ISSN 0974-3235
ProbStat Forum, Volume 03, January 2010, Pages 11–24

Max-semistability: a survey

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Paper received on 24 March 2009; revised, 04 November 2009; accepted, 04 December 2009.

Abstract. In the classical limit theory for normalized sums of independent random variables we change the operation "sum" by the operation "maximum". How does it change the classical structure of the limit probability laws? The max-model enriches Probability Theory with interesting non-classical phenomena. Several of them will be discussed here. As an example for characterizing a limit class of probability distributions we consider the class of max-semistable df's. It is interesting to observe that the max-semistability property is a characteristic property of the univariate distributions of semi-selfsimilar extremal processes with stationary max-increments.

Mathematics Subject Classification. 62E20 ; 60G51 .
Keywords. extreme values; max-semistable laws; extremal processes; non-linear normalization.

1 Introduction

Below I expose my talk at CEAUL (Centre for Statistics and its Applications at the University of Lisboa), given on 10.07.2008. The talk is based on my former papers [5], [1], [6] and [7].

In Section 2 we discuss the so called max-model (the model of extreme values) in which one deals with the properties and the asymptotic behavior of maxima of independent random variables (rv's). The characteristic features of the operation maximum essentially impact phenomena of the classical limit theory for sum of independent rv's (cf [9]). Especially the norming mappings, proper to the max-operation, appear to be monotone rather than linear.

In Section 3 we characterize a max-semistable distribution function (df) in three different ways: a) by a certain functional equation; b) as limit distribution in a particular max-model; c) by its explicit form. At the end several examples are given.

Section 4 is devoted to the semi-selfsimilar extremal processes (SSEP's). Here we recall briefly the notion of an extremal process and particularly of a G-extremal process. We state the characteristic functional equation and the properties of SSEP's. It turns out that a G-extremal process is semi-selfsimilar with respect to (w.r.t.) a time-space change $\eta(t,x) = (\tau(t), L(x))$ if and only if $G(.)$ is max-semistable w.r.t. $[1/\tau(1), L(.)]$. Under time-space change we understand a mapping $\eta : (0,\infty)^2 \to (0,\infty)^2,\ \eta(\mathbf{0}) = \mathbf{0}$ strictly increasing and continuous in both coordinates.

2 The max-model

B. Vl. Gnedenko's (1943) paper "Sur la Distribution Limite du Terme Maximum d'une Serie Aleatoire" marked the beginning of a new branch in the modern Stochastics, the Extreme Value Theory (EVT). During my participation at the Saturday-seminars of Gnedenko at MSU I was introduced to this fascinating field and I have remained faithful to it till today. Especially analogies and differences between both max- and sum-model have occupied my mind.

Sure you know the old lithography where three whales support the earth ball. The three whales of the Probability theory are: the notion of independence, the law of large numbers and the central limit theorem. What would happen if one changed the operation sum (+) by the operation maximum ($\vee$) in these celebrated theorems? In fact, this intriguing question led the stochasticians in developing the modern EVT.

Let χ be the set of all independent rv's in $R := (-\infty, +\infty)$ and let $\mathcal{F}$ be the corresponding set of their df's. Below we denote by "$*$" the operation convolution between df's and by "." the multiplication. The sum-model $(\chi, +)$, respectively (resp.) $(\mathcal{F}, *)$, is a semigroup with unit element the zero, resp. the degenerated at zero distribution δ_0, as $X + 0 = X$. For given X and Z, the relation $X + Y = Z$ determines uniquely (in a certain sense) $Y = Z - X$. In the max-model the point $\{-\infty\}$ plays the role of unit element, as $X \vee \{-\infty\} = X$. Let $\bar{\chi} := \chi \cup \{-\infty\}$ and let $\bar{\mathcal{F}}$ denote the enlarged set $\mathcal{F}$ with all distributions assuming mass at $\{-\infty\}$. Now the max-model $(\bar{\chi}, \vee)$, resp. $(\bar{\mathcal{F}}, .)$ is a semigroup with unit element $\{-\infty\}$, resp. $\delta_{-\infty}$. Unfortunately, in the max-model $(\bar{\chi}, \vee)$ there is no inverse operation to "$\vee$" and for given X and Z the relation $X \vee Y = Z$ does not determine uniquely Y. The intrinsic difficulty in the max-model is the non-uniqueness of the max-components in any decomposition $Z = Z_1 \vee Z_2 \vee ... \vee Z_n$. This phenomenon, called Blotting, is discussed in [1].

