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Chover's form of the law of the iterated logarithm for r^{th} moving maxima

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Abstract. Let (X_n) be a sequence of i.i.d random variables and let $\eta_{r,n}$ denote the r^{th} maxima of $(X_{n-a_n}, X_{n-a_n+1}, \ldots X_n)$, where (a_n) is a non-decreasing sequence such that $0 \leq a_n \leq n$ and $\frac{a_n}{n} \sim b_n$, (b_n) is non-increasing. In this paper we obtain the law of the iterated logarithm for $\eta_{r,n}$, properly normalized.

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1 Introduction

Let $\{X_n\}$ be a sequence of independent and identically distributed (i.i.d) random variables (r.v.s) defined over a probability space (Ω, F, P) and let F denote the common distribution function(d.f). Suppose that F is continuous. Then the r^{th} maxima and r^{th} moving maxima are defined as follows.

Definition 1.1 *Let $X_{1,n} \leq X_{2,n} \leq \ldots \leq X_{n-r+1,n} \leq \ldots \leq X_{n,n}$ denote the ordered arrangement of $X_1, X_2, \ldots, X_n$. Then $X_{n-r+1,n}$ is called the r^{th} maxima, as it is the r^{th} highest member among $X_1, X_2, \ldots, X_n$. It is denoted by $M_{r,n}$. In particular, when $r = 1$, $M_{1,n} = \max(X_1, X_2, \ldots, X_n)$ is the partial maxima; when $r = n$, $M_{n,n} = \min(X_1, X_2, \ldots, X_n)$, is the partial minima.*

Definition 1.2 *Let (a_n) be a non-decreasing sequence of integers with $0 \leq a_n < n$. Consider the r.v.s $X_{n-a_n}, X_{n-a_n+1}, \ldots, X_n$ from $X_1, X_2, \ldots, X_n$ and arrange them in the increasing order as $Y_{n-a_n,n} \leq Y_{n-a_n+1,n} \leq \ldots \leq Y_{n-r+1,n} \cdots \leq Y_{n,n}$. Then $Y_{n-r+1,n}$ is the r^{th} highest member among $X_{n-a_n}, \ldots, X_{n-a_n+1}, \cdots, X_n$. It is denoted by $\eta_{r,n}$. In particular, when $r = 1$, $\eta_{1,n} = \max_{n-a_n \leq j \leq n} X_j$, is well known as the moving maxima. In the same spirit, $\eta_{r,n}$ is called r^{th} moving maxima.*

The study of moving maxima has gained importance like that of the delayed sums, since in the process of realization of a phenomenon, some of the initial observations may be missing. In this paper, we obtain the law of the iterated logarithm (L.I.L) for $(\eta_{r,n})$ when the *d.f* $F(.)$ has (i) exponentially fast right tail (ii) regularly varying right tail and (iii) finite right extremity. In particular when the d.f F is Uniform over $(0,1)$, we denote the r^{th} moving maxima by $\eta^*_{r,n}$ and r^{th} maxima by $M^*_{r,n}$, $n \geq 1$. Throughout the paper we introduce a smoothness condition that (a_n) is non-decreasing and $\frac{a_n}{n} \sim b_n$, where (b_n) is non-increasing.

Barndorff-Nielson (1961) has shown that

$$\limsup \frac{n\left(1 - M^*_{1,n}\right)}{\log\log n} = 1 \quad a.s. \tag{1.1}$$

Rothmann-Russo (1991) have extended the result in (1.1) to moving maxima for certain classes of (a_n). Under the setup of this paper, Vasudeva(1999) has shown that

$$\limsup \frac{a_n\left(1 - \eta^*_{1,n}\right)}{\beta_n} = 1 \quad a.s. \tag{1.2}$$

where $\beta_n = \log \frac{n}{a_n} + \log\log n$. For sequences in (1.1) and (1.2), limit inferior trivially follows to be zero. Since F is Uniform $(0,1)$, one can show that $n\left(1 - M^*_{1,n}\right)$ or $a_n\left(1 - \eta^*_{1,n}\right)$ converge to an exponential distribution. As such, for any sequence θ_n tending to ∞, one can show that $\frac{n(1-M^*_{1,n})}{\theta_n} \to 0 \left(\frac{a_n(1-\eta^*_{1,n})}{\theta_n} \to 0\right)$ in probability or

$$\liminf \frac{n(1 - M^*_{1,n})}{\theta_n} = \liminf \frac{a_n(1 - \eta^*_{1,n})}{\theta_n} = 0 \quad a.s.$$

A precise lower bound for $(1 - M^*_{r,n})$ can be obtained from Kiefer (1971). Note that $M_{n,n} = \min(X_1, X_2, \dots X_n), n \geq 1$. Then Kiefer established that (Theorem 6)

$$\liminf \frac{\log(nM_{n,n})}{\log\log n} = -1 \quad a.s. \tag{1.3}$$

The fact that X is Uniform $(0,1)$ implies that $Y = 1-X$ will again be Uniform $(0,1)$. Consequently $M_{n,n} = 1 - M^*_{1,n}$ and (1.3) implies that

$$\liminf \frac{\log(n(1 - M^*_{1,n}))}{\log\log n} = -1 \quad a.s.$$

which can be equivalently written as

$$\liminf (n(1 - M^*_{1,n}))^{\frac{1}{\log\log n}} = e^{-1} \quad a.s. \tag{1.4}$$

When (X_n) is a sequence of i.i.d symmetric stable r.v.s with exponent α, $0 < \alpha < 2$, Chover (1966) obtained the L.I.L for partial sum $S_n = \sum_{j=1}^{n} X_j$,

$n \geq 1$, by taking $(\log \log n)^{-1}$ in the power. To be precise, he established that

$$\limsup \mid \frac{S_n}{n^{\frac{1}{\alpha}}} \mid^{\frac{1}{\log \log n}} = e^{\frac{1}{\alpha}} \quad a.s.$$

As such, we call the L.I.L results established in this paper, as Chover's form of the L.I.L.

