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Asymptotic Analysis of Reliability in Recursively Defined Networks

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Abstract. Problems of a calculation of a random network reliability has exponential complexity and demand a number of arithmetic operations which increases as a geometric progression by a number of a network arcs. So approaches which accelerate these calculations are actual. In this paper these approaches are based on asymptotic formulas and on a consideration of recursively defined classes of random networks.

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1 Introduction

Problems of a calculation of a random network reliability has exponential complexity and demand a number of arithmetic operations which increases as a geometric progression by a number of a network arcs [1], [2]. So approaches which accelerate these calculations are actual. In this paper these approaches are based on asymptotic formulas and on a consideration of recursively defined classes of random networks.

Asymptotic methods of the calculation of the random network reliability are based on an assumption that all arcs of the network are low reliable or high reliable and their reliability depends on a small parameter h. In these cases asymptotic formulas for the network reliability depends on a minimal way length for low reliable arcs and on minimal ability to handle of cross sections for high reliable arcs. These formulas are transformed to consider a tail of a network life time distribution. Different asymptotics give different definitions of a way length: a sum of arcs lengths for exponential asymptotic and a maximum of arcs lengths for Weibull and Gompertz asymptotic. The asymptotic formulas allow to introduce concepts of a reliability invariance and of a network narrow place. The invariance concept is based on an independence of a network asymptotic parameters on arcs asymptotic parameters. The narrow place concept is based on a definition of a network critical arc.

Recursively defined classes of random networks are constructed from a set of generators using a replacement of their arcs by previously defined networks. Last years these classes call large interest from specialists in the solid state physics and in the nanotechnology. Recursive formulas are constructed to calculate reliabilities of random networks and their main asymptotic parameters. These formulas demand linear number of arithmetic operations. Examples of sequential-parallel recursively defined random networks are considered.

This paper is based on the previous articles of the author [3, 4, 5, 6].

2 Main designations

Define the port as the oriented graph $\Gamma = \{U, W\}$ with the finite set of nodes U, the final set of arcs $W = \{w = (u, v),\ u, v \in U\}$ and the initial and final nodes $u_0, v_0 \in U$. Assume that $\mathcal{R}$ is the set of all ways without circles in the graph Γ from the node u_0 to the node v_0. Introduce the sets

$$\mathcal{A} = \{A \subset U,\ u_0 \in A,\ v_0 \notin A\},\ L = L(A) = \{w = (u, v):\ u \in A,\ v \in U \setminus A\},$$

and put that $\mathcal{L} = \{L(A),\ A \in \mathcal{A}\}$ is the set of all cross sections in the graph Γ. Suppose that arcs $w \in W$ work independently and are characterized by positive numbers: the probabilities of a work $p_w,\ 0 < p_w < 1,\ \overline{p}_w = 1 - p_w$, the length d_w, the ability to handle s_w and the weight c_w.

Denote $D_R = \sum\limits_{w \in R} d_w$ the length of the way $R,\ R \in \mathcal{R}$, $D_L = \sum\limits_{w \in L} s_w$ the ability to handle of the cross section $L,\ L \in \mathcal{L}$, $\widehat{D}_R = \max\limits_{w \in R} d_w$ the pseudo length of the way $R,\ R \in \mathcal{R}$, $\widehat{D}_L = \max\limits_{w \in L} s_w$ the pseudo ability to handle of the cross section $L,\ L \in \mathcal{L}$,

$$D_\Gamma(\mathcal{R}) = \min_{R \in \mathcal{R}} D_R,\ D_\Gamma(\mathcal{L}) = \min_{L \in \mathcal{L}} D_L,\ \widehat{D}_\Gamma(\mathcal{R}) = \min_{R \in \mathcal{R}} \widehat{D}_R,\ \widehat{D}_\Gamma(\mathcal{L}) = \min_{L \in \mathcal{L}} \widehat{D}_L.$$

Call the way I_Γ (the way $\widehat{I}_\Gamma$) minimal (pseudo minimal) if $D_{I_\Gamma} = D_\Gamma(\mathcal{R})$ (if $\widehat{D}_{I_\Gamma} = \widehat{D}_\Gamma(\mathcal{R})$). Call the cross section J_Γ (the cross section $\widehat{J}_\Gamma$) minimal (pseudo minimal) if $D_{J_\Gamma} = D_\Gamma(\mathcal{L})$ (if $\widehat{D}_{J_\Gamma} = \widehat{D}_\Gamma(\mathcal{L})$). Put $\mathcal{R}_2 = \mathcal{R} \backslash \mathcal{R}_1,\ \mathcal{L}_2 = \mathcal{L} \backslash \mathcal{L}_1$,

$$\mathcal{R}_1 = \{R:\ R \in \mathcal{R},\ \widehat{D}_R = \widehat{D}_\Gamma(\mathcal{R})\},\ \mathcal{L}_1 = \{L:\ L \in \mathcal{L},\ \widehat{D}_L = \widehat{D}_\Gamma(\mathcal{L})\},$$

$P_\Gamma = P_\Gamma(p_w,\ w \in W)$ - the probability of a working way existence in the graph Γ, $\overline{P}_\Gamma = 1 - P_\Gamma$.

