ISSN 0974-3235

ProbStat Forum, Volume 04, October 2011, Pages 63–77

ROBUSTNESS OF DOUBLY BALANCED INCOMPLETE BLOCK DESIGNS AGAINST UNAVAILABILITY OF TWO BLOCKS

R. Shunmugathai and M.R. Srinivasan
Department of Statistics
University of Madras

e-mail: *shunmuga_77@yahoo.co.in*

Received 14 February 2011 and accepted 05 October 2011.

Abstract. Robustness of Doubly Balanced Incomplete Block Design is investigated when two blocks are lost, in terms of efficiency of the residual design. In a Doubly Balanced Incomplete Block Design when $r \geq 7$ and $2 \leq k \leq 8$, the efficiencies of 32 Doubly Balanced Incomplete Block Designs were worked out. For all, the design satisfies $e(s) \geq 0.90$. The investigation shows that Doubly Balanced Incomplete Block Designs are fairly robust in terms of efficiency. As a special case, we can also show the robustness of Doubly Balanced Incomplete Block Design when two blocks are lost.

Keywords. Doubly Balanced Incomplete Block Design, Efficiency of residual design, Youden Square Design and Latin Square Design.

1. Introduction

Yates (1936) introduced Balanced Incomplete Block Design (BIBD) in agriculture experiments. Bose (1939) developed the construction of Balanced Incomplete Block Design and its properties. Consequently several authors discussed various properties from the point of view of application. There is no reason to exclude the possibility that a BIBD would contain repeated blocks. Indeed the statistical optimality of BIBD is unaffected by the presence of repeated blocks. A Balanced Incomplete Block Design is defined as an arrangement of v treatments satisfying the following conditions.

1. The blocks shall be of a constant size, k,
2. All v treatments shall be replicated an equal number of times, r,

3. No treatment shall occur more than once in any block $n_{ij} = 1$ or 0 and

4. Each pair of treatments shall occur together in the q blocks and equal number of times λ.
   The conditions defined above is, the usual condition for Balanced Design. Calvin (1954) has introduced one more condition on the arrangement.

5. Each triplet of treatments occurs together in a block μ times.

Calvin has named such a class of BIBD as Doubly Balanced Incomplete Block Design (DBIBD) because of the restrictions on the balancing the pairs as well as triplets. He has also given the analysis of such designs (Ponnuswamy and Srinivasan, 1991).

Das and Kageyama (1992) showed that Balanced Incomplete Block Designs and extended Balanced Incomplete Block Designs are fairly robust against the unavailability of $s(s \leq k)$ observations in any block. Youden Square design and Latin Square design are found to be fairly robust against the loss of any one column.

This study looks into the robustness of Doubly Balanced Incomplete Block Design when two blocks are lost from design. C^* matrix and its non-zero eigenvalues are computed with its corresponding multiplicity and its efficiency. It shows that Doubly Balanced Incomplete Block Design is fairly robust against loss of two blocks from same treatment and from a different treatment. Here, the efficiencies of 32 Doubly Balanced Incomplete Block Designs are worked out. In fact, all design satisfies $e(s) \geq 0.90$. Thus, it shows that design is fairly robust against loss of two blocks.

The robustness criteria against the unavailability of data are: (i) to get the connectedness of the residual design; (ii) to have the variance balance of the residual design; (iii) to consider the A-efficiency of residual design.

This investigation considers a Doubly Balanced Incomplete Block Design d. Let D^* be the residual design obtained when two blocks be lost and assume D^* to be connected. In this case, the criterion of robustness against the unavailability of two blocks in Doubly Balanced Incomplete Block Design is the overall A-efficiency of the residual design D^*, which is given by,

$$e(s) = \frac{\text{Sum of reciprocals of non-zero eigenvalues of } C}{\text{Sum of reciprocals of non-zero eigenvalues of } C^*}$$

$$e(s) = \frac{\phi_2(s)}{\phi_1(s)} \tag{1.1}$$

2. Robustness of Doubly Balanced Incomplete Block Designs against Unavailability of Two Blocks

Consider a Doubly Balanced Incomplete Block Design d having parameters $v = p,\ b,\ r,\ k,\ \lambda,\ \delta$. Suppose two blocks of a Doubly Balanced Incomplete

Block Design with two blocks are lost. Under this situation, the following three cases will occur:

Case i: Unavailability of two blocks where the number of common treatments between two blocks are zero.

Case ii: Unavailability of two blocks where the number of common treatments between two blocks are one.

Case iii: Unavailability of two blocks where the number of common treatments between two blocks are two.