There is a full analogy in characterizing classes of distributions in both models $(\chi, +)$ and $(\bar{\chi}, \vee)$. Let us mention only the class ID of the infinitely divisible df's and the class S of the stable df's. Recall, a rv X with df F is id (infinitely divisible) if for all $n \geq 2$, X can be decomposed in n iid sum-components $X \stackrel{d}{=} X_{n1} + ... + X_{nn}$, i.e. $F = F_n^{*n}$. Here F_n is df of X_{n1}. In the same way a rv X is max-id if for all $n \geq 2$, X can be decomposed in n iid max-components $X \stackrel{d}{=} X_{n1} \vee ... \vee X_{nn}$, i.e. $F = F_n^n$. Recall, the stable df's are characterized by the functional equation $F(x) = F^{*n}(a_n x + b_n)\ \forall n$ and with norming sequences $a_n > 0,\ b_n \in R$, whereas the max-stable df's satisfy $F(x) = F^n(a_n x + b_n)\ \forall n$.

Just here one can put the heretical question: Do the norming mappings in $(\bar{\chi}, \vee)$ have to be linear like in $(\chi, +)$? Our definite answer is "not" and we choose max-automorphisms $L : R \leftrightarrow R$ as norming mappings (see the discussion in [5]). The max-automorphisms are strictly increasing and continuous, so they preserve the max-operation, i.e. $L(X \vee Y) = L(X) \vee L(Y)$, there exists the inverse mapping L^{-1} and they form a group w.r.t. the composition "$\circ$" . We denote it by GMA.

The new norming mappings call for a new understanding of the notion *type*(F) and a new formulation of the convergence to type theorem (CTT). We say G belongs to *type*(F) if there exists $T \in GMA$ such that $G = F \circ T$. A convergence to type takes place if both convergences $F_n \stackrel{w}{\to} F$ and $F_n \circ T_n \stackrel{w}{\to} G$ imply $G \in$ *type*(F), where $T_n \in GMA$. Using here max-automorphisms we are confronted with similar difficulties as if we were working in a space of infinite dimension. Let $\mathcal{R}$ be a subset of GMA closed w.r.t. the pointwise convergence τ. Take $\{T_n\} \subset \mathcal{R}$. Then the new CTT claims: *The τ-compactness of the sequence $\{T_n\}$ is necessary and sufficient for a convergence to type.*

CTT is the main tool for limit theorems for cumulative sums and extremes. Its new formulation in the max-model makes it difficult for application. In order to overcome it, we RESTRICT our investigation to REGULAR norming sequences only.

Definition 2.1 *A sequence $\{L_n\} \subset GMA$ is called regular if there exists a continuous one-parameter group (c.o.g.) $\{\mathbf{L}_t : t > 0\}$, i.e. $\mathbf{L}_s \circ \mathbf{L}_t = \mathbf{L}_{st}\ \forall s, t > 0$ and the correspondence $t \to \mathbf{L}_t(x)$ is continuous $\forall x \in R$, such that pointwise*

$$L^{-1}_{[nt]} \circ L_n(x) \to \mathbf{L}_t(x) \tag{2.1}$$

uniformly on compact subsets of $\{t > 0\}$. Further, $\{L_n\}$ is called semi-regular if it can be embedded into a regular sequence.

The main advantage of restricting to (semi)regular norming sequences is that instead of using the CTT we use the continuity of the composition to obtain a limit max-(semi)stable distribution.

Here and further on we consider only non-degenerated limit df's. We denote by *SuppG* (resp. *SuppX*) the support of a df G (resp. X).

Theorem 2.1 *The following statements are equivalent and characterize the max-stable distributions on R.*
a) G satisfies the functional equation

$$G^t(x) = G(\mathbf{L}_t(x)), \quad x \in R, \tag{2.2}$$

w.r.t. a c.o.g. $\mathcal{L} = \{\mathbf{L}_t : t > 0\} \subset GMA$.
b) There exists a strictly increasing continuous mapping $h : SuppG \leftrightarrow R$ *such that*

$$G(x) = \exp\{-e^{-h(x)}\} \tag{2.3}$$

and $\mathbf{L}_t(x) = h^{-1}(h(x) - \log t)$.

c) There are iid rv's $X_1, X_2, ...$ with df F and a regular norming sequence $\{L_n\}$ satisfying (2.1) such that

$$F^n(L_n(x)) \xrightarrow{w} G(x), \quad x \in R.$$

In case c) we say that F belongs to the generalized max-domain of attraction of G w.r.t. $\mathcal{L}$, briefly $F \in GDA_{\mathcal{L}}(G)$. Clearly $GDA_{\mathcal{L}}(G) \subseteq GDA(G)$ for every subgroup $\mathcal{L} \subseteq GMA$. Here "generalized" stays for reminding that we use non-linear mappings for normalization. Let us note also that, in view of expression (2.3), any continuous strictly increasing df is max-stable.

We like to underline here that representation (2.3) can be expressed also in the parametric form

$$G(x) = \exp\{-ae^{-bh_1(x)}\}$$

for aiming a parametrization of the class MS of the max-stable df's. Indeed, since $h(x) = bh_1(x) - \log a$, then $\mathbf{L}_t(x) = h^{-1}(h(x) - \log t) = h_1^{-1}(h_1(x) - \frac{1}{b}\log t)$.