3 Asymptotic formulas

Denote U_R the random event that all arcs in the way R work and V_L the random event that all arcs in the cross section L do not work,

$$P_\Gamma^i = \sum_{R\in\mathcal{R}_i} P(U_R),\ \overline{P}_\Gamma^i = \sum_{L\in\mathcal{L}_i} P(V_L),\ i=1,2$$

.

Suppose that $D'_\Gamma(\mathcal{R})$ is the next by a quantity after $D_\Gamma(\mathcal{R})$ ($D'_\Gamma(\mathcal{R}) > D_\Gamma(\mathcal{R})$) member of the set $\{D_R:\ R\in\mathcal{R}\}$, $\widehat{D}'_R$ is the next by a quantity after $\widehat{D}_R$ ($\widehat{D}'_R < \widehat{D}_R$) member of the set $\{d_w:\ w\in R\}$. Put

$$C(\mathcal{R}) = \sum_{R:\ R\in\mathcal{R},\ D_R=D_\Gamma(\mathcal{R})} \prod_{w\in R} c_w,\ C'(\mathcal{R}) = \sum_{R:\ R\in\mathcal{R},\ D_R=D'_\Gamma(\mathcal{R})} \prod_{w\in R} c_w,$$

$$\widehat{C}(R) = \sum_{w\in R:\ d_w=\widehat{D}_R} c_w,\ \widehat{C}'(R) = \sum_{w\in R:\ d_w=\widehat{D}'_R} c_w,\ \widehat{C}(\mathcal{R}) = \min_{R:\ R\in\mathcal{R}_1} \widehat{C}(R),$$

$$\widehat{D}'_\Gamma(\mathcal{R}) = \min_{R:\ R\in\mathcal{R}_1,\ \widehat{C}(R)=\widehat{C}(\mathcal{R})} \widehat{D}'_R,\ \widehat{C}'(\mathcal{R}) = \min_{R:\ R\in\mathcal{R}_1,\ \widehat{C}(R)=\widehat{C}(\mathcal{R}),\ \widehat{D}'_R=\widehat{D}'_\Gamma(\mathcal{R})} \widehat{C}'(R).$$

Theorem 3.1 *1. If $p_w = p_w(h) \sim c_w h^{d_w},\ h\to 0,\ w\in W$, then*

$$P_\Gamma \sim C(\mathcal{R}) h^{D_\Gamma(\mathcal{R})}, \tag{3.1}$$

if $p_w = c_w h^{d_w},\ h>0,\ w\in W$, then $P_\Gamma - C(\mathcal{R})h^{D_\Gamma(\mathcal{R})} \sim C'(\mathcal{R})h^{D'_\Gamma(\mathcal{R})}$.
2. If $\overline{p}_w = \overline{p}_w(h) \sim c_w h^{s_w},\ h\to 0,\ w\in W$, then

$$\overline{P}_\Gamma \sim C(\mathcal{L}) h^{D_\Gamma(\mathcal{L})}, \tag{3.2}$$

if $\overline{p}_w = c_w h^{s_w},\ h>0,\ w\in W$, then $\overline{P}_\Gamma - C(\mathcal{L})h^{D_\Gamma(\mathcal{L})} \sim C'(\mathcal{L})h^{D'_\Gamma(\mathcal{L})}$.
3. If $p_w = p_w(h) \sim \exp(-c_w h^{-d_w}),\ h\to 0,\ w\in W$, then

$$P_\Gamma \sim P_\Gamma^1 + P_\Gamma^2 \sim P_\Gamma^1,\ \ln P_\Gamma \sim -\widehat{C}(\mathcal{R})h^{-\widehat{D}_\Gamma(\mathcal{R})}, \tag{3.3}$$

$$\ln P_\Gamma + \widehat{C}(\mathcal{R})h^{-\widehat{D}_\Gamma(\mathcal{R})} \sim -\widehat{C}'(\mathcal{R})h^{-\widehat{D}'_\Gamma(\mathcal{R})}.$$