For all three cases when two blocks are lost from Doubly Balanced Incomplete Block Design, efficiency factor is depending upon the common number of treatments between two lost blocks. The efficiency for all three cases when common number of treatments between two lost blocks are $0, 1, 2, 3, \ldots, (k-1), k$ respectively are studied. Here, the robustness criterion of Doubly Balanced Incomplete Block Design was further discussed for the different value of common number of treatments between two blocks.

Case (i): Unavailability of two blocks where the number of common treatments between two blocks are zero

Consider a Doubly Balanced Incomplete Block Design D with parameters $v = p,\ b,\ r,\ k,\ \lambda,\ \delta$. It follows that C matrix of design D is always given by $C = \theta\left(I_v - \frac{1}{vE_{vv}}\right)$, where $\theta = \lambda\nu/k$ is the eigenvalues of C matrix of design D with multiplicity $(v-1)$.

Let two blocks be lost. Call this design as a residual design assuming a residual design D^* is a connected design. Let the blocks be b_i and b_j, let their zero treatment be common between two lost blocks i.e. $\eta(b_i \cap b_j) = 0$. Each treatment that is present in the two lost blocks will be replicated $(r-1)$ times. All remaining treatment will be replicated r times in design.

Let C^* be the information matrix of design D^*. For this design D^*, the diagonal element of C^* matrix are as follows,

1. $C_{jj} = \frac{(r-1)(k-1)}{k}$, where j denotes those treatments which are present in both the lost blocks but are distinct.

2. $C_{ll} = \frac{r(k-1)}{k}$, where l denotes the remaining treatments.

Similarly, in the residual design, pair of treatments occurs together in following ways, which we say λ_1, λ_2. Pattern of $\lambda_i,\ (i = 1, 2)$ are as follows,

1. $\lambda_1 = (\lambda - 1)$, for those treatments, which are present in two lost blocks.

2. $\lambda_2 = \lambda$, for remaining pair of treatments.

The C^* matrix of design D can be written as,

$$kC^* = \begin{bmatrix} (\lambda\nu-k)I_k-(\lambda-1)J_{kk} & -\lambda J_{kk} & -\lambda J_{k(\nu-2k)} \\ -\lambda J_{kk} & (\lambda\nu-k)I_k-(\lambda-1)J_{kk} & -\lambda J_{k(\nu-2k)} \\ -\lambda J_{k(\nu-2k)} & -\lambda J_{k(\nu-2k)} & \lambda\nu(I_{(\nu-2k)}-\nu^{-1}J_{(\nu-2k)(\nu-2k)}) \end{bmatrix}$$

The non zero eigenvalues of C^* matrix with their corresponding multiplicities are,

1. $\frac{(\lambda\nu-k)}{k}$, with multiplicity $2(k-1)$.
2. $\frac{\lambda\nu}{k}$, with multiplicity $(v-2k+1)$.

Theorem 1. *Doubly Balanced Incomplete Block Designs and with parameters $v = p, b, k, r, \lambda, \delta$ are fairly robust against the unavailability of two blocks, where the number of common treatment between two lost blocks are zero, provided the overall efficiency of the residual design is given by,*

$$e(s) = \frac{(\lambda\nu-k)(\nu-1)}{(\lambda\nu-k)(\nu-2k+1)+2\lambda\nu(k-1)} \tag{2.1}$$

Proof : Without loss of generality, let two blocks be lost from design D where the number of common treatment between two blocks is zero i.e., matrix of the residual design is given by,

$$kC^* = \begin{bmatrix} (\lambda\nu-k)I_k-(\lambda-1)J_{kk} & -\lambda J_{kk} & -\lambda J_{k(\nu-2k)} \\ -\lambda J_{kk} & (\lambda\nu-k)I_k-(\lambda-1)J_{kk} & -\lambda J_{k(\nu-2k)} \\ -\lambda J_{k(\nu-2k)} & -\lambda J_{k(\nu-2k)} & \lambda\nu(I_{(\nu-2k)}-\nu^{-1}J_{(\nu-2k)(\nu-2k)}) \end{bmatrix}$$

The non zero eigenvalues of C^* matrix with their corresponding multiplicities are,

1. $\frac{(\lambda\nu-k)}{k}$, with multiplicity $2(k-1)$.
2. $\frac{\lambda\nu}{k}$, with multiplicity $(v-2k+1)$.

Further, overall A-efficiency is calculated as,

$$e(s) = \frac{\phi_2(s)}{\phi_1(s)} \tag{2.2}$$

Where, $\phi_2(s)$ = sum of reciprocals of non-zero eigenvalues of C matrix of design D and $\phi_1(s)$ = sum of reciprocals of non-zero eigenvalues of C^* matrix of design D*.