Let us end this section with underlining one more difference between $(\chi, +)$ and $(\bar{\chi}, \vee)$, namely the notion of asymptotic negligibility (AN). Assume $\{X_{nk} : k = 1, 2, ..., k_n\},\ n > 1,\ k_n \to \infty$ are row-wise independent rv's. In the sum-model the condition

$$(AN) \qquad \max_k P(|X_{nk}| > \epsilon) \to 0, \quad n \to \infty$$

means an asymptotic closeness to zero (the semigroup unit element). Then the convergence $\sum_{k=1}^{k_n} X_{nk} \xrightarrow{d} Y,\ n \to \infty$ implies Y is id. Let l be the left endpoint of $SuppY$. In the max-model the convergence $\bigvee_{k=1}^{k_n} X_{nk} \xrightarrow{d} Y$ together with the assumption

$$(max - AN) \qquad \max_k P(X_{nk} > x) \to 0, \quad n \to \infty, \quad \forall x > l$$

implies Y is max-id. Thus, we speak here of an asymptotic negligible contribution of the individual X_{nk} to the limit behavior of their row-wise maxima rather than of asymptotic negligible size of X_{nk}.

3 Max-stability

There are many approaches to characterize a class $\mathcal{A}$ of df's. The most popular ones are:

(a) to determine arbitrary df $G \in \mathcal{A}$ as satisfying a certain functional equation;

(b) to give the explicit form characteristic of arbitrary df $G \in \mathcal{A}$;

(c) to obtain arbitrary df $G \in \mathcal{A}$ as limit distribution in a special max-model.

In case we succeed in proving $(a) \Longleftrightarrow (b) \Longleftrightarrow (c)$ we may then think of $\mathcal{A}$ as being completely characterized. In order to demonstrate this scheme let $\mathcal{A}$ be the class MSS of all max-semistable df's on R, see [2] and [6].

(a) A df G is referred to as max-semistable (max-ss) if it satisfies the functional equation

$$G^{\alpha}(x) = G(L(x)), \quad x \in R, \tag{3.1}$$

for a pair of parameters $[\alpha, L],\ \alpha \in (0,1],\ L \in GMA$.

We write briefly $G \in MSS([\alpha, L])$. Below we observe the most important consequences of characteristic equation (3.1):

a1) The n-times iteration of (3.1) results in

$$G^{\alpha^n}(x) = G(L^{\circ n}(x))$$

for $n = 0, \pm 1, \pm 2, ...$where $L^{\circ 0}$ is the identical mapping, $L^{\circ n} = L \circ L^{\circ(n-1)}$, and $L^{\circ(-n)} := (L^{-1})^{\circ n}$. Thus, if $G \in MSS$ then G can not have mass at the left (l) and the right (r) endpoints of $SuppG$.

a2) The set $\Gamma(L) := \{L^{\circ n} : n = 0, \pm 1, \pm 2, ...\} \subset GMA$ forms a cyclic group w.r.t. the composition and the following boundary condition

$$(BC) \qquad L^{\circ n}(x) \to r, \quad L^{\circ(-n)}(x) \to l$$

is satisfied for $n \to \infty$ and for all continuity points $x \in \{0 < G < 1\}$.

For known G let us look at (3.1) as an equation for L. Then one can prove (e.g. [7]) the following property.

a3) The cyclic group $\Gamma(L)$ can be embedded in a c.o.g. $\mathcal{L} = \{\mathbf{L}_t(.) = h^{-1}(h(.) + c^{-1}\log t) : t > 0\} \subset GMA,\ c > 0$ so that $L^{\circ n}(.) = \mathbf{L}_{(\frac{1}{\alpha})^n}(.) = h^{-1}(h(.) + n\frac{1}{c}\log\frac{1}{\alpha})$. Here $h : (l, r) \leftrightarrow R$ is continuous and strictly increasing. So, we may use the notation $G \in MSS([\alpha, h])$ if necessary.

Let us denote the semi-invariant group of G by $SIG(G) := \{T \in GMA : \exists t > 0,\ G^t(x) = G(T(x))\}$, and state (cf [5]) the following proposition.

Proposition 3.1 *A df G belongs to MSS iff its SIG contains a cyclic group $\Gamma(L),\ L \in GMA$, satisfying (BC).*

a4) Max-semistability is a type-property: $G \in MSS([\alpha, L]),\ H = G \circ T$ imply $H \in MSS([\alpha, T^{-1} \circ L \circ T])$.

a5) $G^t \in type(G)$ for countably many $t \in \{\alpha^n,\ n = \pm 1, \pm 2, ...\}$.

a6) A max-semistable df G may be discontinuous, see Example 3.1 at the end of this section. The following statement is easily checked.

