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A Note on Tail Behaviour of Distributions in the Max Domain of Attraction of the Frechét/ Weibull Law under Power Normalization

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Abstract. A simple von-Mises type sufficient condition is derived for a distribution function (df) to belong to the max domain of attraction of the Frechét / Weibull law under power normalization. It is shown that every df in the max domain of attraction of the Frechét / Weibull law under power normalization is tail equivalent to some df satisfying the von-Mises type condition. Several examples illustrating the variety of tail behaviours of df's attracted to the max domain of attraction of the Frechét law are given.

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1 Introduction

Let $X_1, X_2, \ldots,$ be independent identically distributed (iid) random variables with common df F. A df F is said to belong to the max domain of attraction of a nondegenerate df H under power normalization, denoted by $F \in \mathcal{D}_p(H)$, if

$$\lim_{n\to\infty} P\left(\left|\frac{M_n}{\alpha_n}\right|^{\frac{1}{\beta_n}} \operatorname{sign}(M_n) \le x\right) = \lim_{n\to\infty} F^n\left(\alpha_n |x|^{\beta_n} \operatorname{sign}(x)\right) = H(x)$$

for all $x \in \mathcal{C}(H)$, the set of all continuity points of H, for some normalizing constants $\alpha_n > 0$, $\beta_n > 0$, $n \ge 1$, where $M_n = \max\{X_1, X_2, \ldots, X_n\}$,

$\text{sign}(x) = -1,\ 0$ or 1 according as $x < 0,\ = 0$, or > 0. The limit laws H are called p-max stable laws. For the sake of completeness, these are given in the Appendix along with l-max stable laws, the limit laws of linearly normalized partial maxima of iid random variables and a couple of other results used in the article. We refer to Mohan and Ravi (1993), Falk et al.(2004), for details about $F \in \mathcal{D}_p(H)$.

In this article, a simple von-Mises type sufficient condition for the df F to belong to $\mathcal{D}_p(\Phi)$ is given, where Φ is the Frechét limit law given by $\Phi(x) = e^{-\frac{1}{x}}, x > 0, = 0$ otherwise. It is shown that every df in the max domain of attraction of Φ under power normalization is tail equivalent to some df satisfying the von-Mises type condition. Similar results for the Weibull law given by $\Psi(x) = e^{x}, x \leq 0, = 0$ otherwise, are stated and proofs are omitted as they are similar. Several examples illustrating the variety of tail behaviours of df's attracted to the max domain of attraction of the Frechèt law are given. The results obtained here are analogous to results on the max domain of attraction of the Gumbel law under linear normalization, given in Proposition 1.1 (b) in Resnick (1987), Proposition 3.3.25 in Embrechts et al. (1997) and Balkema and de Haan (1972).

The following sufficient condition for $F \in \mathcal{D}_l(\Lambda)$ is due to von-Mises (1936). Let $r(F) = \sup\{x : F(x) < 1\}$ denote the right extremity of the df F. Suppose that F is absolutely continuous with f as the probability density function (pdf). Let f be positive and differentiable on a left neighborhood of $r(F)$. If

$$\lim_{x \uparrow r(F)} \frac{d}{dx}\left(\frac{1 - F(x)}{f(x)}\right) = 0 \tag{1.1}$$

then $F \in D_l(\Lambda)$. A df F satisfying (1.1) is called a *von-Mises function.*
von-Mises type function: We will call df F a *von-Mises type function* if

$$\lim_{x \uparrow r(F)} x\frac{d}{dx}\left(\frac{1 - F(x)}{xf(x)}\right) = 0. \tag{1.2}$$

Representation of von-Mises type function:

Lemma 1.1 *A df F is a von-Mises type function if it has the representation:*

$$1 - F(x) = c\exp\left\{-\int_{z_0}^{x}\left(\frac{1}{ug(u)}\right)du\right\} \tag{1.3}$$

where $c > 0,\ z_0 < r(F),$ are constants, $g(x) > 0,\ z_0 < x < r(F),$ is an auxiliary function, absolutely continuous on $(z_0, r(F))$ with density g' and $\lim_{u \to r(F)} ug'(u) = 0$.

Proof: Let df F has the representation (1.3). Then its pdf is given by,

$$f(x) = \frac{c}{xg(x)} \exp\left\{ - \int_{z_0}^{x} \left(\frac{1}{ug(u)} \right) du \right\}.$$

We have

$$\begin{aligned} \lim_{x \to r(F)} x \frac{d}{dx} \left(\frac{1 - F(x)}{xf(x)} \right) &= \lim_{x \to r(F)} x \frac{d}{dx} \left(\frac{c \exp\left\{ - \int_{z_0}^{x} \left(\frac{1}{ug(u)} \right) du \right\}}{x \frac{c}{xg(x)} \exp\left\{ - \int_{z_0}^{x} \left(\frac{1}{ug(u)} \right) du \right\}} \right) \\ &= \lim_{x \to r(F)} xg'(x) = 0. \end{aligned}$$

Therefore F satisfies (1.2) and hence is a von-Mises type function.