That is,

$$\phi_2(s) = \frac{k(\nu-1)}{\lambda\nu} \tag{2.3}$$

and

$$\phi_1(s) = \frac{k(v-2k+1)}{\lambda\nu} + \frac{2(k-1)k)}{(\lambda\nu-k)}. \tag{2.4}$$

Finally, A- efficiency is given by,

$$e(s) = \frac{(\lambda\nu - k)(\nu - 1)}{(\lambda\nu - k)(v - 2k + 1) + 2\lambda\nu(k - 1)k} \tag{2.5}$$

Example 1: Let D represent the Doubly Balanced Incomplete Block Design with parameters $v = p = 8,\ b = 14,\ r = 7,\ k = 4,\ \lambda = 3,\ \delta = 1$. Design D is given by,

Table 1: 8 Treatments of DBIB Design of Lyle D. Calvin (1954)

Block	Treatments			
1	1	2	3	6
2	5	6	7	8
3	**1**	**2**	**7**	**8**
4	**3**	**4**	**5**	**6**
5	1	3	4	8
6	2	4	5	7
7	1	4	6	7
8	2	3	5	8
9	1	2	5	6
10	3	4	7	8
11	1	3	5	7
12	2	4	6	8
13	1	4	5	8
14	2	3	6	7

Two blocks containing treatments (1 2 7 8) and (3 4 5 6) are lost and number of treatment common two blocks is zero. C^* matrix of the residual design is given by,

$$4C^* = \begin{bmatrix} 18 & -2 & -2 & -2 & -3 & -3 & -3 & -3 \\ -2 & 18 & -2 & -2 & -3 & -3 & -3 & -3 \\ -2 & -2 & 18 & -2 & -3 & -3 & -3 & -3 \\ -2 & -2 & -2 & 18 & -3 & -3 & -3 & -3 \\ -3 & -3 & -3 & -3 & 18 & -2 & -2 & -2 \\ -3 & -3 & -3 & -3 & -2 & 18 & -2 & -2 \\ -3 & -3 & -3 & -3 & -2 & -2 & 18 & -2 \\ -3 & -3 & -3 & -3 & -2 & -2 & -2 & 18 \end{bmatrix}$$

The non-zero eigenvalues with their corresponding multiplicities are,

1. $\dfrac{20}{4}$, with multiplicities 6.
2. $\dfrac{24}{4}$, with multiplicities 1.

The overall A- efficiency of the design is, $e(s) = 0.853659$.

Case (ii): Unavailability of two blocks where the number of common treatment between two blocks are one.

Consider a Doubly Balanced Incomplete Block Design D with parameters $v = p,\ b,\ k,\ r,\ \lambda,\ \delta$. The C matrix of the design is given by $C = \theta\left(I_v - \frac{1}{vE_{vv}}\right)$, Where $\theta = (\lambda\nu/k)$ is the eigenvalues of C matrix of design D with multiplicity $(v-1)$. Let two blocks be lost. Call this design as a residual design and assume that the residual design D^* is a connected design. Let the blocks be b_i and b_j, let one treatment be common between two lost blocks i.e.$\eta(b_i \cap b_j) = 1$. Here, this treatment is repeated $(r-2)$ times. Similarly, those treatments that are present in the two lost blocks but are not common will be replicated $(r-1)$ times. The remaining treatments will be replicated r times in design.

Let C^* be the information matrix of design D^*. For this design D^*, the diagonal element of C^* matrix are as follows,

1. $C_{ii} = \frac{(r-2)(k-1)}{k}$, where i denotes those treatments which are present in both the lost blocks but are distinct.

2. $C_{jj} = \frac{(r-1)(k-1)}{k}$, where j denotes those treatments which are present in both the lost blocks but are distinct.

3. $C_{ll} = \frac{r(k-1)}{k}$, where l denotes the remaining treatments.

Similarly, in the residual design, pair of treatments occurs together in following three ways, which we say $\lambda_1, \lambda_2, \lambda_3$. Pattern of $\lambda_i (i = 1, 2, 3)$ are as follows,

1. $\lambda_1 = (\lambda - 1)$, for those treatments that are present in two lost blocks.

2. $\lambda_2 = (\lambda - 1)$, for those treatments that are present in two lost blocks.

3. $\lambda_3 = \lambda$, for remaining treatment.