Proposition 3.2 *If $I(a) := \{x : G(x) = a\}$ is an interval of constancy of G, then such are countably many disjoint intervals $I(a^{\alpha^n}) = L(I(a^{\alpha^{n-1}}))$.*

Let us say some words on the structure of $SuppG$:

a7) A max-semistable df may be (i) absolutely continuous, or (ii) discrete, or (iii) singular. For an example in the last case we refer to R.Salem (1943) who gave an example of a singular df which is strictly increasing. There are examples for $G \in MSS$ having one component of kind (i) and another component of kind (ii) as well as one component of kind (i) and another component of kind (iii) (in I.Grinevich (1994), Ph.D.Thesis).

Just on place here is the following

Problem 3.1 *Give an example for a max-semistable df having components of all the three kinds.*

a8) Any df G satisfying (3.1) is max-id, since G is limiting df for row-wise maxima in a triangular array of independent rv's obeying condition (max-AN). Indeed, put $k_n := [\frac{1}{\alpha}]^n, \quad L_n := L^{\circ n}, \quad X_{nk} := L_n^{-1}(X_k)$. Here X_k are iid with df G. Then the characteristic equation (4) reads

$$G(x) = G^{\frac{1}{\alpha^n}}(L^{\circ n}(x)) \sim G^{k_n}(L_n(x)) = P(X_{n1} \vee ... \vee X_{nk_n} < x)$$

where $P(X_{nk} > x) = P(X_k > L_n(x)) \to 0,\ n \to \infty,\ x > l$. Furthermore, the norming sequence $\{L_n\}$ satisfies the relation $L_n^{-1} \circ L_{n+1} = L$, i.e. $\{L_n\}$ is semi-regular.

Let us denote the class of all max-id df's by MID, the class of all max-semi-selfdecomposable df's by $MSSD$, the class of all max-selfdecomposable df's by MSD and the class of all max-stable df's by MS. Figure 1 below gives roughly the relations among these classes.

(b) The explicit form of a df $G \in MSS([\alpha, h])$ is given by

$$G(x) = \exp\{-e^{-ch(x)} p_\alpha(h(x))\},\ c > 0, \tag{3.2}$$

where $p_\alpha(.)$ is a periodic function of period $T = \frac{1}{c} \log \frac{1}{\alpha}$.

More precisely, (3.2) is the solution of (3.1). To see this we substitute $L(x) = h^{-1}(h(x) + \frac{1}{c} \log \frac{1}{\alpha})$ in characteristic equation (3.1) and get

$$G(x) = G^{1/\alpha}(h^{-1}(h(x) + \log \phi)), \quad \phi > 1, \quad \alpha \phi^c = 1.$$

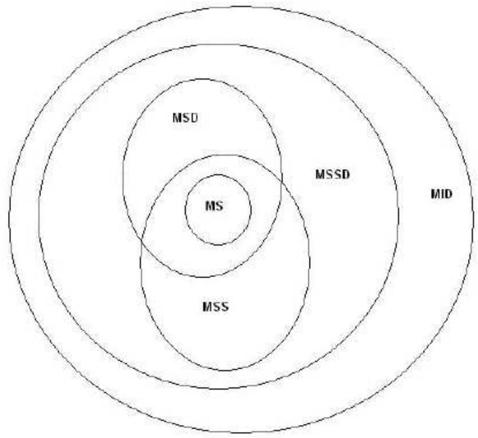

Figure 1: Classes DFs

Put $H_G(x) := -\log G \circ h^{-1}$ and $y := h(x)$. Now (3.1) implies the following chain of equations

$$H_G(y) = \frac{1}{\alpha} H_G(y + \log\phi) = ... = \frac{1}{\alpha^n} H_G(y + \log\phi^n).$$

We multiply all sites by e^{cy} and recall that $\alpha^n = \exp\{-cn \log\phi\}$. So we get

$$e^{cy} H_G(y) = e^{c(y+\log\phi)} H_G(y + \log\phi) = ... = e^{c(y+n\log\phi)} H_G(y + n\log\phi).$$

Hence the function $p_\alpha(y) := e^{cy} H_G(y)$ is periodic with a period $T = \log\phi = \frac{1}{c}\log\frac{1}{\alpha}$. From here one gets finally (3.2).

The periodic function $p_\alpha(.)$ is not arbitrary but possesses the following properties (P):

p1) $p_\alpha(y) > 0$;

p2) $cp_\alpha(y) \geq p'_\alpha(y)$ where $p'_\alpha(y)$ is the first derivative ;

p3) $e^{-cy} p_\alpha(y) \downarrow 0$ for $y \to \infty$;

p4) $p_\alpha(.)$ is a bounded function.

