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On moments of lower generalized order statistics from Frechet-type extreme value distribution and ITS characterization

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Abstract. In this paper we establish explicit expressions for single and product moments of lower generalized order statistics from Frechet-type extreme value distribution. The results include as particular cases the above relations for moments of order statistics and lower records. Further, using a recurrence relation for single moments we obtain characterization of Frechet-type extreme value distribution.
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1 Introduction

The concept of generalized order statistics (gos) was introduced by Kamps (1995) as a unified approach to different model e.g. usual order statistics, sequential order statistics, Stigler's order statistics, record values. They can be easily applicable in practice problems except when $F()$ is so called inverse distribution function. For this, when $F()$ is an inverse distribution function, we need a concept of lower generalized order statistics ($lgos$), which is given as:

Let $n \in N$, $k \geq 1$, $m \in \Re$, be such that

$\gamma_r = k + (n-r)(m+1) > 0, \quad$ for all $1 \leq r \leq n$.

By the $lgos$ from an absolutely continuous distribution function (df) $F(x)$ with the probability density function pdf $f(x)$ we mean random variables $X'(1,n,m,k),\ldots,X'(n,n,m,k)$ having joint pdf of the form

$$k\Big(\prod_{j=1}^{n-1}\gamma_j\Big)\Big(\prod_{i=1}^{n-1}[F(x_i)]^m f(x_i)\Big)[F(x_n)]^{k-1}f(x_n) \tag{1.1}$$

for

$$F^{-1}(1) > x_1 \geq x_2 \geq \ldots \geq x_n > F^{-1}(0).$$

For simplicity we shall assume $m_1 = m_2 = \ldots = m_{n-1} = m$.
The *pdf* of the $r-$th *lgos*,is given by

$$f_{X'(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}[F(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)) \tag{1.2}$$

and the joint *pdf* of $r-$th and $s-$th *lgos*, $1 \leq r < s \leq n$ is

$$f_{X'(r,n,m,k),X'(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!}[F(x)]^m f(x) g_m^{r-1}(F(x))$$
$$\times [h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s - 1} f(y), \quad x > y, \tag{1.3}$$

where

$$C_{r-1} = \prod_{i=1}^{r} \gamma_i \ , \quad \gamma_i = k + (n-i)(m+1),$$

$$h_m(x) = \begin{cases} -\frac{1}{m+1}x^{m+1}, & m \neq -1 \\ -lnx, & m = -1 \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(1), \quad x \in [0,1).$$

We shall also take $X'(0,n,m,k) = 0$. If $m = 0$, $k = 1$, then $X'(r,n,m,k)$ reduced to the $(n-r+1)-$th order statistics, $X_{n-r+1:n}$ from the sample $X_1, X_2, \ldots, X_n$ and when $m = -1$, then $X'(r,n,m,k)$ reduced to the $r-$ th lower k record value [Pawlas and Szynal, (2001)]. The work of Burkschat *et al*. (2003) may also refer for lower generalized order statistics.

Recurrence relations for single and product moments of lower generalized order statistics from the inverse Weibull distribution are derived by Pawlas and Szynal (2001). Khan *et al*. (2008) and Khan and Kumar (2010, 2011a, b) have established recurrence relations for moments of lower generalized order statistics from exponentiated Weibull, Pareto, gamma and generalized exponential distributions. Ahsanullah (2004) and Mbah and Ahsanullah (2007) characterized the uniform and power function distributions based on distributional properties of lower generalized order statistics, respectively.
Kamps (1998) investigated the importance of recurrence relations of order statistics in characterization.

In this paper, we establish explicit expressions for single and product moments of *lgos* from Frechet-type extreme value distribution. Results for order statistics and lower record values can be deduced as special cases and a characterization of Frechet-type extreme value distribution has been obtained on using a recurrence relation for single moments.