When (X_n) is Uniform $(0,1)$ one gets for any $c > 0$,

$$P\left(\frac{\log(n(1-X_n))}{\log \log n} < -c\right) = P\left(n(1-X_n) < \frac{1}{(\log n)^c}\right)$$
$$= P\left(1 - X_n < \frac{1}{n(\log n)^c}\right) = \frac{1}{n(\log n)^c}.$$

From the fact that $\sum \frac{1}{n(\log n)^c} < \infty$ if $c > 1$, $= \infty$ if $c \leq 1$, by Borel-Cantelli lemma one can show that

$$\liminf \frac{\log(n(1-X_n))}{\log \log n} = -1 \quad a.s.$$

or

$$\liminf(n(1-X_n))^{\frac{1}{\log \log n}} = e^{-1} \quad a.s. \tag{1.5}$$

From the relation $X_n \leq \eta^*_{1,n} \leq M^*_{1,n}$, and from (1.4) and (1.5) one can get the L.I.L

$$\liminf(n(1-\eta^*_{1,n}))^{\frac{1}{\log \log n}} = e^{-1} \quad a.s. \tag{1.6}$$

In section 2, we show that when $a_n = [n^p], 0 < p < 1$,

$$\limsup(n(1-\eta^*_{1,n}))^{\frac{1}{\log \log n}} = \infty \quad a.s.$$

Consequently, when $a_n = [n^p], 0 < p < 1$, the norming in (1.6) fails to give a precise upper bound. We establish that (lemma 2.2) for any $r \geq 1$, if $\xi^*_{r,n} = (a_n(1-\eta^*_{r,n}))^{\frac{1}{\beta_n}}$, where $\beta_n = \log\left(\frac{n}{a_n} \log n\right)$ then $\liminf \xi^*_{r,n} = e^{\frac{-1}{r}}$ $a.s.$ and $\limsup \xi^*_{r,n} = 1$ $a.s.$ In sections 3 and 4, we establish L.I.L for $(\eta_{r,n})$ when the right tail of the d.f F is exponentially fast and regularly varying. In the next section we give the L.I.L for $(\eta_{r,n})$ when F has finite right extremity ie., $\omega(F) = \sup\{x : F(x) < 1\}$. The associated boundary crossing results are studied in the last section. For any $x > 0, [x]$ means the greatest integer $\leq x$, c and k (integer) with or without a suffix, stand for generic constants.

2 Lemmas

Lemma 2.1 *When $a_n = [n^p], 0 < p < 1$,*

$$\limsup(n(1-\eta^*_{1,n}))^{\frac{1}{\log \log n}} = \infty \quad a.s.$$

Proof The lemma is proved once we show that for any $M > 0$, however large it may be,

$$P\left((n(1-\eta^*_{1,n}))^{\frac{1}{\log\log n}} > e^M \quad i.o\right) = 1.$$

Note that $\left((n(1-\eta^*_{1,n}))^{\frac{1}{\log\log n}} > e^M\right) = \left(\eta^*_{1,n} < 1 - \frac{(\log n)^M}{n}\right)$.
Define $A_n = \left(\eta^*_{1,n} < 1 - \frac{(\log n)^M}{n}\right)$. Then we have $P(A_n) = \left(1 - \frac{(\log n)^M}{n}\right)^{a_n}$. Since $\left(1 - \frac{(\log n)^M}{n}\right)^{\frac{n}{(\log n)^M}} \to e^{-1}$ as $n \to \infty$, for a given $\delta > 0$, one can find a N_0 such that for all $n \geq N_0$, $\left(1 - \frac{(\log n)^M}{n}\right)^{\frac{n}{(\log n)^M}} > e^{-(1+\delta)}$. Recalling that $a_n = [n^p], 0 < p < 1$, for all $n \geq N_0$, one gets $P(A_n) \geq e^{-(1+\delta)\frac{a_n(\log n)^M}{n}}$, where $\frac{a_n(\log n)^M}{n} \to 0$ as $n \to \infty$. Consequently, $P(A_n) \to 1$ as $n \to \infty$. We have $P(A_n \quad i.o) = P\left(\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} A_m\right) = \lim_{n\to\infty} P\left(\bigcup_{m=n}^{\infty} A_m\right) \geq \lim_{n\to\infty} P(A_n) = 1$, which completes the proof of the lemma. ■

In Vasudeva (1999), Barndorff-Nielson's form of the L.I.L. has been extended to the moving maxima, by considering (a_n) in place of (n) and $\left(\beta_n = \log\left(\frac{n}{a_n}\log n\right)\right)$ in place of $(\log\log n)$. In the literature, L.I.L have been obtained for the delayed sums, with normalizing sequences (a_n) and (β_n). As such, for r^{th} moving maxima also, it is natural to expect that the L.I.L holds with (a_n) in place of (n) and (β_n) in place of $(\log\log n)$. In lemmas 2.2 and 2.3 we show that the assertion is true.

Lemma 2.2

$$\liminf \left(a_n\left(1-\eta^*_{r,n}\right)\right)^{\frac{1}{\beta_n}} = e^{\frac{-1}{r}} \quad a.s.$$

where $\beta_n = \log\left(\frac{n}{a_n}\log n\right), n \geq 3.$

Proof With no loss of generality we prove the result for $r = 2$ ie., we show that

$$\liminf \left(a_n\left(1-\eta^*_{2,n}\right)\right)^{\frac{1}{\beta_n}} = e^{-\frac{1}{2}} \quad a.s.$$

Equivalently we establish that for $\epsilon\varepsilon(0,1)$

$$P\left(\left(a_n\left(1-\eta^*_{2,n}\right)\right)^{\frac{1}{\beta_n}} < e^{\frac{-1+\epsilon}{2}} \quad i.o\right) = 1 \tag{2.1}$$

and

$$P\left(\left(a_n\left(1-\eta^*_{2,n}\right)\right)^{\frac{1}{\beta_n}} < e^{\frac{-1-\epsilon}{2}} \quad i.o\right) = 0 \tag{2.2}$$

Let n_1 be the smallest integer such that $a_{n_1} > 1$.
Define $n_{k+1} = min\{n : n - a_n > n_k\}$. Note that

$$n_{k+1} - a_{n_k+1} > n_k \quad and \quad n_{k+1} - 1 - a_{(n_{k+1}-1)} \leq n_k$$

and hence $n_{k+1}-1-a_{(n_{k+1}-1)} \leq n_k < n_{k+1}-a_{n_{k+1}}$ or

$$1-\frac{1}{n_{k+1}}-\frac{a_{(n_{k+1}-1)}}{n_{k+1}} \leq \frac{n_k}{n_{k+1}} < 1-\frac{a_{n_{k+1}}}{n_{k+1}} \tag{2.3}$$

Since $\frac{a_n}{n} \sim b_n$, where (b_n) is non-increasing, one can find a $\rho, 0 \leq \rho \leq 1$ such that $\lim \frac{a_n}{n} = \rho$.