Expression (3.2) justifies the name "semistable", since the double exponent function characterizes a max-stable df. Denote by $\mathcal{P}$ the set of all periodic functions on R satisfying conditions (P). So we state that the class MSS is generated in the following way

$$MSS = \{G^p : G \in MS,\ p \in \mathcal{P}\}.$$

In Figure 1 let us denote $A := MSS \backslash MSD$, and $B := (MSD \cap MSS) \backslash MS$. Obviously if $p_\alpha(.) \equiv$ constant then $G \in MS$. Let us recall that the characteristic property of a max-selfdecomposable df G w.r.t. a continuous one-parameter semigroup $\{\mathbf{L}_t : t \in (0,1)\} \subset GMA,\ \mathbf{L}_t(.) = h^{-1}(h(.) - c\log t)$ is that $H_G(.)$ is a convex function (cf [6]). From here we conclude that the periodic function $p_\alpha(.)$ of a df $G \in B$ has to satisfy the inequality:

$$c^2 p_\alpha(y) + p''_\alpha(y) \geq 2c p'_\alpha(y). \tag{3.3}$$

Let us see a few examples of max-ss df's.

Example 3.1 $G(x) = \exp\{-e^{-[x]}\} \in A.$

Indeed, $G(x) = \exp\{-e^{-x}e^{\{x\}}\}$. Here $[x]$ and $\{x\}$ denote the integer and the fractional part of x, resp. Hence G is max-ss w.r.t. $[\alpha = e^{-1}, L(x) = x+1]$. Further $c = 1,\ T = 1,\ h(x) = x$ and the characteristic equation (3.1) reads

$$G^\alpha(x) = \exp\{-e^{-[x]}e^{-1}\} = G(x+1) = G(L(x)).$$

Moreover, G is step function, hence $G \notin MSD$.

Example 3.2 $G(x) = \exp\{-x^{-1}(d - \sin(\log x))\} \in B,\ x > 0,\ d > 2.$

We can rewrite this expression in the form (3.2) where $h(x) = \log x,\ c = 1,\ T = 2\pi,\ p_\alpha(y) = d - \sin(y)$. Hence G is max-ss w.r.t. $[\alpha = e^{-2\pi}, L(x) = x/\alpha]$. Let us check inequality (3.3): it is satisfied for $d > 2$. The same result we receive if we check directly the convexity of $H_G(x)$. One can see that $H''_G(x) > 0$ for $d > 2$. Then $G \in MSD$. As far as $\alpha \neq 1,\ G \notin MS$.

Example 3.3 *(see Grinevich (1993))*

Let $p_\alpha(.)$ be periodic function with a period $T = -\log q,\ 0 < q < 1$, such that $x^{-\alpha}p_\alpha(\log x)$ is non-increasing. Then

$$G(x) = \exp\{-(\log \frac{r}{x})^{-\alpha} p_\alpha(\log\log \frac{r}{x})\},\ x \in (0, r)$$

is max-ss w.r.t. $[\alpha = -\frac{\log r}{\log q},\ L(x) = r^{1-q}|x| sign x]$.

Now we go over to the next characterization of a max-ss df.

(c) Any df $G \in MSS$ is limiting df in the following max-model: For a sequence $\{X_n\}$ of iid rv's with df F we assume that there exists a sequence of integers $k_n \to \infty$ and a norming sequence $\{L_n\} \subset GMA$ such that the following conditions (I) are fulfilled:

$$\begin{array}{ll} (i) & L_n^{-1} \circ L_{n+1} \to L \in GMA \\ (ii) & F^{k_n}(L_n(x)) = P(L_n^{-1}(X_1 \vee ... \vee X_{k_n}) < x) \xrightarrow{w} G(x). \end{array}$$

As direct consequences of conditions (I) we observe the following properties of G:

c1) G is max-id, since it is limit df for a triangular array of row-wise iid rv's $\{X_{nk} = L_n^{-1}(X_k),\ k = 1...k_n\},\ n \geq 1$ which satisfy the max-AN condition, i.e.

$$P(X_{nk} < x) = F(L_n(x)) \sim G^{1/k_n}(x) \to 1,\ n \to \infty$$

for all continuity points $x \in \{0 < G < 1\}$.

c2) There exists $\lim \frac{k_n}{k_{n+1}} =: \alpha \in [0, 1]$.

The case $\alpha = 0$ implies G is degenerate, $\alpha = 1$ implies G is max-stable and $k_n \sim n, \quad \alpha \in (0, 1)$ implies that the characteristic functional equation (3.1), namely $G^\alpha = G \circ L$, holds.

Caution: (c2)+(ii) do not in general imply (i), as it is the case when using linear normalization $L_n(x) = a_n x + b_n$.

c3) The characteristic asymptotic relations (I) can be read also as "df F belongs to the partial general domain of attraction of the max-ss df G w.r.t. the pair $[\alpha, L]$", briefly $F \in PGDA(G[\alpha, L])$.

Here is on place to put the next

Problem 3.2 *Find necessary and sufficient conditions for* $F \in PGDA(G[\alpha, L])$ *using semi-regular sequences* $\{L_n\} \subset GMA$.

