$$

Notice that (5.2) is a multivariate power distribution that reduces to (2.7) for $p = 2$ and the method of proof is the same as in Theorem 3.1. In a similar manner, the linearity of $m_i(\boldsymbol{x})$ in x_i viz.

$$m_i(\boldsymbol{x}) = \mathbf{B}_i(x_{1,i-1}, x_{i+1,p}) x_i, \quad i = 1, 2, \dots p$$

among absolutely continuous distributions in $\mathbb{R}_p$ is satisfied only by the multivariate power distribution (5.2). This generalises the result in Theorem 2.1. Finally, it is evident from the method of proof in Theorem 4.1 that the multivariate proportional reversed mean residual life model defined by

$$\boldsymbol{Y}(\boldsymbol{x}) = (c_1 m_1(\boldsymbol{x}), c_2 m_2(\boldsymbol{x}), \dots, c_p m_p(\boldsymbol{x}))$$

and the corresponding proportional reversed hazard model

$$\boldsymbol{a}(\boldsymbol{x}) = (\theta_1 h_1(\boldsymbol{x}), \theta_2 h_2(\boldsymbol{x}), \dots, \theta_p h_p(\boldsymbol{x}))$$

will yield identical distributions if and only if the baseline distribution is of multivariate power type with independent marginals having shape parameters

$$\left(\frac{c_1 - 1}{1 - c_1\theta_1}, \dots, \frac{c_p - 1}{1 - c_p\theta_p}\right).$$

6 Conclusion

Our discussions in the preceding sections were centred around the definition and properties of reversed mean residual life in higher dimensions, a simple relationship with reversed hazard rate that characterizes the power distribution and some results in connection with proportional reversed mean residual life models. Although, the definition of the BRMRL is similar to that of the usual bivariate mean residual life (BMRL), the properties of the former appeared to differ from the latter, especially in characterizing specific probability distributions. While characterizations by constancy (bivariate distributions with independent marginals), local constancy (Gumbel's distribution) linearity (Lomax and beta models) of the BMRL function hold in the entire first octant of $\mathbb{R}_2$, no such results seem to be true in the case of BRMRL except in the finite range case (see Theorems 2.1 and 2.3). Similarly, the local constancy of the product of the components of BMRL and bivariate hazard rate characterizes the Gumbel, Lomax or beta models (Asadi (1999)) where as, there is only the power distribution that shares analogous property in the reversed case as seen from Theorem 3.1. The general nature of the results indicated by the present study and the fact that the concepts in reversed time are more appropriate than those truncated from below, when the observations are predominantly from the left tail, point out the relevance and usefulness of studying reversed hazard rates and reversed mean residual life in the context of modeling and analysing data.

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