The C* matrix of design D can be written as,

$kC^* =$

$$\begin{bmatrix} (\lambda\nu-2k+1)I_1-(\lambda-1)J_{11} & -(\lambda-1)J_{1(k-1)} & -(\lambda-1)J_{1(k-1)} & -\lambda J_{1(\nu-2k+1)} \\ -(\lambda-1)J_{1(k-1)} & (\lambda\nu-k)I_{k-1}-(\lambda-1)J_{(k-1)(k-1)} & -\lambda J_{(k-1)(k-1)} & -\lambda J_{(k-1)(\nu-2k+1)} \\ -(\lambda-1)J_{1(k-1)} & -\lambda J_{(k-1)(k-1)} & (\lambda\nu-k)I_{k-1}-(\lambda-1)J_{(k-1)(k-1)} & -\lambda J_{(k-1)(\nu-2k+1)} \\ -\lambda J_{1(\nu-2k+1)} & -\lambda J_{(k-1)(\nu-2k+1)} & -\lambda J_{(k-1)(\nu-2k+1)} & \lambda\nu(I_{\nu-2k+1}-\frac{1}{\nu}J_{(\nu-2k+1)(\nu-2k+1)}) \end{bmatrix}$$

The non-zero eigenvalues of C^* matrix with their corresponding multiplicities are,

1. $\frac{(\lambda\nu-2k+1)}{k}$, with multiplicity 1.

2. $\frac{(\lambda\nu-k)}{k}$, with multiplicity $2(k-2)$.

3. $\frac{\lambda\nu}{k}$, with multiplicity $(v-2k+1)$.

4. $\frac{(\lambda\nu-1)}{k}$, with multiplicity 1.

Theorem 2. *Doubly Balanced Incomplete Block Designs and with parameters $v = p,\ b,\ k,\ r,\ \lambda,\ \delta$ are fairly robust against the unavailability of two blocks, where the number of common treatment between two blocks is one, provided the overall efficiency of the residual design is given by*

$$e(s) = \frac{(\nu-1)(\lambda\nu-2k)(\lambda\nu-1)(\lambda\nu-2k+1)}{(\lambda\nu-2k)(\lambda\nu-1)((\nu-2k+1)(\lambda\nu-2k+1)+\lambda\nu)+(\lambda\nu-2k+1)\lambda\nu(2(k-2)(\lambda\nu-1)+(\lambda\nu-2k))} \tag{2.6}$$

Proof : Without loss of generality, let two blocks be lost from design D where the number of common treatment between two blocks is one i.e., $\eta(b_i \cap b_j) = 1$, C^* matrix of the residual design is given by,

$kC^* =$

$$\begin{bmatrix} (\lambda\nu-2k+1)I_1-(\lambda-1)J_{11} & -(\lambda-1)J_{1(k-1)} & -(\lambda-1)J_{1(k-1)} & -\lambda J_{1(\nu-2k+1)} \\ -(\lambda-1)J_{1(k-1)} & (\lambda\nu-k)I_{k-1}-(\lambda-1)J_{(k-1)(k-1)} & -\lambda J_{(k-1)(k-1)} & -\lambda J_{(k-1)(\nu-2k+1)} \\ -(\lambda-1)J_{1(k-1)} & -\lambda J_{(k-1)(k-1)} & (\lambda\nu-k)I_{(k-1)}-(\lambda-1)J_{(k-1)(k-1)} & -\lambda J_{(k-1)(\nu-2k+1)} \\ -\lambda J_{1(\nu-2k+1)} & -\lambda J_{(k-1)(\nu-2k+1)} & -\lambda J_{(k-1)(\nu-2k+1)} & \lambda\nu(I_{(\nu-2k+1)}-\nu^{-1}J_{(\nu-2k+1)(\nu-2k+1)}) \end{bmatrix}$$

The non-zero eigenvalues of their corresponding multiplicities are,

1. $\frac{(\lambda\nu-2k+1)}{k}$, with multiplicity 1.

2. $\frac{(\lambda\nu-k)}{k}$, with multiplicity $2(k-2)$.

3. $\frac{\lambda\nu}{k}$, with multiplicity $(v-2k+1)$.

4. $\frac{(\lambda\nu-1)}{k}$, with multiplicity 1.

Further, overall A-efficiency is calculated as,

$$e(s) = \frac{\phi_2(s)}{\phi_1(s)} \tag{2.7}$$

Where, $\phi_2(s)$ = sum of reciprocals of non-zero eigenvalues of C matrix of design D and $\phi_1(s)$ = sum of reciprocals of non-zero eigenvalues of C^* matrix of design D*.