At the end of this section we discuss the normal max-domain of attraction of a max-ss distribution. As known, any max-ss df belongs to its own $PGDA$, i.e. for $n \to \infty$

$$G(x) = G^{1/\alpha^n}(L^{\circ n}(x)) \sim G^{k_n}(L_n(x))$$

where $k_n = [\frac{1}{\alpha}]^n, \ L_n(x) = L^{\circ n}(x) = h^{-1}(h(x) + n \log \phi), \ \alpha\phi^c = 1$. We say that F belongs to the normal generalized max-domain of attraction of $G \in MSS$, briefly $F \in NGDA(G)$, if $F \in PGDA(G)$ w.r.t. the same sequences $\{k_n\}$ and $\{L_n\}$ as G itself. The asymptotic relation

$$F^{k_n}(L_n(x)) \to G(x) = \exp\{-e^{-ch(x)} p_\alpha(h(x))\}$$

is equivalent to

$$1 - F(L_n(x)) \sim k_n^{-1} e^{-ch(x)} p_\alpha(h(x))$$

for $n \to \infty$. We put here $y := h(x), \ z := y + n \log \phi$ and get further

$$1 - F \circ h^{-1}(z) \sim k_n^{-1} \phi^{cn} e^{-cz} p_\alpha(z - n \log \phi).$$

Hence we state the following

Proposition 3.3 *Let G be max-semistable w.r.t. the pair $[\alpha, h]$. A df F belongs to the $NGDA(G[\alpha, h])$ iff $1 - F \circ h^{-1}(z) = e^{-cz} p_\alpha(z)[1 + o(1)]$.*

4 Semi-selfsimilar extremal processes

Let $X = \{X(t) : t \geq 0\}$ be a random process in R. By the celebrated Kolmogoroff's theorem if the family of all finite dimensional distributions (fdd's) $P(X(t_1) \leq x_1, ..., X(t_m) \leq x_m)$ is known, then we may think X known. Assume that X has independent additive increments, i.e. if for arbitrary finite sequence $0 = t_0 < t_1 < ... < t_m = t$ we denote here $U_k = U(t_{k-1}, t_k] := X(t_k) - X(t_{k-1}),\ k = 1, 2, ..., m$ then the rv's $U_1, U_2, ..., U_m$ are independent and

$$(X(t_1), X(t_2), ..., X(t_m)) \stackrel{d}{=} (U_1, U_1 + U_2, ..., U_1 + ... + U_m)$$

Thus, $X(t)$ can be decomposed in sum of independent increments $X(t) = \sum_{k=1}^{m} U_k$.

Analogously, random processes with independent max-increments we call extremal processes.

Definition 4.1 *An extremal process* $Y : [0, \infty) \to [0, \infty)$ *has the properties:*
1. the sample paths are right continuous increasing functions,
2. for $0 = t_0 < t_1 < ... < t_m = t$ *there exist independent rv's* $U_1, ...U_m$ *(called max-increments) such that*

$$(Y(t_1), Y(t_2), ..., Y(t_m)) \stackrel{d}{=} (U_1, U_1 \vee U_2, ..., U_1 \vee ... \vee U_m).$$

Obviously, $Y(t)$ can be decomposed in maximum of independent max-increments $Y(t) = \bigvee_{k=1}^{m} U_k$. The multidimensional distributions of an extremal process are completely determined by the family of df's $\{F_t(.)\}$ of the random variables $\{Y(t) : t \geq 0\}$ since these determine the df's of the max-increments: for $Y(t) = Y(s) \vee U(s, t]$ the df $H(.)$ of $U(s, t]$ is just the quotient $H(.) = F_t(.)/F_s(.)$. One has to be a little careful if $F_s(x)$ vanishes for certain $x > 0$. Hence, define the increasing function $C(t) := \inf\{x \geq 0 : F_t(x) > 0\}$. This is the lower endpoint of the df $F_t(.)$, and the curve $C : [0, \infty) \to [0, \infty)$ is the so called lower curve of the process $Y(.)$, see [1]. The df of the max-increment $U(s, t]$ above the lower curve is unique if we impose the condition $U(s, t] \geq C(t)$ a.s., for all $0 \leq s < t$. Below we count several distributional properties (E) of an extremal process $Y : [0, \infty) \to [0, \infty)$.

E1) The df of the process $P(Y(t) < x) =: f(t, x)$ (for fixed t, $f(t, x) =: F_t(x)$) is increasing and left-continuous in x, decreasing and right-continuous in t.

E2) The df of the max-increment $U(s, t]$ over a time interval $(s, t]$ is given by $P(U(s, t] < x) = \frac{F_t(x)}{F_s(x)}$ for $s < t$ and $x > C(t)$.

E3) For $0 < t_1 < ... < t_n$, $0 < x_1 < ... < x_n$, $(t_k, x_k) \in [0, C]^c$ the multivariate df of the process is determined by

$$F_{t_1,...,t_n}(x_1, ..., x_n) = F_{t_1}(x_1) \frac{F_{t_2}(x_2)}{F_{t_1}(x_2)} ... \frac{F_{t_n}(x_n)}{F_{t_{n-1}}(x_n)}.$$

Here $[0, C]^c$ is the set of all points x which lie above the lower curve C.