The two-parameter Frechet-type extreme value distribution know as Type II Extreme value distribution if its *pdf* is of the form

$$f(x) = \alpha\beta^{\alpha}x^{-(\alpha+1)}e^{-(\beta/x)^{\alpha}}, \quad x > 0, \alpha > 0, \beta > 0 \ \alpha, \lambda > 0 \tag{1.4}$$

and the corresponding *df* is

$$F(x) = e^{-(\beta/x)^{\alpha}}, \quad x > 0, \alpha > 0, \beta > 0, \tag{1.5}$$

where α and β are shape and scale parameters respectively. More details on this distribution and their applications can be found in Maswadah (2005).

2 Relations for single moments

We shall first establish the explicit expression for $E[X'(r,n,m,k)]$. Using (1.2), we have when $m \neq -1$

$$E[X'^j(r,n,m,k)] = \frac{C_{r-1}}{(r-1)!}\int_0^{\infty} x^j (F(x))^{\gamma_r - 1} f(x) g_m^{r-1}(F(x))dx \tag{2.1}$$

By using binomial expansion, we can rewrite (2.1) as

$$E[X'^j(r,n,m,k)] = \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}}\sum_{a=0}^{r-1}(-1)^a \begin{pmatrix} r-1 \\ a \end{pmatrix} \int_0^{\infty} x^j (F(x))^{\gamma_{r-a}-1}dx \tag{2.2}$$

By setting $t = -lnF(x)$ in (2.2), we obtain

$$E[X'^j(r,n,m,k)] = \frac{\beta^j C_{r-1}}{(r-1)!(m+1)^{r-1}}\sum_{a=0}^{r-1}(-1)^a \begin{pmatrix} r-1 \\ a \end{pmatrix} \times \Gamma(1-(j/\alpha))(\gamma_{r-a})^{(j/\alpha)-1} \tag{2.3}$$

and when $m = -1$ that

$$E[X'^j(r,n,-1,k)] = \frac{(k)^{j/\alpha}\beta^j}{(r-1)!}\Gamma(r-(j/\alpha)). \tag{2.4}$$

Special cases

i) Putting $m = 0, \ k = 1$ in (2.3), the explicit formula for single moments of order statistics of the Frechet-type extreme value distribution can be obtained as

$$E(X^j_{n-r+1:n}) = \beta^j\Gamma(1-(j/\alpha))C_{r:n}\sum_{a=0}^{r-1}(-1)^a \begin{pmatrix} r-1 \\ a \end{pmatrix} (n-r+1+a)^{(j/\alpha)-1},$$

where,

$$C_{r:n} = \frac{n!}{(r-1)!(n-r)!}.$$

ii) Putting $k = 1$ in (2.4), we deduce the explicit expression for the moments of lower record values for the Frechet-type extreme value distribution as

$$E(X_{L(r)}^{j}) = \frac{\beta^{j}}{(r-1)!}\Gamma(r-(j/\alpha)).$$

3 Relations for product moments

On using (1.3) and binomial expansion, the the explicit expressions for the product moments of *lgos* $X'^{i}(r,n,m,k)$ and $X'^{j}(s,n,m,k)$, $1 \leq r < s \leq n$ can be obtained when $m \neq -1$ as

$$E[X'^{i}(r,n,m,k)X'^{j}(s,n,m,k)] = \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}$$
$$\times \sum_{a=0}^{r-1}\sum_{b=0}^{s-r-1}(-1)^{a+b}\binom{r-1}{a}\binom{s-r-1}{b}\int_0^{\infty} y^{j}(F(y))^{\gamma_{s-b}-1}f(y)I(y)dy \tag{3.1}$$

where

$$I(y) = \int_y^{\infty} x^{i}(F(x))^{(s-r+a-b)(m+1)-1}f(x)dx. \tag{3.2}$$

by setting $t = -lnF(x)$ in (3.2), we obtain

$$I(y) = \beta^{i}[(s-r+a-b)(m+1)]^{(i/\alpha)-1}$$
$$IG(1-(i/\alpha),(s-r+a-b)(m+1)(-lnF(y))), \tag{3.3}$$

where $IG(.,.)$ denotes the incomplete gamma function defined by

$$IG(l,z) = \int_0^{z} u^{l-1}e^{-u}du.$$

Using the series expansion

$$IG(1-(i/\alpha),(s-r+a-b)(m+1)(-lnF(y)))$$
$$= [(s-r+a-b)(m+1)(-lnF(y))]^{1-(i/\alpha)}$$
$$\times \sum_{c=0}^{\infty}(-1)^{c}\frac{[(s-r+a-b)(m+1)(-lnF(y))]^{c}}{c!(c+1-(i/\alpha))}$$