4. If $\overline{p}_w = \overline{p}_w(h) \sim \exp(-c_w h^{-s_w}),\ h\to 0,\ w\in W$, then

$$\overline{P}_\Gamma \sim \overline{P}_\Gamma^1 + \overline{P}_\Gamma^2 \sim \overline{P}_\Gamma^1,\ \ln \overline{P}_\Gamma \sim -\widehat{C}(\mathcal{L})h^{-\widehat{D}_\Gamma(\mathcal{L})}, \tag{3.4}$$

$$\ln \overline{P}_\Gamma + \widehat{C}(\mathcal{L})h^{-\widehat{D}_\Gamma(\mathcal{L})} \sim -\widehat{C}'(\mathcal{L})h^{-\widehat{D}'_\Gamma(\mathcal{L})},$$

$$\widehat{D}_\Gamma(\mathcal{L}) = \max_{R\in\mathcal{R}} \min_{w\in R} s_w. \tag{3.5}$$

In the items *1, 3* models of low reliable arcs are considered and in the items *2, 4* models of high reliable arcs are considered.
Proof. 1. It is easy to obtain for $R \neq R'$

$$\sum_{R\in\mathcal{R}} P(U_R) - \sum_{R,R'\in\mathcal{R},\ R\neq R'} P(U_R U_{R'}) \leq P_\Gamma \leq \sum_{R\in\mathcal{R}} P(U_R),\ P(U_R U_{R'}) = o(P(U_R)),$$

consequently

$$P_\Gamma \sim \sum_{R\in\mathcal{R}} P(U_R),\ h \to 0. \tag{3.6}$$

From the formula (3.6) we have (3.1). The case $p_w = c_w h^{d_w}$ may be considered analogously.
2. The item 2 proof practically word by word repeats the item 1 proof and is based on the asymptotic formula

$$\overline{P}_\Gamma \sim \sum_{L\in\mathcal{L}} P(V_L),\ h \to 0. \tag{3.7}$$

3. The probability P_Γ satisfies the formula

$$P_\Gamma \sim P_\Gamma^1 + P_\Gamma^2,\ P_\Gamma^i = \sum_{R\in\mathcal{R}_i} exp(-\widehat{C}(R)h^{-\widehat{D}_R} - \widehat{C}'(R)h^{-\widehat{D}'_R}(1+o(1))),\ i = 1,2.$$

By a definition obtain that $P_\Gamma^2 = o(P_\Gamma^1)$, so

$$P_\Gamma \sim P_\Gamma^1 \sim \sum_{R\in\mathcal{R}_1} exp(-\widehat{C}(R)h^{-\widehat{D}_\Gamma(\mathcal{R})} - \widehat{C}'(R)h^{-\widehat{D}'_R}(1+o(1))) \sim$$

$$\sim \exp(-\widehat{C}(\mathcal{R})h^{-\widehat{D}_\Gamma(\mathcal{R})}) \sum_{R\in\mathcal{R}_1,\ \widehat{C}(R)=\widehat{C}(\mathcal{R})} exp(-\widehat{C}'(R)h^{-\widehat{D}'_R}(1+o(1))) \sim$$

$$\sim \exp(-\widehat{C}(\mathcal{R})h^{-\widehat{D}_\Gamma(\mathcal{R})}) \sum_{R\in\mathcal{R}_1,\ \widehat{C}(R)=\widehat{C}(\mathcal{R}),\ \widehat{D}'_R=\widehat{D}'_\Gamma(\mathcal{R})} exp(-\widehat{C}'(R)h^{-\widehat{D}'_\Gamma(\mathcal{R})}(1+o(1))),$$

consequently

$$P_\Gamma \sim \exp(-\widehat{C}(\mathcal{R})h^{-\widehat{D}_\Gamma(\mathcal{R})} - \widehat{C}'(\mathcal{R})h^{-\widehat{D}'_\Gamma(\mathcal{R})}(1+o(1))).$$

Moving to logarithms in the last formula obtain (3.3).
4. The formula (3.4) proof is based on the formula (3.7) and repeats the item 2 proof. The equality (3.5) is based on the obvious formulas

$$\bigcup_{R\in\mathcal{R}} U_R \supseteq \bigcap_{L\in\mathcal{L}} \overline{V}_L,\ \bigcap_{R\in\mathcal{R}} \overline{U}_R \supseteq \bigcup_{L\in\mathcal{L}} V_L \Longrightarrow \bigcup_{R\in\mathcal{R}} U_R = \bigcap_{L\in\mathcal{L}} \overline{V}_L.$$

Theorem 3.1 is proved.

4 Asymptotic invariance

Define the graph Γ_w^0, $w = (u', u'') \in W$, by an exclusion of the arc w from the graph Γ and the graph Γ_w^1 by an exclusion of the arc w from the graph Γ and a gluing of the nodes u', u''.