E4) Any real-valued extremal process Y is generated by a Poisson point process (P.p.p.) $\mathcal{N} = \{(T_k, X_k) : k \geq 1\}$, and a lower curve C such that

$$Y(t) = C(t) \vee \{\vee X_k : 0 < T_k \leq t\}.$$

There is a simple connection between the df of the extremal process and the mean measure μ of the generating P.p.p.

$$F_t(x) = P(Y(t) < x) = P(\mathcal{N}([0,t] \times [x,\infty)) = 0) = e^{-\mu([0,t]\times[x,\infty))}.$$

On Figure 2 below one possible sample path of an extremal process is depicted.

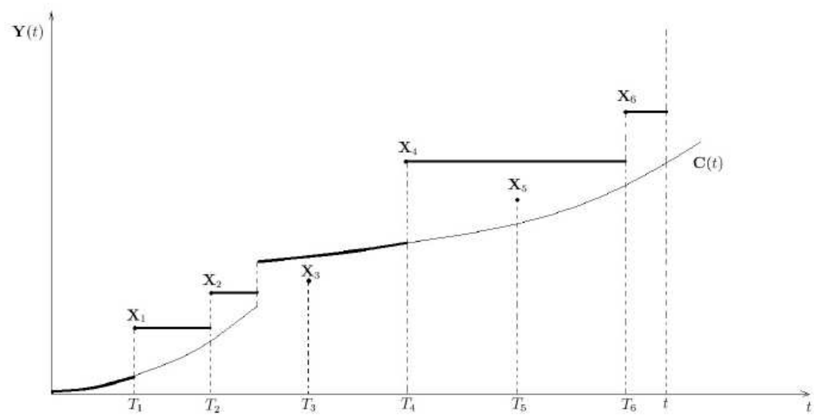

Figure 2: Extremal Process

A particular subclass of extremal processes is the class of the so called G-extremal processes, introduced in [4]. Let G be a df on R and choose $t_1 < ... < t_n,\ x_1 < ... < x_n$. The family of fdd's

$$F_{t_1,...,t_n}(x_1,...,x_n) = G^{t_1}(x_1).G^{t_2-t_1}(x_2)...G^{t_n-t_{n-1}}(x_n)$$

defines an extremal process Y with df $f(t,x) = P(Y(t) < x) = G^t(x)$ where $G(x) = P(Y(1) < x)$ and with df of the max-increments $P(U(s,t] < x) = G^{t-s}(x)$. Hence $C(t) \equiv l = \inf\{G > 0\}$. The generating p.p. $\mathcal{N} = \{(T_k, X_k)\}$ is time-homogeneous P.p.p. with mean measure

$$\mu([0,t] \times [x,\infty)) = t\nu([x,\infty))$$

where the exponent measure ν of $Y(1)$ is determined by

$$\nu([x,\infty)) = -\log G(x), \quad x \geq l.$$

Another subclass of extremal processes is the class of the semi-selfsimilar extremal processes (SSEP's). Here our interest in SSEP's is caused by their relation to the max-semistability given in Proposition 4 below. The SSEP's are introduced and studied in [7]. Here we give a short overview of their properties.

Let $Y:[0,\infty)\to[0,\infty)$ have df g and let $\eta(t,x)=(\tau(t),L(x))$ be a time-space change in $GMA((0,\infty)^2)$ such that its cyclic group $\Gamma(\eta)=\{\eta^{\circ n}:n=0,\pm1,\pm2,...\}$ satisfies the boundary conditions:

$$\eta^{\circ n}\to(-\infty,\infty),\ \eta^{\circ(-n)}\to(0,0),\ n\to\infty\ .$$

We call Y semi-selfsimilar w.r.t. η if

$$Y(\tau(t))\stackrel{fdd}{=}L(Y(t))\ . \tag{4.1}$$

Hence, its df satisfies the characteristic functional equation

$$g(\tau(t),x)=g(t,L^{-1}(x))\quad\forall t>0\ . \tag{4.2}$$

The n-times iteration in (4.1) leads to $Y(\tau^{\circ n}(t))\stackrel{d}{=}L^{\circ n}(Y(t))$. Moreover, we observe that the semi-selfsimilar extremal process Y has the properties:

1. g is invariant w.r.t. $\Gamma(\eta)$.

2. $Y(0)=C(0)$ a.s.

3. $\Gamma(\eta)$ can be embedded in a c.o.g. $\{\xi_s(z)=h^{-1}(h(z)+e.\log s):s>0\}$, $e=(1,1)$, such that $\eta=\xi_\phi$ for some $\phi>1$ and $\eta^{\circ n}=\xi_{\phi^n}$.

4. Denote $h(t,x)=(h_0(t),h_1(x))$ in the above expression of $\{\xi_s(z)$. Then the process $X(t):=h_1\circ Y\circ h_0^{-1}(t)$ is periodically stationary, i.e. $X(t+s)\stackrel{d}{=}X(t)\quad\forall s\in\{\log\phi^n:n\quad\text{integer}\}$.