(see Gradshteyn and Ryzhik (2000)),$I(y)$ in (3.3) can be expressed as

$$I(y)=\beta^{i}\sum_{c=0}^{\infty}(-1)^{c}\frac{[(s-r+a-b)(m+1)]^{c}}{c!(c+1-(i/\alpha))}(-lnF(y)))^{c+1-(i/\alpha)}.$$

On substituting the above expression of $I(y)$ in (3.1), we find that

$$E[X'^{i}(r,n,m,k)X'^{j}(s,n,m,k)]$$

$$=\frac{\beta^{i}C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{a=0}^{r-1}\sum_{b=0}^{s-r-1}\sum_{c=0}^{\infty}(-1)^{a+b+c}$$
$$\times\binom{r-1}{a}\binom{s-r-1}{b}\frac{[(s-r+a-b)(m+1)]^{c}}{c!(c+1-(i/\alpha))}$$
$$\times\int_{0}^{\infty}y^{j}(F(y))^{\gamma_{s-b}-1}f(y)(-lnF(y))^{c+1-(i/\alpha)}dy. \tag{3.4}$$

Again by setting $t=-lnF(x)$ in (3.4) and simplifying the resulting equation, we get

$$E[X'^{i}(r,n,m,k)X'^{j}(s,n,m,k)]$$

$$=\frac{\beta^{i+j}C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{a=0}^{r-1}\sum_{b=0}^{s-r-1}\sum_{c=0}^{\infty}(-1)^{a+b+c}$$
$$\times\binom{r-1}{a}\binom{s-r-1}{b}\frac{[(s-r+a-b)(m+1)]^{c}}{c!(c+1-(i/\alpha))}$$
$$\times(\gamma_{s-b})^{((i+j)/\alpha)-c-2}\Gamma(c+2-(i+j)/\alpha) \tag{3.5}$$

and when$m=-1$ that

$$E[X'^{i}(r,n,-1,k)X'^{j}(s,n,-1,k)]$$

$$=\frac{(k)^{(i+j)/\alpha}\beta^{i+j}}{(r-1)!(s-r-1)}\sum_{a=0}^{s-r-1}(-1)^{s-r-a-1}\binom{s-r-1}{a}$$
$$\times\frac{\Gamma(s-(i+j)/\alpha)}{(s-a-1-(i/\alpha))}. \tag{3.6}$$

Special cases:

i) Putting $m=0,\ k=1$ in (3.5), the explicit formula for the product moments of order statistics of the Frechet-type extreme value distribution can be obtained as

$$E(X^i_{n-r+1:n}X^j_{n-s+1:n}) = \beta^{i+j}C_{r,s:n}\sum_{a=0}^{r-1}\sum_{b=0}^{s-r-1}\sum_{c=0}^{\infty}(-1)^{a+b+c}$$
$$\times\begin{pmatrix} r-1 \\ a \end{pmatrix}\begin{pmatrix} s-r-1 \\ b \end{pmatrix}$$
$$\times\frac{(s-r+a-b)^c}{c!(c+1-(i/\alpha))}(n-s+1+b)^{((i+j)/\alpha)-c-2}\Gamma(c+2-(i+j)/\alpha),$$

where,

$$C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}.$$

ii)Putting $k = 1$ in (3.6), we deduce the explicit expression for the product moments of lower record values for the Frechet-type extreme value distribution as

$$E(X^i_{L(r)}X^j_{L(s)}) = \frac{\beta^{i+j}}{(r-1)!(s-r-1)}\sum_{a=0}^{s-r-1}(-1)^{s-r-a-1}\begin{pmatrix} s-r-1 \\ a \end{pmatrix}$$
$$\times\frac{\Gamma(s-(i+j)/\alpha)}{(s-a-1-(i/\alpha))}.$$