2 Main Results

Theorem 2.1 *If a df F satisfies the representation (1.3), then $F \in \mathcal{D}_p(\Phi)$ if $r(F) > 0$, otherwise $F \in \mathcal{D}_p(\Psi)$.*

Proof: For fixed $t \in R$ and x sufficiently close to $r(F)$, (1.3) implies that

$$\begin{aligned} \frac{1 - F(x.e^{tg(x)})}{1 - F(x)} &= \exp\left(- \int_{x}^{xe^{tg(x)}} \frac{1}{ug(u)} du \right) \\ &= \exp\left(- \int_{0}^{t} \frac{xe^{sg(x)} g(x)}{xe^{sg(x)} g(xe^{sg(x)})} ds \right) \\ &= \exp\left(- \int_{0}^{t} \frac{g(x)}{g(xe^{sg(x)})} ds \right). \qquad (2.1) \end{aligned}$$

We show that the integrand converges locally uniformly to 1. For arbitrary $\epsilon > 0$, $x_0 < r(F)$, $x_0 \leq x$ close to $r(F)$, $0 < s < t$,

$$\begin{aligned} \left| g(xe^{sg(x)}) - g(x) \right| &= \left| \int_{x}^{xe^{sg(x)}} g'(v) dv \right| \\ &= \left| \int_{0}^{sg(x)} xe^{t} g'(xe^{t}) dt \right| \\ &\leq \epsilon s g(x) \\ &\leq \epsilon t g(x) \end{aligned}$$

since $xg'(x) \to 0$ as $x \to r(F)$. So

$$\left| \frac{g(xe^{sg(x)})}{g(x)} - 1 \right| \leq \epsilon t, \quad \epsilon > 0, \quad x_0 \leq x, \quad 0 < s < t.$$

Since ϵ is arbitrary, we have

$$\lim_{x\to r(F)} \frac{g(x)}{g(xe^{sg(x)})} = 1,$$

uniformly on bounded s-intervals. This together with (2.1) yields

$$\lim_{x\to r(F)} \frac{1-F(xe^{tg(x)})}{1-F(x)} = e^{-t}, \quad t>0.$$

Therefore $F \in \mathcal{D}_p(\Phi)$ by Theorem A.1. If $r(F) \leq 0$, then the proof that $F \in \mathcal{D}_p(\Psi)$ is similar and is omitted.

Theorem 2.2 *If F is a von-Mises type function then $F \in \mathcal{D}_p(\Phi)$ or $F \in \mathcal{D}_p(\Psi)$ according as $r(F) > 0$ or ≤ 0.*

Proof: Let $r(F) > 0$. Define $R(x) = -\log(1-F(x))$, $0 < x < r(F)$, and $g(x) = \frac{1-F(x)}{xf(x)}$, $0 < x < r(F)$. Then $R'(x) = \frac{f(x)}{1-F(x)} = \frac{1}{xg(x)}$, $0 < x < r(F)$ and

$$R(x) = \int_{z_0}^{x} \frac{du}{ug(u)}, \quad 0 < z_0 < x.$$

Since F is a von-Mises type function, using (1.2), we get

$$\lim_{x\to r(F)} xg'(x) = \lim_{x\to r(F)} x\frac{d}{dx}\left(\frac{1-F(x)}{xf(x)}\right) = 0.$$

Also,

$$1-F(x) = \exp\left(-R(x)\right) = \exp\left(-\int_{z_0}^{x}\frac{du}{ug(u)}\right), \quad 0 < z_0 < x.$$

Hence F satisfies representation (1.3) and by Theorem 2.1, $F \in \mathcal{D}_p(\Phi)$.

We omit the proof when $r(F) \leq 0$ in which case $F \in \mathcal{D}_p(\Psi)$, as it is similar.