5. If additionally Y has stationary max-increments then $g(t,x)=[g(1,x)]^t$ $=:G^t(x)$. We observe that

$$g(t,x)=g(\tau(t),L(x))=G^{\tau(t)}(L(x))=G^t(x)$$

   hence $(G^t)^{1/\tau(t)}(x)=G(L(x))$, i.e. all univariate marginals G^t of Y are max-semistable .

Proposition 4.1 *A G-extremal process is semi-selfsimilar w.r.t.* $\Gamma(\eta=(\tau,L))$ *iff* $G\in MSS([1/\tau(1),L])$.

6. In view of the n-times iteration of (4.1) we see that any semi-selfsimilar extremal process Y appears to be limiting for the sequence of extremal processes $Y_n:=L^{\circ(-n)}\circ Y\circ\tau^{\circ n}(t)$. Conversely, let $X(t)=C(t)\vee\{\vee X_k:t_k\le t\}$ be extremal process generated by the p.p. $\mathcal{N}=\{(t_k,X_k):k\ge1\}$ with deterministic time points $0<t_1<t_2<...$ and $\{X_k\}$ independent rv's. Assume that there exists a sequence of time-space changes $\{\xi_n=(\tau_n,L_n)\}\subset GMA$

satisfying the boundary conditions and such that $\xi_n^{-1} \circ \xi_{n+1} \to \eta = (\tau, L)$. Suppose further that

$$\begin{aligned} Y_n(t) &:= L_n^{-1} \circ X \circ \tau_n(t) \\ &= C_n(t) \vee \{\vee L_n^{-1}(X_k) : \tau_n^{-1}(t_k) \leq t\} \xrightarrow{d} Y(t). \end{aligned}$$

Then the df g of the limiting extremal process Y satisfies the characteristic equation (4.2), i.e. Y is semi-selfsimilar.

Finally, let us construct examples for SSEP's :

Example 4.1 $g(t,x) = \exp\{-t.e^{-[x]}\} = G^t(x)$ *with* $G \in MSS,\ \alpha = e^{-1},\ L(x) = x+1,\ \eta(t,x) = (\frac{t}{\alpha}, x+1)$.

Example 4.2 $g(t,x) = \exp\{-\frac{t}{x}.\{\log x\}\} = G^t(x)$ *with* $G \in MSS,\ \alpha = 1/\phi,\ L(x) = x.\phi,\ \eta(t,x) = (t\phi, x\phi)$.

Example 4.3 $g(t,x) = \exp\{-\frac{t}{x}.(d - \sin(\log x))\} = G^t(x)$ *with* $G \in MSS$, $\alpha = e^{-2\pi},\ L(x) = \frac{x}{\alpha},\ \eta(t,x) = (t\alpha, x\alpha)$.

Example 4.4 *Let* Y *have df* $g(t,x) = 1 - \frac{t}{x},\ x > t,\ t > 0$. *Obviously* $Y \in SSEP$ *w.r.t.* $\eta(t,x) = (\tau(t) = at, L(x) = ax)\ a > 0$. *This is an example for a semi-selfsimilar extremal process with nonhomogeneous max-increments.*

4.1 Acknowledgement

The author thanks CEAUL, Lisboa for its hospitality during her stay there.

References

[1] Balkema A.A., Pancheva E.I. (1996) : Decomposition for multivariate extremal processes. *Commun. Statist.-Theory Meth.*, **25**(4), 737-758.

[2] Grinevich I. (1992) : Max-semistable limit laws corresponding to linear and power normalizations. *Theory Probab. Appl.*, **37**, 720-721.

[3] Grinevich I. (1993) : Domains of attraction of the max-semistable laws under linear and power normalizations. *Theory Probab. Appl.*, **38**, 640-650.

[4] Grinevich I. (1994) : Univariate semistable distributions in the scheme of maximum of independent random variables. Ph.D.Thesis. UDK 519.2. Moscow.

[5] Resnick S., Rubinovich R. (1973) : The structure of Extremal Processes. *Adv. Appl. Probab.* **5**, 287-307.

[6] Pancheva E. (1994) : Extreme value limit theory with nonlinear normalization. In J.Galambos et al. (eds.) *Extreme Value Theory and Applications*, 305-318.

[7] Pancheva E. (1992) : Multivariate max-semistable distributions. *Theory Probab. Appl.*, **37**(4), 794-795.

[8] Pancheva E. (2000) : Semi-selfsimilar extremal processes. *J. Math. Sci.*, **99** (3), 1306-1316.

[9] Salem R. (1943) : On some singular monotonic functions which are strictly increasing. *Trans. Amer. Math. Soc.*, bf 53 (3), 427-439.

[10] Zolotarev V.M. (1997) : *Modern Theory of Summation of Random Variables.* VSP, Utrecht.

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