That is,

$$\phi_2(s) = \frac{k(\nu-1)}{\lambda\nu} \tag{2.8}$$

and

$$\phi_1(s) = \frac{k(v-2k+1)}{\lambda\nu} + \frac{k}{(\lambda\nu-2k+1)} + \frac{2(k-2)k}{(\lambda\nu-k)} + \frac{k}{(\lambda\nu-1)}. \tag{2.9}$$

Finally, A- efficiency is given by,

$$e(s) = \frac{(\nu-1)(\lambda\nu-2k)(\lambda\nu-1)(\lambda\nu-2k+1)}{(\lambda\nu-2k)(\lambda\nu-1)((\nu-2k+1)(\lambda\nu-2k+1)+\lambda\nu)+(\lambda\nu-2k+1)\lambda\nu(2(k-2)(\lambda\nu-1)+(\lambda\nu-2k))} \tag{2.10}$$

Table 2: 8 Treatments of DBIB Design of Lyle D. Calvin (1954)

Block	Treatments			
1	**1**	**2**	**3**	**6**
2	**5**	**6**	**7**	**8**
3	1	2	7	8
4	3	4	5	6
5	1	3	4	8
6	2	4	5	7
7	1	4	6	7
8	2	3	5	8
9	1	2	5	6
10	3	4	7	8
11	1	3	5	7
12	2	4	6	8
13	1	4	5	8
14	2	3	6	7

Example 2: Let D represent the Doubly Balanced Incomplete Block Design with parameters $v = p = 8, b = 14, r = 7, k = 4, \lambda = 1, \delta = 3$. Design D is given by,

Two blocks containing treatments (1 2 3 6) and (5 6 7 8) are lost and number of treatment common two blocks is one. C^* matrix of the residual design is given by,

$$4C^* = \begin{bmatrix} 15 & -2 & -2 & -2 & -2 & -2 & -2 & -3 \\ -2 & 18 & -2 & -2 & -3 & -3 & -3 & -3 \\ -2 & -2 & 18 & -2 & -3 & -3 & -3 & -3 \\ -2 & -2 & -2 & 18 & -3 & -3 & -3 & -3 \\ -2 & -3 & -3 & -3 & 18 & -2 & -2 & -3 \\ -2 & -3 & -3 & -3 & -2 & 18 & -2 & -3 \\ -2 & -3 & -3 & -3 & -2 & -2 & 18 & -3 \\ -3 & -3 & -3 & -3 & -3 & -3 & -3 & 21 \end{bmatrix}$$

The non-zero eigenvalues with their corresponding multiplicities are,

1. $\frac{17}{4}$, with multiplicities 1.
2. $\frac{20}{4}$, with multiplicities 4.
3. $\frac{23}{4}$, with multiplicities 1.
4. $\frac{24}{4}$, with multiplicities 1.

The overall A- efficiency of the design is, $e(s) = 0.74033$.

Case (iii): Unavailability of two blocks where the number of common treatments between two blocks are two.

Consider a Doubly Balanced Incomplete Block Design D with parameters $v = p,\ b,\ k,\ r,\ \lambda,\ \delta$. The C matrix of the design is given by $C = \theta\left(I_v - \frac{1}{vE_{vv}}\right)$, Where, $\theta = (\lambda\nu/k)$ is the eigenvalues of C matrix of design D with multiplicity $(v-1)$. When two blocks are lost, call this design as a residual design and assume that the residual design D^* is a connected design. Let the blocks be b_i and b_j, let number of common treatment between two blocks be two, i.e.$\eta(b_i \cap b_j) = 2$. Here, these two blocks of the same treatment are repeated $(r-2)$ times. Similarly, those treatments which are present in the two lost blocks but are not common will be replicated $(r-1)$ times. The remaining treatments will be replicated r times in design.

Let C^* be the information matrix of design D^*. For this design D^*, the diagonal element of C^* matrix are as follows,

1. $C_{ii} = \frac{(r-2)(k-1)}{k}$, where i denotes those treatments which are present in both the lost blocks but are distinct.

2. $C_{jj} = \frac{(r-1)(k-1)}{k}$, where j denotes those treatments which are present in both the lost blocks but are distinct.

3. $C_{ll} = \frac{r(k-1)}{k}$, where l denotes the remaining treatments.

Similarly, in the residual design, pair of treatments occurs together in following three ways, which we say $\lambda_1, \lambda_2, \lambda_3$. Pattern of $\lambda_i (i = 1, 2, 3)$ are as follows,

1. $\lambda_1 = (\lambda - 2)$, for those treatments that are present in two lost blocks.

2. $\lambda_2 = (\lambda - 1)$, for those treatments that are present in two lost blocks.

3. $\lambda_3 = \lambda$, for remaining treatment.