Case 1: $0 \leq \rho < 1$.
To prove (2.2), let $u_n = 1-\frac{1}{a_n\left(\frac{n}{a_n}\log n\right)^{\frac{1+\epsilon}{2}}}$. Note that

$$P\left(\left(a_n\left(1-\eta^*_{2,n}\right)\right)^{\frac{1}{\log\log n}} < e^{\frac{-1-\epsilon}{2}}\right) = P\left(\eta^*_{2,n} > u_n\right).$$

Define $A_n = \left(\eta^*_{2,n} > u_n\right)$, $B_k = \left(second\max_{n_k \leq n \leq n_{k+1}} \eta^*_{2,n} > u_{n_k}\right)$, $C_k = \left(second\max_{n_k-a_{n_k} \leq j \leq n_{k+1}} X_j > u_{n_k}\right)$, and observe that

$$(A_n \quad i.o) \subseteq (B_k \quad i.o) \subseteq (C_k \quad i.o).$$

We have

$$\begin{aligned}
P(C_k) &= P\left(second \max_{n_k-a_{n_k} \leq j \leq n_{k+1}} X_j > u_{n_k}\right)\\
&= 1-F^{n_{k+1}-n_k+a_{n_k}}\left(u_{n_k}\right)+(n_{k+1}-n_k+a_{n_k})(1-u_{n_k})F^{n_{k+1}-n_k+a_{n_k}-1}\left(u_{n_k}\right)\\
&\simeq \frac{\left(n_{k+1}-n_k+a_{n_k}\right)^2}{2a^2_{n_k}\left(\frac{n_k}{a_{n_k}}\log n_k\right)^{1+\epsilon}} = \left(\frac{n_{k+1}-n_k}{a_{n_k}}+1\right)^2 \frac{1}{2\left(\frac{n_k}{a_{n_k}}\log n_k\right)^{1+\epsilon}}.
\end{aligned}$$

From (2.3) one gets, $\lim \frac{n_k}{n_{k+1}} = 1-\rho$ or $\lim \frac{n_{k+1}}{n_k} = (1-\rho)^{-1}$. From the definition of (n_k) and from the relation

$$\frac{n_{k+1}-n_k}{a_{n_k}} \leq \frac{a_{n_{k+1}}}{a_{n_k}} \leq \frac{n_{k+1}}{n_k},$$

one gets for n_k large,

$$\begin{aligned}
P(C_k) &\leq c_1\frac{a^{1+\epsilon}_{n_k}}{\left(n_k\log n_k\right)^{1+\epsilon}} \leq c_1\frac{\left(n_k-n_{k-1}\right)^{1+\epsilon}}{n_k^{1+\epsilon}(\log n_k)^{1+\epsilon}}\\
&= c_1\left(\frac{n_k-n_{k-1}}{n_k}\right)^{\epsilon}\frac{n_k-n_{k-1}}{n_k(\log n_k)^{1+\epsilon}} \leq \frac{c_2(n_k-n_{k-1})}{n_k(\log n_k)^{1+\epsilon}}.
\end{aligned}$$

Note that

$$\begin{aligned}
\infty > \int\frac{dx}{x(\log x)^{1+\epsilon}} = \sum_k\int_{n_{k-1}}^{n_k}\frac{dx}{x(\log x)^{1+\epsilon}} &\geq \sum_k\int_{n_{k-1}}^{n_k}\frac{1}{n_k\left(\log n_k\right)^{1+\epsilon}}\\
&= \sum_k\frac{n_k-n_{k-1}}{n_k\left(\log n_k\right)^{1+\epsilon}}.
\end{aligned}$$

Using this and Borel-Cantelli lemma, one gets $P(C_k \ \ i.o) = 0$ which implies that $P(A_n \, i.o) = 0$. Hence (2.2) is established.
To prove (2.1), let $v_n = 1 - \frac{1}{a_n\left(\frac{n}{a_n}\log n\right)^{\frac{1-\epsilon}{2}}}$. Then

$$P\left(\left(a_n\left(1-\eta^*_{2,n}\right)\right)^{\frac{1}{\beta_n}} < e^{\frac{-1+\epsilon}{2}}\right) = P\left(\eta^*_{2,n} > v_n\right).$$

Define $A^*_n = \{\eta^*_{2,n} > v_n\}$ and $D^*_k = \{second \max_{n_k - a_{n_k} \le j \le n_k} X_j > v_{n_k}\}$. Note that $(A^*_n \ \ i.o) \supseteq (D^*_k \ \ i.o)$ and that (D^*_k) are mutually independent. Also,

$$\begin{aligned} P(D^*_k) &= 1 - \left(F^{a_{n_k}}\left(v_{n_k}\right) + a_{n_k}(1-v_{n_k})F^{a_{n_k}-1}\left(v_{n_k}\right)\right) \\ &\simeq \frac{c_3}{\left(\frac{n_k}{a_{n_k}}\log n_k\right)^{1-\epsilon}} \ge \frac{c_4 a_{n_k}}{n_k(\log n_k)^{1-\epsilon}} \end{aligned}$$

and

$$(n_k - 1) - a_{(n_k-1)} < n_{k-1} \Rightarrow n_k - n_{k-1} < a_{(n_k-1)} + 1 < a_{n_k} + 1 < 2a_{n_k}.$$

Using this, one gets

$$P(D^*_k) \ge \frac{c_4(n_k - n_{k-1})}{n_k(\log n_k)^{1-\epsilon}}.$$

Note that

$$\begin{aligned} \infty = \int \frac{dx}{x(\log x)^{1-\epsilon}} = \sum_k \int_{n_{k-1}}^{n_k} \frac{dx}{x(\log x)^{1-\epsilon}} &\le \sum_k \frac{n_k - n_{k-1}}{n_{k-1}(\log n_{k-1})^{1-\epsilon}} \\ &\le \sum_k c_5 \frac{n_k - n_{k-1}}{n_k(\log n_k)^{1-\epsilon}}. \end{aligned}$$

$$(\lim \frac{n_k}{n_{k-1}} = (1-\rho)^{-1} \Rightarrow n_k \le n_{k-1}(1-\rho)^{-1} \Rightarrow c_5\frac{1}{n_k} \ge \frac{1}{n_{k-1}} \ for \ k \ large).$$

Since (D^*_k) are mutually independent, appealing to Borel-Cantelli lemma we get $P(D^*_k \ \ i.o) = 1$, which implies that $P(A^*_n \ \ i.o) = 1$.