For the next result, we need the well known concept of tail equivalence (see, for example, Resnick (1987)):
Df's F and G are tail equivalent if $r_0 = r(F) = r(G)$ and for some $A > 0$,

$$\lim_{x\uparrow r_0} \frac{1-F(x)}{1-G(x)} = A. \tag{2.2}$$

Theorem 2.3 *$F \in \mathcal{D}_p(\Phi)$ iff there exists a von-Mises type function F_* satisfying (1.2) with $r(F) = r(F_*) = r_0$, $r_0 > 0$, and tail equivalent to F.*

Proof: If F_* is a von-Mises type function satisfying (1.2) then $F_* \in \mathcal{D}_p(\Phi)$ by Theorem 2.2. Further, if F and F_* are tail equivalent, then by Ravi (1991), $F \in \mathcal{D}_p(\Phi)$.
Conversely, let $F \in \mathcal{D}_p(\Phi)$ with $r(F) = r_0$. We will show that there exists a von-Mises type function F_* tail equivalent to F. Define the sequence $U_0, U_1, ...$ by

$$\begin{aligned} U_0(x) &= 1 - F(x), \quad x > 0, \\ U_{n+1}(x) &= \int_x^{r_0} \frac{U_n(t)}{t} dt, \quad n = 0, 1, 2, \ldots, \quad x > 0. \end{aligned}$$

By Lemma 2.2 in Subramanya (1994), the df F_n defined by $F_n(x) = \max(0, 1 - U_n(x))$ belongs to $\mathcal{D}_p(\Phi)$ if F_{n-1} does, $n = 1, 2, \ldots$. In particular, the integral above converges. So, $F_n \in \mathcal{D}_p(\Phi)$, for $n = 0, 1, 2, ...$ and by Theorem 2.2 in Subramanya (1994), we have

$$\lim_{x \to r_0} \frac{U_{n-1}(x) U_{n+1}(x)}{U_n^2(x)} = 1. \tag{2.3}$$

We now define the function U_* on $(0, r_0)$ by

$$U_*(x) = (U_3(x))^4 (U_4(x))^{-3}, \quad x > 0.$$

Then U_* is twice differentiable on a left neighbourhood of r_0 and

$$\frac{d}{dx} \log U_* = -4 \frac{U_2}{xU_3} + 3 \frac{U_3}{xU_4} = \frac{1}{x} \left(\frac{3 - 4U_2 U_3^{-2} U_4}{U_4 U_3^{-1}} \right). \tag{2.4}$$

Then

$$\frac{U_4 U_3^{-1}}{3 - 4U_2 U_3^{-2} U_4} = \frac{U_*}{xU_*'}.$$

Here, the denominator tends to -1 as $x \to r_0$ using (2.3) and both $(\frac{d}{dx}) U_4 U_3^{-1}$ and $x U_4 U_3^{-1} (\frac{d}{dx})(3 - 4U_2 U_3^{-2} U_4)$ tend to 0 as $x \to r_0$. Hence

$$\lim_{x \to r_0} x \frac{d}{dx} \left(\frac{U_*(x)}{xU_*'(x)} \right) = 0. \tag{2.5}$$

Observe that

$$U_0 = \frac{U_0 U_2}{U_1^2} \left(\frac{U_1 U_3}{U_2^2} \right)^2 \left(\frac{U_2 U_4}{U_3^2} \right)^3 U_*.$$

Hence by (2.3), we obtain

$$\lim_{x \to r_0} \frac{U_0(x)}{U_*(x)} = 1. \tag{2.6}$$

Then $\lim_{x\to r_0} U_*(x) = 0$, and since by (2.4), U_* is decreasing on a left neighborhood of r_0, there exists a twice differentiable df F_* which coincides with $1 - U_*$ on a left neighborhood of r_0. Now, by (2.5), F_* satisfies (1.2) and is a von-Mises type function, and (2.6) shows that F_* is tail equivalent to F. Hence the proof.

Now we state a similar result for $F \in \mathcal{D}_p(\Psi)$ omitting the proof which is similar.

Theorem 2.4 *A df* $F \in \mathcal{D}_p(\Psi)$ *iff there exists a von-Mises type function* F_* *with* $r(F) = r(F_*) = r_0$, $r_0 \leq 0$, *tail equivalent to* F.

Examples:

- Example of a df F (Galambos (1978)) with $r(F) < \infty$ satisfying sufficient conditions (1.1) and (1.2) for $D_l(\Lambda)$ and $D_p(\Phi)$ respectively:

$$F(x) = \begin{cases} 0 & \text{if} \quad x < 0, \\ 1 - \exp\left(\frac{-x}{1-x}\right) & \text{if} \quad 0 \leq x < 1, \\ 1 & \text{if} \quad 1 < x. \end{cases}$$

  Note that $F \in \mathcal{D}_l(\Lambda)$ and $F \in \mathcal{D}_p(\Phi)$, the latter result is also true by Theorem A.2(b).