The C* matrix of design D can be written as,

$$kC^* = \begin{bmatrix} (\lambda\nu-k+1)I_2-(\lambda)J_{22} & -(\lambda-1)J_{2(k-1)} & -\lambda J_{2(\nu-k-1)} \\ -(\lambda-1)J_{2(k-1)} & (\lambda\nu-2k)I_{(k-1)}-(\lambda-2)J_{(k-1)(k-1)} & -\lambda J_{(k-1)(\nu-k-1)} \\ -\lambda J_{2(\nu-k-1)} & -\lambda J_{(k-1)(\nu-k-1)} & \lambda\nu(I_{(\nu-k-1)}-\nu^{-1}J_{(\nu-k-1)(\nu-k-1)}) \end{bmatrix}$$

The non-zero eigenvalues of their corresponding multiplicities are,

1. $\frac{(\lambda\nu-k+1)}{k}$, with multiplicity 1.

2. $\frac{(\lambda\nu-2k)}{k}$, with multiplicity $(k-2)$.

3. $\frac{\lambda\nu-k-1}{k}$, with multiplicity 1.

4. $\frac{\lambda\nu}{k}$, with multiplicity $(v-k-1)$.

Theorem 3. *Doubly Balanced Incomplete Block Designs D and with parameters $v=p,\ b,\ k,\ r,\ \lambda,\ \delta$ are fairly robust against the unavailability of two blocks, where the number of common treatment between two blocks are two, i.e.$\eta(b_i \cap b_j)=2$, provided the overall efficiency of the residual design is given by,*

$$e(s)=\frac{(\nu-1)(\lambda\nu-2k)(\lambda\nu-k+1)(\lambda\nu-k-1)}{(\lambda\nu-k+1)(\lambda\nu-k-1)((\nu-k-1)(\lambda\nu-2k)+\lambda\nu(k-2))+2\lambda\nu(\lambda\nu-k)(\lambda\nu-2k)} \tag{2.11}$$

Proof: Without loss of generality, let two blocks be lost from design D where the number of common treatment between two blocks is two i.e. $\eta(b_i \cap b_j)=2$, C^* matrix of the residual design is given by,

$$kC^*=\begin{bmatrix} (\lambda\nu-k+1)I_2-(\lambda)J_{22} & -(\lambda-1)J_{2(k-1)} & -\lambda J_{2(\nu-k-1)} \\ -(\lambda-1)J_{2(k-1)} & (\lambda\nu-2k)I_{(k-1)}-(\lambda-2)J_{(k-1)(k-1)} & -\lambda J_{(k-1)(\nu-k-1)} \\ -\lambda J_{2(\nu-k-1)} & -\lambda J_{(k-1)(\nu-k-1)} & \lambda\nu(I_{(\nu-k-1)}-\nu^{-1}J_{(\nu-k-1)(\nu-k-1)}) \end{bmatrix}$$

The non-zero eigenvalues of their corresponding multiplicities are,

1. $\frac{(\lambda\nu-k+1)}{k}$, with multiplicity 1.

2. $\frac{(\lambda\nu-2k)}{k}$, with multiplicity $(k-2)$.

3. $\frac{\lambda\nu-k-1}{k}$, with multiplicity 1.

4. $\frac{\lambda\nu}{k}$, with multiplicity $(v-k-1)$.

Further, overall A-efficiency is calculated as,

$$e(s)=\frac{\phi_2(s)}{\phi_1(s)} \tag{2.12}$$

Where, $\phi_2(s)$ = sum of reciprocals of non-zero eigenvalues of C matrix of design D and $\phi_1(s)$ = sum of reciprocals of non-zero eigenvalues of C^* matrix of design D*.

That is,

$$\phi_2(s)=\frac{k(\nu-1)}{\lambda\nu} \tag{2.13}$$

and

$$\phi_1(s)=\frac{k(v-k-1)}{\lambda\nu}+\frac{k}{(\lambda\nu-k+1)}+\frac{(k-2)k}{(\lambda\nu-2k)}+\frac{k}{(\lambda\nu-k-1)} \tag{2.14}$$

Finally, A- efficiency is given by,

$$e(s) = \frac{(\nu-1)(\lambda\nu-2k)(\lambda\nu-k+1)(\lambda\nu-k-1)}{(\lambda\nu-k+1)(\lambda\nu-k-1)((\nu-k-1)(\lambda\nu-2k)+\lambda\nu(k-2))+2\lambda\nu(\lambda\nu-k)(\lambda\nu-2k)} \tag{2.15}$$

Example 3: Let D represent the Doubly Balanced Incomplete Block Design with parameters $v = p = 8, b = 14, r = 7, k = 4, \delta = 1, \lambda = 3$. Design D is given by,

Table 3: 8 Treatments of DBIB Design of Lyle D. Calvin (1954)