Case 2: $\rho = 1$.
When $\rho = 1$ ie., $\lim \frac{a_n}{n} = 1$ one has $\beta_n \sim \log\log n$. Then one gets

$$P(A_n) \le P\left(\eta^*_{2,n} > 1 - \frac{1}{n\left(\log n\right)^{\frac{1+\frac{\epsilon}{2}}{2}}}\right).$$

Note that $\eta^*_{2,n} \le M^*_{2,n}$ and the fact that

$$P\left(M^*_{2,n} > 1 - \frac{1}{n(\log n)^{\frac{1+\frac{\epsilon}{2}}{2}}} \ i.o\right) = 0 \Rightarrow P\left(\eta^*_{2,n} > 1 - \frac{1}{n(\log n)^{\frac{1+\frac{\epsilon}{2}}{2}}} \ i.o\right) = 0.$$

This proves (2.2).
To prove (2.1) we proceed as follows:
Let $a_n^* = [np], 0 < p < 1$. Note that $\frac{a_n^*}{n} \to p$. Define $M_{2,n}^{'(m)} = \max_{n-a_n^* \le j \le n} X_j$ and observe that $M_{2,n}^{'(m)} \le \eta_{2,n}^*$. Let $u_n^* = 1 - \frac{1}{n(\log n)^{\frac{1-\epsilon}{2}}}$. and let (m_k) be sequence such that $a_{m_1}^* > 1$ and $m_{k+1} = \min\{n : n - a_n^* > m_k\}$.
Then

$$P\left(M_{2,m_k}^{'(m)} > \eta_{n_k}\right) = 1 - \left(F^{a_{m_k}}(\eta_{n_k}) + (a_{m_k}(1-\eta_{n_k}))F^{a_{m_k}-1}(\eta_{n_k})\right)$$

$$\simeq \frac{m_k p - 1}{2m_k(\log m_k)^{1-\epsilon}} = \frac{c_6}{(\log m_k)^{1-\epsilon}}.$$

Note that

$$m_{k+1} - a_{m_{k+1}}^* > m_k \;\; and \;\; m_{k+1} - 1 - a_{(m_{k+1}-1)}^* \le m_k$$

$$\Rightarrow m_{k+1} - 1 - a_{(m_{k+1}-1)}^* \le m_k \le m_{k+1} - a_{m_{k+1}}^*$$

$$\Rightarrow m_{k+1} - 1 - pm_{k+1} + 1 \le m_k \le m_{k+1} - pm_{k+1}$$

$$\Rightarrow \lim \frac{m_k}{m_{k+1}} = 1 - p \Rightarrow m_k \simeq (1-p)^k, \;\; for \;\; k \ge k_0,$$

where k_0 is some constant. Hence one gets $P\left(M_{2,m_k}^{'(m)} > \eta_{m_k}\right) \ge \frac{c_6}{k^{1-\epsilon}}$, for all $k \ge k_0$. Since $\sum \frac{1}{k^{1-\epsilon}} = \infty$ and $\left(M_{2,m_k}^{'(m)}\right)$ are mutually independent, from Borel-Cantelli lemma one gets $P\left(M_{2,m_k}^{'(m)} > u_{m_k}\; i.o\right) = 1$. Since $M_{2,n}^{'(m)} \le \eta_{2,n}^*$, $P\left(\eta_{2,n}^* > 1 - \frac{1}{n(\log n)^{\frac{1-\epsilon}{2}}} \;\; i.o\right) = 1$. Hence the proof. ■

Lemma 2.3 *Let* $\lim \frac{a_n}{\log n} = \infty$. *Then*

$$\limsup \left(a_n\left(1-\eta_{r,n}^*\right)\right)^{\frac{1}{\beta_n}} = 1 \;\; a.s.$$

Proof With no loss of generality we prove the result for $r = 2$.
Equivalently we show that for $\epsilon > 0$, but small,

$$P\left(\left(a_n\left(1-\eta_{2,n}^*\right)\right)^{\frac{1}{\beta_n}} > e^{\epsilon} \;\; i.o\right) = 0 \tag{2.4}$$

and

$$P\left(\left(a_n\left(1-\eta_{2,n}^*\right)\right)^{\frac{1}{\beta_n}} > e^{-\epsilon} \;\; i.o\right) = 1 \tag{2.5}$$

Define $A_n = \left(1 - \eta_{2,n}^* > \frac{1}{a_n}\left(\frac{n}{a_n}\log n\right)^{\epsilon}\right)$ and note that

$$\frac{\log\left(\frac{n}{a_n}\log n\right)}{\left(\frac{n}{a_n}\log n\right)^{\epsilon}} \to 0 \Rightarrow \frac{\left(\frac{n}{a_n \log n}\right)^{\epsilon}}{a_n} > \frac{c_7 \log\left(\frac{n}{a_n}\log n\right)}{a_n}.$$

Let $A'_n = \left(1 - \eta^*_{2,n} > \frac{c_7 \log(\frac{n}{a_n} \log n)}{a_n}\right)$ for $c_7 > 1, n \geq 2$. Observe that $(A'_n \ \ i.o) \supseteq (A_n \ \ i.o)$. We have

$$P(A'_n) = P\left(\eta^*_{2,n} < 1 - \frac{c_7 \log(\frac{n}{a_n} \log n)}{a_n}\right)$$

$$= F^{a_n}\left(1 - \frac{c_7 \log(\frac{n}{a_n} \log n)}{a_n}\right) + c_7 \log(\frac{n}{a_n} \log n) F^{a_n - 1}\left(1 - \frac{c_7 \log(\frac{n}{a_n} \log n)}{a_n}\right)$$

$$= \left(1 - \frac{c_7 \log(\frac{n}{a_n} \log n)}{a_n}\right)^{a_n} + c_7 \log(\frac{n}{a_n} \log n)\left(1 - \frac{c_7 \log(\frac{n}{a_n} \log n)}{a_n}\right)^{a_n - 1}$$

$$= \left(1 - \frac{c_7 \log(\frac{n}{a_n} \log n)}{a_n}\right)^{a_n - 1}\left(1 - \frac{c_7 \log(\frac{n}{a_n} \log n)}{a_n} + c_7 \log(\frac{n}{a_n} \log n)\right).$$