4 Characterization

Before coming to the main characterization theorem, we shall produce the following relation proved by Pawlas and Szynal (2001) which will be used in sequel.

$$E[X'^{j+\alpha+1}(r,n,m,k)] = \frac{\alpha\beta^{\alpha}\gamma_r}{j+1}(E[X'^{j+1}(r-1,n,m,k))]-E[X'^{j+1}(r,n,m,k))]). \tag{4.1}$$

Theorem 4.1: Let X be a non-negative random variable having an absolutely continuous distribution function $F(x)$ with $F(0) = 0$ and $0 < F(x) < 1$ for all $x > 0$, then

$$E[X'^{j+\alpha+1}(r,n,m,k)] = \frac{\alpha\beta^{\alpha}\gamma_r}{j+1}(E[X'^{j+1}(r-1,n,m,k))]-E[X'^{j+1}(r,n,m,k))]) \tag{4.2}$$

if and only if

$$F(x) = e^{-(\beta/x)^{\alpha}}, \;\; x > 0, \alpha > 0, \beta > 0.$$

Proof: The necessary part follows immediately from equation (4.1). On the other hand if the recurrence relation in equation (4.2) is satisfied, then on using equation (1.2), we have

$$\frac{C_{r-1}}{(r-1)!}\int_0^\infty x^{j+\alpha+1}[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx$$

$$=\frac{\alpha\beta^\alpha(r-1)C_{r-1}}{(j+1)(r-1)!}\int_0^\infty x^{j+1}[F(x)]^{\gamma_r+m}f(x)g_m^{r-2}(F(x))dx$$

$$-\frac{\alpha\beta^\alpha\gamma_r C_{r-1}}{(j+1)(r-1)!}\int_0^\infty x^{j+1}[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx$$

$$=\frac{\alpha\beta^\alpha C_{r-1}}{(j+1)(r-1)!}\int_0^\infty x^{j+1}[F(x)]^{\gamma_r}f(x)g_m^{r-2}(F(x))$$

$$\times\{(r-1)(F(x))^m-\frac{\gamma_r g_m(F(x))}{F(x)}\}dx. \tag{4.3}$$

Let

$$h(x)=-(F(x))^{\gamma_r}g_m^{r-1}(F(x)) \tag{4.4}$$

and

$$h'(x)=[F(x)]^{\gamma_r}f(x)g_m^{r-2}(F(x))\{(r-1)(F(x))^m-\frac{\gamma_r g_m(F(x))}{F(x)}\}$$

Thus,

$$\frac{C_{r-1}}{(r-1)!}\int_0^\infty x^{j+\alpha+1}[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx$$

$$=\frac{\alpha\beta^\alpha C_{r-1}}{(j+1)(r-1)!}\int_0^\infty x^{j+1}h'(x)dx. \tag{4.5}$$

Now integrating RHS in (4.5) by parts and using the value of $h(x)$ from (4.4), we obtain

$$\frac{C_{r-1}}{(r-1)!}\int_0^\infty x^{j+\alpha+1}[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx$$

$$=\frac{\alpha\beta^\alpha C_{r-1}}{(j+1)(r-1)!}\int_0^\infty x^{j}[F(x)]^{\gamma_r}f(x)g_m^{r-1}(F(x))dx$$

which reduces to

$$\frac{C_{r-1}}{(r-1)!}\int_0^\infty x^{j}[F(x)]^{\gamma_r-1}g_m^{r-1}(F(x))\{\alpha\beta^\alpha F(x)-x^{\alpha+1}f(x)\}dx=0. \tag{4.6}$$

Now applying a generalization of the Müntz-Szász Theorem (Hwang and Lin, 1984) to equation (4.6), we get

$$\frac{f(x)}{F(x)} = \alpha\beta^{\alpha}x^{-(\alpha+1)}$$

which prove that

$$F(x) = e^{-(\beta/x)^{\alpha}}, \ x > 0, \alpha > 0, \beta > 0 \ ,$$

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