Corollary 4.1 *In the conditions of Theorem 3.1*

$$D_\Gamma(\mathcal{R}) = \min[\max(d_w, D_{\Gamma_w^1}(\mathcal{R})), D_{\Gamma_w^0}(\mathcal{R})],\ D_{\Gamma_w^1}(\mathcal{R}) \le D_{\Gamma_w^0}(\mathcal{R}),$$

$$\widehat{D}_\Gamma(\mathcal{R}) = \min[\max(d_w, \widehat{D}_{\Gamma_w^1}(\mathcal{R})), \widehat{D}_{\Gamma_w^0}(\mathcal{R})],\ \widehat{D}_{\Gamma_w^1}(\mathcal{R}) \le \widehat{D}_{\Gamma_w^0}(\mathcal{R}).$$

The constant $D_\Gamma(\mathcal{R})$ (the constant $\widehat{D}_\Gamma(\mathcal{R})$) does not depend on d_w if and only if $D_{\Gamma_w^1}(\mathcal{R}) = D_{\Gamma_w^0}(\mathcal{R})$ (if and only if $\widehat{D}_{\Gamma_w^1}(\mathcal{R}) = \widehat{D}_{\Gamma_w^0}(\mathcal{R})$). Analogous statements are true for cross sections.

All equalities for the fixed arc $w \in W$ are based on the complete probability formula and on its corollary [2, §7.4]:

$$P_\Gamma = p_w P_{\Gamma_w^1} + \overline{p}_w P_{\Gamma_w^1} \tag{4.1}$$

Example 4.1 *Denote Γ the port with the set of nodes $U = \{u_0, u_1, u_2, u_3\}$ and the set of arcs (figure. 1) $W = \{w_1, w_2, w_3, w_4, w = w_5\}$, with the initial node u_0 and the final node u_3. This port is called the bridge scheme in the reliability theory [7] and the arc w is called the bridge element.*

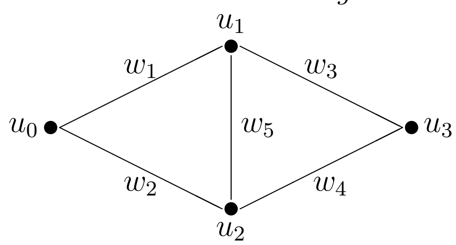

Figure 1. The bridge scheme Γ.

It is simple to calculate

$$D_{\Gamma_w^0}(\mathcal{R}) = \min((d_{w_1} + d_{w_3}), (d_{w_2} + d_{w_4})), D_{\Gamma_w^1}(\mathcal{R}) = \min(d_{w_1}, d_{w_2}) + \min(d_{w_3}, d_{w_4}),$$

$$\widehat{D}_{\Gamma_w^0}(\mathcal{R}) = \min(\max(d_{w_1}, d_{w_3}), \max(d_{w_2}, d_{w_4})),$$

$$\widehat{D}_{\Gamma_w^1}(\mathcal{R}) = \max(\min(d_{w_1}, d_{w_2}), \min(d_{w_3}, d_{w_4})).$$

The equality $D_{\Gamma_w^0}(\mathcal{R}) = D_{\Gamma_w^1}(\mathcal{R})$ is true if and only if one of the following two conditions takes place:

$$a)\ d_{w_4} \ge d_{w_3},\ d_{w_2} \ge d_{w_1},\ b)\ d_{w_3} \ge d_{w_4},\ d_{w_1} \ge d_{w_2}.$$

The equality $\widehat{D}_{\Gamma_w^0}(\mathcal{R}) = \widehat{D}_{\Gamma_w^1}(\mathcal{R})$ is true if and only if one of the following eight conditions takes place:

$$a)\ d_3 \ge d_1 > d_2,\ b)\ d_3 \ge d_1 = d_2,\ c)\ d_4 \ge d_1 = d_2,\ d)\ d_4 \ge d_2 > d_1,$$

$$e)\ d_1 \ge d_3 > d_4,\ f)\ d_1 \ge d_3 = d_4,\ g)\ d_2 \ge d_3 = d_4,\ h)\ d_2 \ge d_4 > d_3.$$

5 Applications to life time models

Suppose that $\tau(w)$ are independent random variables - the life times of arcs $w \in W$. Denote $p_w(h) = P(\tau(w) > t)$ and put the life time of the graph Γ equal to

$$\tau(\Gamma) = \min_{R \in \mathcal{R}} \max_{w \in R} \tau(w).$$

If $h = h(t)$ is the monotonically decreasing and continuous function and $h \to 0$ for $t \to \infty$ then Theorem 3.1 is true (the items 1,3) when P_Γ is replaced by $P(\tau(\Gamma) > t)$. Particularly if $h = \exp(-t)$ then for $P(\tau(w) > t) \sim c_w \exp(-d_w t)$ obtain from the item 1 that

$$P(\tau(\Gamma) > t) \sim C(\mathcal{R}) \exp(-D_\Gamma(\mathcal{R})t), \ t \to \infty.$$