Using the fact that $a_n - 1 \simeq a_n$, we get for n large and $c_8, c_9 > 0$

$$P(A_n^{'}) = \left(1 - \frac{c_7 \log(\frac{n}{a_n} \log n)}{a_n}\right)^{a_n} \frac{c_7 \log(\frac{n}{a_n} \log n)}{a_n}$$

$$\simeq \left(e^{-c_8 \log(\frac{n}{a_n} \log n)}\right)\left(\frac{c_7 \log(\frac{n}{a_n} \log n)}{a_n}\right) \leq c_9 \left(\frac{a_n}{n \log n}\right)^{\frac{c_8}{2}}.$$

Hence $P(A_n^{'}) \to 0$ as $n \to \infty$. We now show that $P(A'_n \bigcap A'^{c}_{n+1} \ \ i.o) = 0$, which proves (2.4). We have

$$P(A'_n \bigcap A'^{C}_{n+1}) \leq P(A'_n) P\left(X_{n+1} > 1 - \frac{c_{10} \log\left(\frac{n+1}{a_{n+1}} \log(n+1)\right)}{a_{n+1}}\right)$$

$$= P(A'_n) \frac{c_{10} \log\left(\frac{n+1}{a_{n+1}} \log(n+1)\right)}{a_{n+1}} \leq c_{11} \frac{\log\left(\frac{n+1}{a_{n+1}} \log(n+1)\right)}{a_{n+1}} \left(\frac{a_n}{n \log n}\right)^{c_{12}}$$

$$\leq c_{13} \left(\frac{a_n}{n \log n}\right)^{c_{12}} \frac{(n+1)^{\epsilon}}{a^{\epsilon}_{n+1}} (\log(n+1))^{\epsilon} \frac{1}{a_n} \leq c_{14} \left(\frac{a_n}{n \log n}\right)^{c_{12}} \frac{n^{\epsilon}}{a^{\epsilon}_n} (\log n)^{\epsilon} \frac{1}{a_n}$$

$$\leq \frac{c_{15}}{n(\log n)^{c_{12} - \epsilon}},$$

where $c_{12} > 1$ is such that $c_{12} - \epsilon > 1$. Since $\sum \frac{1}{n(\log n)^{c_{12} - \epsilon}} < \infty$, using Borel-Cantelli lemma, we get $P(A'_n \bigcap A'^{C}_{n+1} \ \ i.o) = 0$ which implies $P(A'_n \ \ i.o) = 0$ and in turn $P(A_n \ \ i.o) = 0$.

With no loss of generality we show (2.5) for $r = 2$.

Define $B_n = \left(1 - \eta^*_{2,n} > \frac{1}{a_n}\left(\frac{a_n}{n(\log n)}\right)^{\epsilon}\right)$. Now

$$P(B_n) = P\left(\eta^*_{2,n} < 1 - \frac{1}{a_n}\left(\frac{a_n}{n \log n}\right)^{\epsilon}\right)$$

$$= \left(1 - \frac{1}{a_n}\left(\frac{a_n}{n\log n}\right)^{\epsilon}\right)^{a_n} + \left(\frac{a_n}{n\log n}\right)^{\epsilon}\left(1 - \frac{1}{a_n}\left(\frac{a_n}{n\log n}\right)^{\epsilon}\right)^{a_n-1}$$

Using the fact that $a_n - 1 \simeq a_n$ for n large, we get

$$P(B_n) = \left(1 - \frac{1}{a_n}\left(\frac{a_n}{n\log n}\right)^{\epsilon}\right)^{a_n}\left(1 + \left(\frac{a_n}{n\log n}\right)^{\epsilon}\right) \sim e^{-\frac{a_n}{n\log n}},$$

which tends to 1 as $n \to \infty$.
Hence $P(B_n \quad i.o) = 1$ and the result is proved. ■

Remark 2.1 *When $a_n = [np], 0 < p < 1$, $\beta_n \sim \log\log n$. Hence from the above two lemmas one can get*

$$\liminf (n(1 - \eta^*_{r,n}))^{\frac{1}{\log\log n}} = e^{\frac{-1}{r}} \quad a.s.$$

and

$$\limsup (n(1 - \eta^*_{r,n}))^{\frac{1}{\log\log n}} = 1 \quad a.s.$$

3 L.I.L for distributions with exponentially fast tails.

Peter Hall(1976) obtained Kiefer's results for a class of distributions which include those with tail $1 - F(x) \simeq e^{-x^{\gamma}L(x)}, \gamma > 0$, where $L(.)$ is a slowly varying. In this section, we present L.I.L for $(\eta_{r,n})$ under the setup of Hall(1976).
Define $U(x) = -\log(1 - F(x))$ and denote its inverse function by V. As in Peter Hall(1976), suppose that for all functions $a(.)$ with $0 \neq a(x) \to 0$ as $x \to \infty$

$$\frac{V\left(x(1 + a(x))\right) - V(x)}{a(x)V(x)} \to \gamma^{-1} \;\; as \;\; x \to \infty \tag{3.1}$$

which implies that V is continuous for all large x, and regularly varying with exponent γ^{-1}. If V is eventually differentiable, then the condition above is equivalent to $x\frac{d}{dx}\log(V(x)) \to \gamma^{-1} \;\; as \;\; x \to \infty$ or alternatively, to

$$x\frac{d}{dx}\log\log(1 - F(x))^{-1} \to \gamma \;\; as \;\; x \to \infty.$$

Hence d.f s with $1 - F(x) \simeq exp\left(-x^{\gamma}L(x)\right)$, where $\gamma > 0$ and L is slowly varying at ∞ belong to this class. Let (X_n) be i.i.d with d.f F of this class and let $\eta_{r,n}$ be the r^{th} moving maxima. We have the following theorem.