- Example of the exponential df satisfying sufficient conditions (1.1) and (1.2) for $D_l(\Lambda)$ and $D_p(\Phi)$ respectively:

$$F(x) = \begin{cases} 0 & \text{if} \quad x \leq 0, \\ 1 - \exp(-x) & \text{if} \quad x > 0. \end{cases}$$

  Here $r(F) = \infty$. Note that $F \in \mathcal{D}_l(\Lambda)$ and $F \in \mathcal{D}_p(\Phi)$, the latter result also true by Theorem A.2(a).

- The following df's (Mohan and Ravi (1993)) satisfy sufficient condition (1.2) for $\mathcal{D}_p(\Phi)$ but not (1.1) for $\mathcal{D}_l(\Lambda)$:

$$\begin{aligned} F_1(x) &= \begin{cases} 0 & \text{if} \quad x < 1, \\ 1 - \exp\left(-\sqrt{\log x}\right) & \text{if} \quad 1 \leq x. \end{cases} \\ F_2(x) &= \begin{cases} 0 & \text{if} \quad x < 1, \\ 1 - \frac{1}{x} & \text{if} \quad 1 \leq x. \end{cases} \\ F_3(x) &= \begin{cases} 0 & \text{if} \quad x < 1, \\ 1 - \exp\left(-(\log x)^2\right) & \text{if} \quad 1 \leq x. \end{cases} \end{aligned}$$

  Here $r(F) = \infty$ and note that $F_1 \in \mathcal{D}_p(\Phi)$, the Pareto df $F_2 \in \mathcal{D}_l(\Phi_\alpha) \subset \mathcal{D}_p(\Phi)$ by Theorem A.2(a) and $F_3 \in \mathcal{D}_p(\Phi)$. In fact, the df's F_1 and F_3 do not belong to $\mathcal{D}_l(\Lambda)$.

- The following df (Galambos (1978)) does not satisfy sufficient conditions (1.2) and (1.1) for both $D_p(\Phi)$ and $D_l(\Lambda)$:

$$F_4(x) = \begin{cases} 0 & \text{if} \quad x < e, \\ 1 - \frac{1}{\log x} & \text{if} \quad e \leq x. \end{cases}$$

Remark 2.5 $\mathcal{D}_p(\Phi)$ *contains df's whose tail behaviours are diverse.*

- $\mathcal{D}_l(\Phi_\alpha) \subset \mathcal{D}_p(\Phi)$ *by Theorem A.2(a) and hence* $\mathcal{D}_p(\Phi)$ *contains df's whose tails are regularly varying.*
- $\mathcal{D}_l(\Lambda) \subset \mathcal{D}_p(\Phi)$ *by Theorem A.2(a), (b). Hence* $\mathcal{D}_p(\Phi)$ *contains df's with exponential-like light tails.*
- $\mathcal{D}_p(\Phi)$ *contains df's with slowly varying tails.*

3 Appendix

A df F is said to belong to the max domain of attraction of a nondegenerate df G under linear normalization (notation $F \in \mathcal{D}_l(G)$) if there exist constants $a_n > 0$ and b_n real, $n \geq 1$, such that

$$\lim_{n\to\infty} P\left(\frac{M_n - b_n}{a_n} \leq x\right) = \lim_{n\to\infty} F^n(a_n x + b_n) = G(x)$$

for all $x \in \mathcal{C}(G)$, the set of continuity points of G. Then it is known that G can be one of only three types of extreme value df's, called l-max stable laws by Mohan and Ravi (1993), namely,

$$\Phi_\alpha(x) = \begin{cases} 0 & \text{if} \quad x \leq 0, \\ \exp(-x^{-\alpha}) & \text{if} \quad 0 < x, \end{cases}$$

$$\Psi_\alpha(x) = \begin{cases} \exp(-(-x)^{\alpha}) & \text{if} \quad x \leq 0, \\ 1 & \text{if} \quad 0 < x, \end{cases}$$

$$\Lambda(x) = \exp(-e^{-x}), \qquad -\infty < x < \infty,$$

where $\alpha > 0$ is a parameter.