Block	Treatments			
1	1	2	3	6
2	**5**	**6**	**7**	**8**
3	**1**	**2**	**7**	**8**
4	3	4	5	6
5	1	3	4	8
6	2	4	5	7
7	1	4	6	7
8	2	3	5	8
9	1	2	5	6
10	3	4	7	8
11	1	3	5	7
12	2	4	6	8
13	1	4	5	8
14	2	3	6	7

Two blocks containing treatments (5 6 7 8) and (1 2 7 8) are lost and number of treatment common two blocks is two. C^* matrix of the residual design is given by,

$$4C^* = \begin{bmatrix} 18 & -3 & -2 & -2 & -2 & -3 & -3 & -3 \\ -3 & 18 & -2 & -2 & -2 & -3 & -3 & -3 \\ -2 & -2 & 15 & -1 & -1 & -3 & -3 & -3 \\ -2 & -2 & -1 & 15 & -1 & -3 & -3 & -3 \\ -2 & -2 & -1 & -1 & 15 & -2 & -2 & -3 \\ -3 & -3 & -3 & -3 & -3 & 21 & -3 & -3 \\ -3 & -3 & -3 & -3 & -3 & -3 & 21 & -3 \\ -3 & -3 & -3 & -3 & -3 & -3 & -3 & 21 \end{bmatrix}$$

The non-zero eigenvalues with their corresponding multiplicities are,

1. $\frac{19}{4}$, with multiplicities 1.

2. $\frac{16}{4}$, with multiplicities 2.

3. $\frac{21}{4}$, with multiplicities 1.
4. $\frac{24}{4}$, with multiplicities 3.

The overall A - efficiency of the design is, $e(s) = 0.832737$.

Table 4: Efficiency table when two blocks is lost from a Doubly Balanced Incomplete Block Design when $r \geq 7$ and $2 \leq k \leq 8$

D.No	$v = p$	k	b	r	λ	δ	Case(i) e(s)	Case(ii) $e(s)$	Case(iii) $e(s)$
1	8	14	7	4	3	1	0.853659	0.74033	0.832737
2	10	4	30	13	4	1	0.931034	0.879098	0.925551
3	10	5	36	18	8	3	0.944056	0.90031	0.941198
4	10	6	30	18	10	5	0.933775	0.88023	0.930391
5	11	4	165	60	18	4	0.98778	0.979418	0.987605
6	11	5	33	15	6	2	0.938462	0.889307	0.934559
7	11	6	33	18	9	4	0.939394	0.889786	0.936242
8	11	7	165	105	63	35	0.987903	0.977828	0.9878
9	12	4	165	55	15	3	0.987755	0.979336	0.987562
10	12	5	132	55	20	6	0.984762	0.973106	0.984514
11	12	6	22	11	5	2	0.908257	0.830725	0.899944
12	12	7	132	77	42	21	0.984868	0.972231	0.984689
13	12	8	165	110	70	42	0.98791	0.977571	0.987811
14	13	4	143	44	11	2	0.985816	0.975963	0.985531
15	13	5	429	165	55	15	0.995327	0.991797	0.995302
16	13	6	286	132	55	20	0.992997	0.987363	0.992949
17	13	7	286	154	77	35	0.993007	0.987167	0.992966
18	13	8	429	264	154	84	0.995341	0.991348	0.995325
19	14	4	91	26	6	1	0.977444	0.961253	0.976647
20	14	5	182	65	20	5	0.988935	0.980465	0.988781
21	14	6	91	39	15	5	0.977876	0.95977	0.977348
22	14	7	52	26	12	5	0.961415	0.928641	0.960006
23	14	8	91	52	28	14	0.978056	0.959198	0.977664
24	15	5	273	91	26	6	0.992634	0.987022	0.99256
25	15	6	455	182	65	20	0.995597	0.992055	0.995575
26	15	7	195	91	39	15	0.989726	0.981105	0.989622
27	15	8	195	104	56	24	0.990476	0.982292	0.990397
28	16	4	140	35	7	1	0.985401	0.975079	0.985021
29	16	5	336	105	28	6	0.994016	0.989464	0.993965
30	16	6	56	21	7	2	0.963636	0.932957	0.961946
31	16	7	80	35	14	5	0.974843	0.953424	0.974159
32	16	8	30	15	7	3	0.933014	0.87394	0.92856

3. Conclusion

There are 32 Doubly Balanced Incomplete Block Designs for different parametric values with unavailability of two blocks. The design with parametric values differ in efficiencies based on

(i) number of common treatments between two blocks is zero

(ii) number of common treatments between two blocks is one and

(iii) number of common treatments between two blocks is two or more.