If $h = 1/t$ then for $P(\tau(w) > t) \sim \exp(-c_w t^{d_w})$ obtain from the item 3 that

$$\ln P(\tau(\Gamma) > t) \sim -\widehat{C}(\mathcal{R}) t^{\widehat{D}_\Gamma(\mathcal{R})}, \ \frac{\ln P(\tau(\Gamma) > t)}{-\widehat{C}(\mathcal{R}) t^{\widehat{D}_\Gamma(\mathcal{R})}} - 1 \sim \frac{\widehat{C}'(\mathcal{R}) t^{\widehat{D}'_\Gamma(\mathcal{R}) - \widehat{D}_\Gamma(\mathcal{R})}}{\widehat{C}(\mathcal{R})}.$$

If $h = \exp(-t)$ then for $P(\tau(w) > t) \sim \exp(-c_w \exp(\widehat{D}_\Gamma(\mathcal{R})t))$ obtain from the item 3 that

$$\ln P(\tau(\Gamma) > t) \sim -\widehat{C}(\mathcal{R}) \exp(\widehat{D}_\Gamma(\mathcal{R})t),$$

$$\frac{\ln P(\tau(\Gamma) > t)}{-\widehat{C}(\mathcal{R}) \exp(\widehat{D}_\Gamma(\mathcal{R})t)} - 1 \sim \frac{\widehat{C}'(\mathcal{R}) \exp((\widehat{D}'_\Gamma(\mathcal{R}) - \widehat{D}_\Gamma(\mathcal{R}))t)}{\widehat{C}(\mathcal{R})}.$$

So for the Weibull distribution we obtain the power rate convergence and for the Gompertz distribution - the exponential rate convergence of $\ln P(\tau(\Gamma) > t)$ to its asymptotic, $t \to \infty$. The Weibull and the Gompertz asymptotics of life times distributions of random arcs and networks are validated on a physical level of a rigidity in [8, 9] by the entropy approach. These distributions are limit in the scheme of a minimum of independent and identically distributed random variables.

If $\overline{p}_w(h) = P(\tau(w) \le t)$, $h = h(t)$ is the monotonically increasing and continious function and $h \to 0$ for $t \to 0$ then Theorem 3.1 is true (the item 2) if $\overline{P}_\Gamma$ is replaced by $P(\tau(\Gamma) \le t)$. Particularly if $h = t$ then for $P(\tau(w) \le t) \sim c_w t^{s_w}$ obtain that

$$P(\tau(\Gamma) \le t) \sim C(\mathcal{L}) t^{D_\Gamma(\mathcal{L})}, \ t \to 0.$$

6 Recursively defined ports

Suppose that $\mathcal{B}_*$ is the set of ports Γ with nonintersected sets of nodes. Define the class $\mathcal{B}$ of ports with the set of generators $\mathcal{B}_*$, $\mathcal{B}_* \subset \mathcal{B}$ by the following condition. If the port $\Gamma = \{U, W\} \in \mathcal{B}_*$ and $W = \{w_1, \dots, w_m\}$ and the ports $\Gamma_1 = \{U_1, W_1\}, \dots, \Gamma_m = \{U_m, W_m\} \in \mathcal{B}$ and $U_1 \cap \dots \cap U_m = \emptyset$ then the port $\Gamma' = \Gamma(\Gamma_1, \dots, \Gamma_m)$ obtained by a replacement in the graph Γ of the arcs $w_1, \dots, w_m$ by the ports $\Gamma_1, \dots, \Gamma_m$ also belongs to the class $\mathcal{B}$.

For the port $\Gamma' = \Gamma(\Gamma_1, \dots, \Gamma_m) \in \mathcal{B}$ algorithms of a calculation of the reliability $P_{\Gamma'}$, of the length $D_{\Gamma'}(\mathcal{R})$, of the ability to handle of cross sections $D_{\Gamma'}(\mathcal{L})$, of the pseudo length $\widehat{D}_{\Gamma'}(\mathcal{R})$, of the pseudo ability to handle of cross sections $\widehat{D}_{\Gamma'}(\mathcal{L})$ are based on the recursive formulas

$$P_{\Gamma'} = P_\Gamma(P_{\Gamma_1}, \dots, P_{\Gamma_m}),$$

$$D_{\Gamma'}(\mathcal{R}) = D_\Gamma(D_{\Gamma_1}(\mathcal{R}_1), ..., D_{\Gamma_m}(\mathcal{R}_m)), \ \ D_{\Gamma'}(\mathcal{L}) = D_\Gamma(D_{\Gamma_1}(\mathcal{L}_1), ..., D_{\Gamma_m}(\mathcal{L}_m)), \tag{6.1}$$

$$\widehat{D}_{\Gamma'}(\mathcal{R}) = \widehat{D}_\Gamma(\widehat{D}_{\Gamma_1}(\mathcal{R}_1), ..., \widehat{D}_{\Gamma_m}(\mathcal{R}_m)), \ \ \widehat{D}_{\Gamma'}(\mathcal{L}) = \widehat{D}_\Gamma(\widehat{D}_{\Gamma_1}(\mathcal{L}_1), ..., \widehat{D}_{\Gamma_m}(\mathcal{L}_m)). \tag{6.2}$$

Denote $m(\Gamma')$ the number of arcs in the graph Γ' and

$$n(P_{\Gamma'}), \ n(D_{\Gamma'}(\mathcal{R})), \ n(D_{\Gamma'}(\mathcal{L})), \ n(\widehat{D}_{\Gamma'}(\mathcal{R})), \ n(\widehat{D}_{\Gamma'}(\mathcal{L}))$$

are the numbers of arithmetic operations necessary to calculate these characteristics.