Theorem 3.1

$$\limsup \frac{\log a_n}{\beta_n}\left(\frac{\eta_{r,n}}{V(\log a_n)} - 1\right) = \frac{1}{r\gamma} \quad a.s.$$

where $\beta_n = \log\left(\frac{n}{a_n}\log n\right), n \geq 3.$

Proof We need to show that for $\epsilon \varepsilon (0,1)$,

$$P\left(r\gamma \frac{\log a_n}{\beta_n}\left(\frac{\eta_{r,n}}{V(\log a_n)}-1\right) > 1+\epsilon \;\; i.o\right) = 0 \tag{3.2}$$

and

$$P\left(r\gamma \frac{\log a_n}{\beta_n}\left(\frac{\eta_{r,n}}{V(\log a_n)}-1\right) > 1-\epsilon \;\; i.o\right) = 1 \tag{3.3}$$

From lemma 2.2, we have

$$P\left(1-\eta^*_{r,n} < \frac{1}{a_n\left(\frac{n}{a_n}\log n\right)^{\frac{1+\epsilon}{r}}} \;\; i.o\right) = 0 \tag{3.4}$$

Note that $\eta^*_{r,n} = F\left(\eta_{r,n}\right)$. Hence
$1-\eta^*_{r,n} < \frac{1}{a_n\left(\frac{n}{a_n}\log n\right)^{\frac{1+\epsilon}{r}}} \Leftrightarrow 1-F\left(\eta_{r,n}\right) < \frac{1}{a_n\left(\frac{n}{a_n}\log n\right)^{\frac{1+\epsilon}{r}}} \Leftrightarrow -\log\left(1-F\left(\eta_{r,n}\right)\right) < -\log\left(\frac{1}{a_n\left(\frac{n}{a_n}\log n\right)^{\frac{1+\epsilon}{r}}}\right) \Leftrightarrow U\left(\eta_{r,n}\right) > \log a_n + \log\left(\frac{n}{a_n}\log n\right)\frac{1+\epsilon}{r} \Leftrightarrow \eta_{r,n} > V\left(\log a_n + \log\left(\frac{n}{a_n}\log n\right)\frac{1+\epsilon}{r}\right) \Leftrightarrow$

$$\eta_{r,n} - V(\log a_n) > V\left(\log a_n\left(1+\frac{\beta_n}{\log a_n}\frac{1+\epsilon}{r}\right)\right) - V(\log a_n) \tag{3.5}$$

From condition (3.1) we have

$$V\left(\log a_n\left(1+\frac{\beta_n}{\log a_n}\frac{1+\epsilon}{r}\right)\right) - V(\log a_n) \sim \frac{(1+\epsilon)}{r\gamma}\frac{\beta_n V(\log a_n)}{\log a_n}.$$

Consequently, from (3.5), for n large,

$$1-\eta^*_{r,n} < \frac{1}{a_n\left(\frac{n}{a_n}\log n\right)^{\frac{1+\epsilon}{r}}} \Leftrightarrow \eta_{r,n} - V(\log a_n) > \frac{(1+\epsilon)}{r\gamma}\frac{\beta_n V(\log a_n)}{\log a_n}$$

$$\Leftrightarrow r\gamma\frac{\log a_n}{\beta_n}\left(\frac{\eta_{r,n}}{V(\log a_n)}-1\right) > 1+\epsilon.$$

From (3.4), we hence have (3.2). Again from lemma 2.2, recalling that

$$P\left(1-\eta^*_{r,n} < \frac{1}{a_n\left(\frac{n}{a_n}\log n\right)^{\frac{1-\epsilon}{r}}} \;\; i.o\right) = 1$$

and proceeding on the above lines, (3.3) can be established. The details are omitted. ■

We consider some of the standard distributions and give the form of the V function and the L.I.L.

Example 1 *When (X_n) is i.i.d unit exponential, one can get $V(x) = x$. Then we have*

$$\limsup \frac{\log a_n}{\beta_n}\left(\frac{\eta_{r,n}}{\log a_n} - 1\right) = \frac{1}{r} \quad a.s$$

Example 2 *When (X_n) is i.i.d with common d.f $F(x) = e^{-e^{-x}}, -\infty < x < \infty$.*
Note that $1 - F(x) \sim e^{-x}$, as $x \to \infty$. Consequently the L.I.L coincides with that of L.I.L obtained in case of unit exponential. Hence we have

$$\limsup \frac{\log a_n}{\beta_n}\left(\frac{\eta_{r,n}}{\log a_n} - 1\right) = \frac{1}{r} \quad a.s$$

Example 3 *When (X_n) is i.i.d standard normal, one gets*
$V(x) = \sqrt{2x - \log x - 2\log\sqrt{2\pi} - \log 2} \simeq \sqrt{2x}$ *for large x. Then we have*

$$\limsup \frac{\log a_n}{\beta_n}\left(\frac{\eta_{r,n}}{\sqrt{2\log n}} - 1\right) = \frac{1}{r} \quad a.s$$

4 L.I.L when the distribution has a regularly varying right tail.

Let (X_n) be i.i.d with d.f F having regularly varying tail and $\eta_{r,n}$ denote the r^{th} moving maxima. Define $U^*(x) = 1 - F(x) \sim x^{-\gamma}L(x), \gamma > 0$, where L is a slowly varying function. Let V^* be the inverse of U^*. Observe that $V^*(y) = y^{-\frac{1}{\gamma}}\ l\left(\frac{1}{y}\right), 0 < y \leq 1$, where l is slowly varying . Note that $U^*(.)$ and $V^*(.)$ are decreasing functions. From the fact that $F(X_n)$ is a Uniform $(0,1)$ r.v, we note that $\eta^*_{r,n} = F(\eta_{r,n}), n \geq 1$. Let B_{a_n} be a solution of the equation $a_n(1 - F(B_{a_n})) \simeq 1$. When $1 - F$ is regularly varying with index γ, we know that F belongs to the domain of attraction of Frechet law denoted by $F\varepsilon DA(H_{1,\gamma})$.

Lemma 4.1 *If $y_n \to \infty, z_n \to \infty$, one can find a $\delta > 0$ such that*

$$\lim z_n^{-\delta}\frac{L(y_n z_n)}{L(y_n)} = 0 \quad and \quad \lim z_n^{\delta}\frac{L(y_n z_n)}{L(y_n)} = \infty$$

Proof For proof, see Seneta(1976). ■

Theorem 4.1 *Let $F\varepsilon DA(H_{1,\gamma}), \gamma > 0$. Then*

$$\limsup \left(\frac{\eta_{r,n}}{B_{a_n}}\right)^{\frac{1}{\beta_n}} = e^{\frac{1}{r\gamma}} \quad a.s$$

where B_{a_n} is a solution of the equation $a_n(1 - F(B_{a_n})) \simeq 1$.