The efficiency of the residual design D^*, $e(s) = \frac{\phi_2(s)}{\phi_1(s)}$ is obtained for $r \geq 7$ and $2 \leq k \leq 8$. The evaluation reveals that except for the design derivable from a Doubly Balanced Incomplete Block Design $(12, 22, 11, 6, 5, 2)$ with $e(s) = 0.908257$, all the designs satisfy $e(s) > 0.90$, that is, efficiency for case (ii) and case (iii) will be more than case (i) with the same parameter. The MATLAB coding for the calculation of efficiency is given in Appendix. It appears that Doubly Balanced Incomplete Block Designs are fairly robust against the unavailability of two blocks corresponding to the same test treatment.

References

[1] Aloke Dey, (1986). *Theory of Block Designs*, Wiley Eastern Ltd., New Delhi.

[2] Bhatt, P., (2008). Robustness of Balanced Incomplete Block Design with repeated blocks and other IBD against the unavailability of observations and blocks, Published Thesis.

[3] Bose, R.C., (1939). On the construction of balanced incomplete block designs, *Ann. Eugene.,* **9**, 353-399.

[4] Calvin, D., (1954). Doubly Balanced Incomplete Block Designs for experiments in which the treatment effects are correlated, *Biometrics,* **10(1)**, 61- 88.

[5] Das, A., and Kageyama, S., (1992). Robustness of BIB and extended BIB designs against the unavailability of any number of observations in a block, *Comput. Statist. Data Ana.,* **14(3)**, 343358.

[6] Ghosh, S., Kageyama, S.,and Mukerjee, R., (1992). Efficiency of connected binary block designs when a single observation is unavailable, *Ann. Inst. Statist. Math.,* **6(3)**, 593603.

[7] Prescott, P., and Mansson, R.A., (2002). Efficiency of pair-wise treatment comparisons in incomplete block experiments subject to the loss of a block of observations, *Comm. in Stat.: Theory Method,* 31.

[8] Prescott, P., and Mansson, R.A., (2001). Robustness of BIB designs to randomly missing observations, *J. Statist. Plann. Inference,* **92 (12)**, 283296.

[9] Ponnuswamy, K.N., and Srinivasan, M.R. (1991). Construction of Partial Triallel crosses (PTC) using a class of balanced incomplete block designs (BIBD), *Comm. Statist.,***A(20)**, 3315 3323.

[10] Yates, F. (1936). Incomplete randomised blocks, *Ann. Eugene.,* **7**, 121 140.

Appendix

```
%DBIBD Two Block missing
%No treatment common
function [eff1,eff2,eff3] = dbibdtwoblock1(n, la, v, k)
for i = 1:n
p1(i) = ((v(i)-1)*(la(i)*v(i)-k(i)));
p2(i) = (((la(i)*v(i)-k(i))*(v(i)-2*k(i)+1))+(2*la(i)*v(i)*(k(i)-1)));
e1(i) = p1(i)/p2(i);
%One treatment common
p3(i) =(((v(i)-1)*(la(i)*v(i)-2*k(i)))*(la(i)*v(i)-1)*(la(i)*v(i)-2*k(i)+1));
p4(i) = ((la(i)*v(i)-2*k(i))*(la(i)*v(i)-1))*((v(i)-2*k(i)+1)*(la(i)*v(i)- 2*k(i)+1)+
la(i)*v(i))+(la(i)*v(i)-2*k(i)+1)*la(i)*v(i)*(2*(k(i)-2)*(la(i)*v(i)-1)+(la(i)*v(i)-
2*k(i)));
e2(i) = p3(i)/p4(i);
%Two treatment common
p5(i) = ((v(i)-1)*(la(i)*v(i)-2*k(i))*(la(i)*v(i)-k(i)+1)*(la(i)*v(i)-k(i)-1));
p6(i) = (la(i)*v(i)-k(i)+1)*(la(i)*v(i)-k(i)-1)*((v(i)-k(i)-1)*(la(i)*v(i)-2*k(i))
+la(i)*v(i)*(k(i)- 2))+2*la(i)*v(i)*((la(i)*v(i)-k(i))*(la(i)*v(i)-2*k(i)));
e3(i) = p5(i)/p6(i);
eff1(i)=e1(i);
eff2(i)=e2(i);
eff3(i)=e3(i);
end
end
```

OUTPUT

[eff1, eff2, eff3] = dbibdtwoblock1 (1, 3, 8, 4)
eff1 = 0.8537
eff2 = 0.7403
eff3 = 0.8327

ProbStat Forum is an e-journal. For details please visit; *www.probstat.org.in.*