Theorem 6.1 *Suppose that* $\inf_{\Gamma \in \mathcal{B}_*} m(\Gamma) > 1$. *For each* $\Gamma' \in \mathcal{B}$ *the following inequailities are true:*

$$n(P_{\Gamma'}) \le (m(\Gamma') - 1) \sup_{\Gamma \in \mathcal{B}_*} n(P_\Gamma), \tag{6.3}$$

$$n(D_{\Gamma'}(\mathcal{R})) \le (m(\Gamma') - 1) \sup_{\Gamma \in \mathcal{B}_*} n(D_\Gamma(\mathcal{R})), n(D_{\Gamma'}(\mathcal{L})) \le (m(\Gamma') - 1) \sup_{\Gamma \in \mathcal{B}_*} n(D_\Gamma(\mathcal{L})),$$

$$n(\widehat{D}_{\Gamma'}(\mathcal{R})) \le (m(\Gamma') - 1) \sup_{\Gamma \in \mathcal{B}_*} n(\widehat{D}_\Gamma(\mathcal{R})), n(\widehat{D'}_\Gamma(\mathcal{L})) \le (m(\Gamma') - 1) \sup_{\Gamma \in \mathcal{B}_*} n(\widehat{D}_\Gamma(\mathcal{L})).$$

Proof. Prove the inequality (6.3) because all other inequalities may be proved analogously. Denote $\sup_{\Gamma \in \mathcal{B}_*} n(P_\Gamma) = n_*$, if the inequality (6.3) is true for the graphs $\Gamma_1, \dots, \Gamma_m$, then from (6.1) obtain

$$n(P_{\Gamma'}) \le \sum_{i=1}^m n(P_{\Gamma_i}) + n(P_\Gamma) \le n_*(\sum_{i=1}^m (m(\Gamma_i) - 1) + (m(\Gamma) - 1)) \le$$

$$\le n_*(\sum_{i=1}^m m(\Gamma_i) + m(\Gamma) - 1) \le n_*(m(\Gamma') - 1).$$

Theorem 6.1 is proved.

From the inequality (6.3) we obtain that to calculate $P_{\Gamma'}, D_{\Gamma'}(\mathcal{R})), D_{\Gamma'}(\mathcal{L})$, $\widehat{D}_{\Gamma'}(\mathcal{R})), \widehat{D}_{\Gamma'}(\mathcal{L})$ it is necessary to use linear by $m(\Gamma')$ numbers of arithmetic operations. It is worthy to note that for ports of a general type $n(D_{\Gamma'}(\mathcal{R}))$ increases as a square of $m(\Gamma')$, $n(D_{\Gamma'}(\mathcal{L}))$ increases as a cube of $m(\Gamma')$, $n(P_{\Gamma'})$ increases in a geometric progression of $m(\Gamma')$ [1, 2, 10].

7 Recursive definitions of minimal ways and cross sections

Using the formulas (6.1),(6.2) it is possible to construct a recursive algorithm of a definition of minimal and pseudominimal ways and cross sections. Denote $\mathcal{R}_i$ the set of ways in the graph $\Gamma_i,\ 1 \le i \le m$.

Theorem 7.1 *1. To define minimal (pseudo minimal) way in the graph $\Gamma' = \Gamma(\Gamma_1, \dots, \Gamma_m)$ it is necessary to calculate $D_{\Gamma_1}(\mathcal{R}_1), \dots, D_{\Gamma_m}(\mathcal{R}_m)$ (to calculate $\widehat{D}_{\Gamma_1}(\mathcal{R}_1), \dots, \widehat{D}_{\Gamma_m}(\mathcal{R}_m)$), to replace $d_{w_1}, \dots, d_{w_m}$ by these quantities and then to define in the graph Γ the minimal way I_Γ (the pseudo minimal way $\widehat{I}_\Gamma$). Then in the way I_Γ (in the way $\widehat{I}_\Gamma$) each arc w_i is to be substituted by the way I_{Γ_i} (by the way $\widehat{I}_{\Gamma_i}$), $1 \le i \le m$. Each minimal (pseudo minimal) way in the graph Γ' is defined by the algorithm of this item.*
2. The numbers $C(\mathcal{R})$, $\widehat{C}(\mathcal{R})$ are defined by the equalities