The p-Max stable laws: Two df's F and G are said to be of the same p-type if $F(x) = G(A \mid x \mid^B sign(x)), x \in R$, for some positive constants A, B. The

p-max stable laws are p-types of one of the following six laws.

$$\begin{aligned}
H_{1,\alpha}(x) &= \begin{cases} 0 & \text{if} \quad x \leq 1, \\ \exp\{-(\log x)^{-\alpha}\} & \text{if} \quad 1 < x; \end{cases} \\
H_{2,\alpha}(x) &= \begin{cases} 0 & \text{if} \quad x < 0, \\ \exp\{-(-\log x)^{\alpha}\} & \text{if} \quad 0 \leq x < 1, \\ 1 & \text{if} \quad 1 \leq x; \end{cases} \\
H_{3,\alpha}(x) &= \begin{cases} 0 & \text{if} \quad x \leq -1, \\ \exp\{-(-\log(-x))^{-\alpha}\} & \text{if} \quad -1 < x < 0, \\ 1 & \text{if} \quad 0 \leq x; \end{cases} \\
H_{4,\alpha} &= \begin{cases} \exp\{-(\log(-x))^{\alpha}\} & \text{if} \quad x < -1, \\ 1 & \text{if} \quad -1 \leq x; \end{cases} \\
\Phi(x) &= \Phi_1(x), \quad -\infty < x < \infty; \\
\Psi(x) &= \Psi_1(x), \quad -\infty < x < \infty;
\end{aligned}$$

where $\alpha > 0$ is a parameter.

Necessary and sufficient conditions for a df F to belong to $\mathcal{D}_p(\Phi)$ and $\mathcal{D}_p(\Psi)$:

Theorem A.1:(Mohan and Ravi (1993))

- $F \in \mathcal{D}_p(\Phi)$ iff $r(F) > 0$ and there exists a positive function g such that
$$\lim_{t \uparrow r(F)} \frac{1 - F\left(t.\exp(x.g(t))\right)}{1 - F(t)} = \exp(-x),\ x > 0.$$
If this condition holds for some g, then
$$\int_a^{r(F)} \frac{\bar{F}(s)}{s} ds < \infty,\ 0 < a < r(F),$$
and the condition holds with the choice
$$g(t) = \frac{1}{\bar{F}(t)} \int_t^{r(F)} \frac{\bar{F}(s)}{s} ds.$$
In this case, one may take $\alpha_n = F^{-}\left(1 - \frac{1}{n}\right) = \inf\left\{x : F(x) \geq 1 - \frac{1}{n}\right\}$ and $\beta_n = g(\alpha_n)$ so that $\lim_{n \to \infty} F^n\left(\alpha_n x^{\beta_n}\right) = \Phi(x),\ x \in R.$

- $F \in \mathcal{D}_p(\Psi)$ iff $r(F) \leq 0$ and there exists a positive function g such that
$$\lim_{t \uparrow r(F)} \frac{1 - F\left(t.\exp(x.g(t))\right)}{1 - F(t)} = e^{x},\ x < 0.$$

If this condition holds for some g, then

$$-\int_a^{r(F)} \frac{\bar{F}(s)}{s} ds < \infty,\ a < r(F),$$

and the condition holds with the choice

$$g(t) = -\left(\frac{1}{\bar{F}(t)}\right) \int_t^{r(F)} \frac{\bar{F}(s)}{s} ds.$$

In this case, one may take $\alpha_n = -F^{-}\left(1 - \frac{1}{n}\right)$ and $\beta_n = g(-\alpha_n)$ so that $\lim_{n\to\infty} F^n\left(\alpha_n \mid x \mid^{\beta_n} sign(x)\right) = \Psi(x),\ x \in R.$

Comparison of Max Domains under Linear and Power normalization:
Theorem A.2:(Mohan and Ravi (1993))

(a) $\left.\begin{array}{l} (i) F \in \mathcal{D}_l(\Phi_\alpha) \\ (ii) F \in \mathcal{D}_l(\Lambda), r(F) = \infty \end{array}\right\} \Longrightarrow F \in \mathcal{D}_p(\Phi);$

(b) $F \in \mathcal{D}_l(\Lambda), 0 < r(F) < \infty \Longleftrightarrow F \in \mathcal{D}_p(\Phi), 0 < r(F) < \infty;$

(c) $F \in \mathcal{D}_l(\Lambda), r(F) < 0 \Longleftrightarrow F \in \mathcal{D}_p(\Psi), r(F) < 0;$

(d) $\left.\begin{array}{l} (i) F \in \mathcal{D}_l(\Lambda), r(F) = 0 \\ (ii) F \in \mathcal{D}_l(\Psi_\alpha), r(F) = 0 \end{array}\right\} \Longrightarrow F \in \mathcal{D}_p(\Psi);$

(e) $F \in \mathcal{D}_l(\Psi_\alpha), r(F) > 0 \Longleftrightarrow F \in \mathcal{D}_p(H_{2,\alpha});$

(f) $F \in \mathcal{D}_l(\Psi_\alpha), r(F) < 0 \Longleftrightarrow F \in \mathcal{D}_p(H_{4,\alpha}).$

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