Proof From lemma 2.2 we have

$$P\left(1-\eta^*_{r,n} < \frac{1}{a_n\left(\frac{n}{a_n}\log n\right)^{\frac{1+\epsilon}{r}}} \quad i.o\right) = 0 \tag{4.1}$$

and

$$P\left(1-\eta^*_{r,n} < \frac{1}{a_n\left(\frac{n}{a_n}\log n\right)^{\frac{1-\epsilon}{r}}} \quad i.o\right) = 1 \tag{4.2}$$

Using the relation $\eta^*_{r,n} = F(\eta_{r,n})$ and $U^*(x) = 1-F(x) \sim x^{-\gamma}L(x)$, (4.1) can be written as

$$P\left(U^*(\eta_{r,n}) < \frac{1}{a_n\left(\frac{n}{a_n}\log n\right)^{\frac{1+\epsilon}{r}}} \quad i.o\right) = 0 \tag{4.3}$$

Note that

$$U^*(\eta_{r,n}) < \frac{1}{a_n\left(\frac{n}{a_n}\log n\right)^{1+\epsilon}} \quad f.o \Leftrightarrow V^*\left(U^*(\eta_{r,n})\right) > V^*\left(\frac{1}{a_n\left(\frac{n}{a_n}\log n\right)^{\frac{1+\epsilon}{r}}}\right) \quad f.o$$

$$\Leftrightarrow \eta_{r,n} > \left(\frac{1}{a_n\left(\frac{n}{a_n}\log n\right)^{\frac{1+\epsilon}{r}}}\right)^{\frac{-1}{\gamma}} l\left(a_n\left(\frac{n}{a_n}\log n\right)^{\frac{1+\epsilon}{r}}\right) \quad f.o$$

$$\Leftrightarrow \eta_{r,n} > a_n^{\frac{1}{\gamma}}\left(\frac{n}{a_n}\log n\right)^{\frac{1+\epsilon}{r\gamma}} l\left(a_n\left(\frac{n}{a_n}\log n\right)^{\frac{1+\epsilon}{r}}\right) \quad f.o$$

By lemma 4.1 for any $\delta > 0$, we have

$$\lim\left(\frac{n}{a_n}\log n\right)^{\delta}\frac{l\left(a_n\left(\frac{n}{a_n}\log n\right)^{\frac{1+\epsilon}{r}}\right)}{l(a_n)} = \infty.$$

Choosing $\delta = \frac{\epsilon}{2r\gamma}$, one can find a N_0 such that for all $n \geq N_0$

$$l\left(a_n\left(\frac{n}{a_n}\log n\right)^{\frac{1+\epsilon}{r}}\right) \geq \frac{l(a_n)}{\left(\frac{n}{a_n}\log n\right)^{\frac{\epsilon}{2r\gamma}}}. \tag{4.4}$$

Hence for $n \geq N_0$,

$$\eta_{r,n} \geq a_n^{\frac{1}{\gamma}}\left(\frac{n}{a_n}\log n\right)^{\frac{1+\epsilon}{r\gamma}}\left(\frac{n}{a_n}\log n\right)^{\frac{-\epsilon}{2r\gamma}} l(a_n) \Leftrightarrow \eta_{r,n} \geq B_{a_n}\left(\frac{n}{a_n}\log n\right)^{\frac{1+\frac{\epsilon}{2}}{r\gamma}}$$

$$\Leftrightarrow \left(\frac{\eta_{r,n}}{B_{a_n}}\right)^{\frac{1}{\beta_n}} \geq e^{\frac{1+\frac{\epsilon}{2}}{r\gamma}} (since, a_n(1-F(B_{a_n})) \simeq 1 \ implies \ that \ B_{a_n} = a_n^{\frac{1}{\gamma}} l(a_n)).$$

Consequently, from (4.3) and (4.4)

$$P\left(\left(\frac{\eta_{r,n}}{B_{a_n}}\right)^{\frac{1}{\beta_n}} \geq e^{\frac{1+\frac{\epsilon}{2}}{r\gamma}} \quad i.o\right) = 0. \tag{4.5}$$

From lemma 4.1, Choosing $\delta = \frac{\epsilon}{2r\gamma}$, one can find a N_1 such that for all $n \geq N_1$

$$l\left(a_n\left(\frac{n}{a_n}\log n\right)^{\frac{1-\epsilon}{r}}\right) \geq l(a_n)\left(\frac{n}{a_n}\log n\right)^{\frac{\epsilon}{2r\gamma}}.$$

Proceeding on lines similar to those used to obtain (4.5), one can get

$$P\left(\left(\frac{\eta_{r,n}}{B_{a_n}}\right)^{\frac{1}{\beta_n}} \geq e^{\frac{1-\frac{\epsilon}{2}}{r\gamma}} \quad i.o\right) = 1. \tag{4.6}$$

From (4.5) and (4.6) we claim the result. ■

5 L.I.L when the distribution has finite right extremity.

Let (X_n) be i.i.d with common d.f F and let $\omega(F) = \sup\{x : F(x) < 1\}$ be finite. In this section we obtain the L.I.L for $\eta_{r,n}$ when $\omega(F) < \infty$ and when F belongs to the domain of attraction of the Weibull law ie., $F\varepsilon DA(H_{2,\gamma}), \gamma > 0$. From the fact that $F(X_n)$ is a Uniform $(0,1)$, we note that $\eta^*_{r,n} = F(\eta_{r,n}), n \geq 1$. Let $\eta'_{r,n}$ be the r^{th} maxima of $(Y_{n-a_n}, Y_{n-a_{n+1}}, \ldots Y_n)$ where (Y_n) are i.i.d r.v's given by $\frac{1}{\omega(F)-X_n}, n \geq 1$. Let F^* denote the d.f of $Y_n, n \geq 1$. Note that $F^*\varepsilon DA(H_{1,\gamma})$.