$$C(\mathcal{R}) = \sum_{I_\Gamma \in \mathcal{R}} \prod_{i:\, w_i \in I_\Gamma,\, 1 \le i \le m} C(\mathcal{R}_i),$$

$$\widehat{C}(\mathcal{R}) = \min_{\widehat{I}_\Gamma \in \mathcal{R}} \sum_{i:\, w_i \in \widehat{I}_\Gamma,\, 1 \le i \le m,\, \widehat{D}_{\Gamma_i}(\mathcal{R}_i) = \widehat{D}'_\Gamma(\mathcal{R})} \widehat{C}(\mathcal{R}_i),$$

where I_Γ are minimal ways in the graph Γ taking into account the replacement of d_{w_i} by $D_{\Gamma_i}(\mathcal{R}_i)$, $1 \le i \le m$, and $\widehat{I}_\Gamma$ are pseudo minimal ways in the graph Γ taking into account the replacement of d_{w_i} by $\widehat{D}_{\Gamma_i}\mathcal{R}_i$, $1 \le i \le m$.
3. The items 1 - 2 may be spread onto minimal (pseudo minimal) cross sections after appropriate replacements.

Remark 7.1 *Assume that $R \in \mathcal{R}$ is some way between the nodes u, v in the graph Γ.*
1. Suppose that the increment $\Delta d_w \le 0,\ w \in R,\ \Delta d_w = 0,\ w \notin R$. Then the appropriate increment $\Delta D_\Gamma(\mathcal{R}) = \sum\limits_{w \in R} \Delta d_w$ if $R = I_\Gamma$, $\Delta D_\Gamma(\mathcal{R}) > \sum\limits_{w \in R} \Delta d_w$ if the way R is not minimal.
2. Denote $\Delta = -\min(|\widehat{D}_\Gamma(\mathcal{R}) - d_w| :\ w \in W,\ d_w \ne \widehat{D}_\Gamma(\mathcal{R}))$ and suppose that the increment $\Delta d_w = \Delta$ if $w \in R$ and $d_w = \widehat{D}_\Gamma(\mathcal{R})$ else $\Delta d_w = 0$. Then the appropriate difference $\Delta \widehat{D}_R = \Delta$, if $R = \widehat{I}_\Gamma$, $\Delta \widehat{D}_R > \Delta$ if the way R is not minimal. Consequently the increment Δ of $d_w,\ w \in \widehat{I}_\Gamma,\ d_w = \widehat{D}_\Gamma(\mathcal{R})$ gives the same increment of $\widehat{D}_\Gamma(\mathcal{R})$. So the set of arcs $w \in \widehat{I}_\Gamma,\ d_w = \widehat{D}_\Gamma(\mathcal{R})$, may be called narrow place in the graph Γ. Analogous statements are true for cross sections.

Example 7.1 *Suppose that the set of generators $\mathcal{B}_*$ consists of sequential $\Gamma_1 \to \Gamma_2$ and parallel $\Gamma_1 \| \Gamma_2$ connections and denote $\mathcal{R}_{\Gamma_1 \to \Gamma_2}$, $\mathcal{L}_{\Gamma_1 \to \Gamma_2}$, $\mathcal{R}_{\Gamma_1 \| \Gamma_2}$, $\mathcal{L}_{\Gamma_1 \| \Gamma_2}$ the sets of ways and cross sections in appropriate graphs.*

If $P_{\Gamma_i} \sim C(\mathcal{R}_i)h^{D_{\Gamma_i}(\mathcal{R}_i)}$, $(\overline{P}_{\Gamma_i} \sim C(\mathcal{L}_i)h^{D_{\Gamma_i}(\mathcal{L}_i)})$, $i = 1, 2,\ h \to 0$, *then*

$$P_{\Gamma_1\to\Gamma_2} \sim C(\mathcal{R}_{\Gamma_1\to\Gamma_2})h^{D_{\Gamma_1\to\Gamma_2}(\mathcal{R}_{\Gamma_1\to\Gamma_2})},\ P_{\Gamma_1\|\Gamma_2} \sim C(\mathcal{R}_{\Gamma_1\|\Gamma_2})h^{D_{\Gamma_1\|\Gamma_2}(\mathcal{R}_{\Gamma_1\|\Gamma_2})},$$

$$(\overline{P}_{\Gamma_1\to\Gamma_2} \sim C(\mathcal{L}_{\Gamma_1\to\Gamma_2})h^{D_{\Gamma_1\to\Gamma_2}(\mathcal{L}_{\Gamma_1\to\Gamma_2})},\ \overline{P}_{\Gamma_1\|\Gamma_2} \sim C(\mathcal{L}_{\Gamma_1\|\Gamma_2})h^{D_{\Gamma_1\|\Gamma_2}(\mathcal{L}_{\Gamma_1\|\Gamma_2})}),$$