Theorem 5.1 *Let $F\varepsilon DA(H_{2,\gamma}), \gamma > 0$. Then*

$$\liminf \left(B_{a_n}\left(\omega(F) - \eta_{r,n}\right)\right)^{\frac{1}{\beta_n}} = e^{-\frac{1}{r\gamma}} \ a.s.$$

Proof Let $F^*(y) = F\left(\omega(F) - \frac{1}{y}\right), y > 0$. We know that $F\varepsilon DA(H_{2,\gamma})$ iff $F^*\varepsilon DA(H_{1,\gamma})$ ie.,

$$P(Y_n \leq y) = P(X_n \leq \omega(F) - \frac{1}{y}) = P\left(\frac{1}{\omega(F) - X_n} \leq y\right)$$

for every y which implies that $Y_n =^d \frac{1}{\omega(F)-X_n}$. Observe that $\eta'_{r,n} = \frac{1}{\omega(F)-\eta_{r,n}}$. Since $F^* \varepsilon DA(H_{1,\gamma})$, from Theorem 3.1 we have

$$\limsup \left(\frac{\eta'_{r,n}}{B_{a_n}} \right)^{\frac{1}{\beta_n}} = e^{\frac{1}{r\gamma}} \quad a.s.$$

Substituting $\eta'_{r,n} = \frac{1}{\omega(F)-\eta_{r,n}}$, one gets the required result. ■

Example 4 *Let F be Weibull with parameter $\gamma > 0$. Then*

$$\liminf \left(a_n^{\frac{1}{\gamma}} (-\eta_{r,n}) \right)^{\frac{1}{\beta_n}} = e^{\frac{-1}{r\gamma}} \quad a.s.$$

Example 5 *Let $F(x) = x^p, 0 \leq x \leq 1, p > 0$. Note that $\omega(F) = 1$. We have the following L.I.L*

$$\liminf \left(a_n (1 - \eta_{r,n}) \right)^{\frac{1}{\beta_n}} = e^{\frac{p}{r}} \ a.s.$$

6 Boundary crossing problem.

Let (ξ_n) be a sequence of r.v.s and (α_n) be a sequence of real constants such that $\xi_n \leq \alpha_n \ a.s$ or $P(\xi_n \geq \alpha_n \ i.o) = 0$. Here α_n is called a.s upper boundary for $\xi_n, n \geq 1$. The number of times $\xi_n > \alpha_n$ is a proper r.v giving the total number of boundary crossings. Define $Z_n = 1$ if $\xi_n > \alpha_n$,=0 otherwise and $N = \sum_{n=1}^{\infty} Z_n$. Then N is a r.v giving the total number of boundary crossings, which is an infinite sum of dependent, non-identically distributed Bernoulli r.v.s. One of the important measures is $E(N)$, which gives some idea of the precision of α_n.

Lemma 6.1 *$E(N) < \infty$ or $= \infty$ according to $\sum P(Z_n = 1) < \infty$ or $= \infty$.*

Proof Define $N_m = \sum_{n=1}^{m} Z_n, m \geq 1$, and note that $N_m \to N$ as $m \to \infty$. We have $E(N_m) = \sum_{n=1}^{m} P(Z_n = 1)$. Since $N \geq N_m, E(N) \geq \lim_{m\to\infty} E(N_m) = \sum_{n=1}^{\infty} P(Z_n = 1)$. Consequently, $E(N) < \infty$ whenever $\sum P(Z_n = 1) = \infty$. Since (N_m) is a sequence of non-negative, non-decreasing measurable functions, we have $\liminf E(N_m) \geq E(\liminf N_m)$ or $\liminf \sum_{n=1}^{m} P(Z_n = 1) \geq E(N)$, which implies that $E(N) \leq \sum_{n=1}^{\infty} P(Z_n = 1)$. In turn, $E(N) < \infty$ whenever $\sum_{n=1}^{m} P(Z_n = 1) < \infty$. ■

Let (X_n) be i.i.d Uniform $(0,1)$. From lemma 2.2 we have for any $\epsilon > 0$, $P\left(\eta^*_{r,n} > \left(1 - \frac{1}{a_n} \left(\frac{a_n}{n \log n} \right)^{1+\epsilon} \right) i.o \right) = 0$. Define $Z_n = 1$ if $\eta^*_{r,n} > \alpha_n, = 0$ otherwise, where $\alpha_n = \left(1 - \frac{1}{a_n} \left(\frac{a_n}{n \log n} \right)^{1+\epsilon} \right)$. Then we have the following lemma.

Lemma 6.2 *When* (X_n) *is i.i.d Uniform* $(0,1)$, $E(N) < \infty$ *if* $a_n = o((\log n)^\delta)$ *for any* $\delta > 0$ *and* $E(n) = \infty$ *if* $\frac{a_n}{n^\delta} \to \infty$ *for some* $\delta > 0$.

Proof With no loss of generality, we give the proof when $r = 2$. We have

$$P(\eta^*_{2,n} > \alpha_n) = 1 - P(\eta^*_{2,n} \leq \alpha_n) = 1 - \alpha_n^{a_n} - a_n \alpha_n^{a_n - 1}(1 - \alpha_n) \sim \left(\frac{a_n}{n \log n}\right)^{(1+\epsilon)}.$$

If $a_n = o(\log n)^\delta$ for any $\delta > 0$, one can show that $\sum \left(\frac{a_n}{n(\log n)}\right)^{1+\epsilon} < \infty$. Consequently, $\sum_n P(Z_n = 1) < \infty$ or $E(N) < \infty$. If $\frac{a_n}{n^\delta} \to \infty$ for some $\delta > 0$, then $a_n > n^\delta$ for all n large. Consequently, for n large $\left(\frac{a_n}{n \log n}\right)^{1+\epsilon} > \frac{1}{(n^{1-\delta} \log n)^{1+\epsilon}}$. Whenever $\epsilon < \frac{\delta}{1-\delta}$, $\sum \left(\frac{a_n}{n \log n}\right) = \infty$, which implies that $E(N) = \infty$. ■

Remark 6.1 *One can similarly discuss the behaviour of* $E(N)$ *under sections* 3, 4 *and* 5. *Recognizing the equivalence of events, one can see that* $E(N) < \infty$ *whenever* $a_n = (\log n)^\delta$ *for any* $\delta > 0$ *and* $E(N) = \infty$ *when* $\frac{a_n}{n^\delta} \to \infty$ *for some* $\delta > 0$ *in all the cases.*

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