where $D_{\Gamma_1\to\Gamma_2}(\mathcal{R}_{\Gamma_1\to\Gamma_2}) = D_{\Gamma_1}(\mathcal{R}_1) + D_{\Gamma_2}(\mathcal{R}_2)$, $C(\mathcal{R}_{\Gamma_1\to\Gamma_2}) = C(\mathcal{R}_1)C(\mathcal{R}_2)$,
$D_{\Gamma_1||\Gamma_2}(\mathcal{L}_{\Gamma_1||\Gamma_2}) = D_{\Gamma_1}(\mathcal{L}_1) + D_{\Gamma_2}(\mathcal{L}_2)$, $C(\mathcal{L}_{\Gamma_1||\Gamma_2}) = C(\mathcal{L}_1)C(\mathcal{L}_2)$,
$D_{\Gamma_1\|\Gamma_2}(\mathcal{R}_{\Gamma_1||\Gamma_2})=\min(D_{\Gamma_1}(\mathcal{R}_1), D_{\Gamma_2}(\mathcal{R}_2))$,
$D_{\Gamma_1\to\Gamma_2}(\mathcal{L}_{\Gamma_1\to\Gamma_2})=\min(D_{\Gamma_1}(\mathcal{L}_1), D_{\Gamma_2}(\mathcal{L}_2))$,

$$C(\mathcal{R}_{\Gamma_1||\Gamma_2}) = \begin{cases} C(\mathcal{R}_1), D_{\Gamma_1}(\mathcal{R}_1) < D_{\Gamma_2}(\mathcal{R}_2), \\ C(\mathcal{R}_2), D_{\Gamma_1}(\mathcal{R}_1) > D_{\Gamma_2}(\mathcal{R}_2), \\ C(\mathcal{R}_1) + C(\mathcal{R}_2),\ else, \end{cases}$$

$$C(\mathcal{L}_{\Gamma_1\to\Gamma_2}) = \begin{cases} C(\mathcal{L}_1), D_{\Gamma_1}(\mathcal{L}_1) < D_{\Gamma_2}(\mathcal{L}_2), \\ C(\mathcal{L}_2), D_{\Gamma_1}(\mathcal{L}_1) > D_{\Gamma_2}(\mathcal{L}_2), \\ C(\mathcal{L}_1) + C(\mathcal{L}_2),\ else. \end{cases}$$

For the minimal ways I *and cross sections* J *we have*

$$I_{\Gamma_1\to\Gamma_2} = I_{\Gamma_1} \to I_{\Gamma_2},\ J_{\Gamma_1\|\Gamma_2} = J_{\Gamma_1} \cup J_{\Gamma_1}, \tag{7.1}$$

$$I_{\Gamma_1\|\Gamma_2}=\begin{cases} I_{\Gamma_1}, D_{\Gamma_1}(\mathcal{R}_1) < D_{\Gamma_2}(\mathcal{R}_2), \\ I_{\Gamma_2}, D_{\Gamma_1}(\mathcal{R}_1) > D_{\Gamma_2}(\mathcal{R}_2), \\ I_{\Gamma_1} or\ I_{\Gamma_2},\ else, \end{cases} J_{\Gamma_1\to\Gamma_2}=\begin{cases} J_{\Gamma_1}, D_{\Gamma_1}(\mathcal{L}_1) < D_{\Gamma_2}(\mathcal{L}_2), \\ J_{\Gamma_2}, D_{\Gamma_1}(\mathcal{L}_1) > D_{\Gamma_2}(\mathcal{L}_2), \\ J_{\Gamma_1} or\ J_{\Gamma_2},\ else. \end{cases} \tag{7.2}$$

If $P_{\Gamma_i} \sim \exp(-\widehat{C}(\mathcal{R}_i)h^{-\widehat{D}_{\Gamma_i}(\mathcal{R}_i)})$, $i = 1, 2,\ h \to 0$, *then*

$$P_{\Gamma_1\to\Gamma_2}\sim \exp(-\widehat{C}(\mathcal{R}_{\Gamma_1\to\Gamma_2})h^{-\widehat{D}_{\Gamma_1\to\Gamma_2}(\mathcal{R}_{\Gamma_1\to\Gamma_2})}),$$

$$P_{\Gamma_1\|\Gamma_2}\sim \exp(-\widehat{C}(\mathcal{R}_{\Gamma_1\|\Gamma_2})h^{-\widehat{D}_{\Gamma_1\|\Gamma_2}(\mathcal{R}_{\Gamma_1\|\Gamma_2})}),$$

where

$$\widehat{D}_{\Gamma_1\to\Gamma_2}(\mathcal{R}_{\Gamma_1\to\Gamma_2}) = \max(\widehat{D}_{\Gamma_1}(\mathcal{R}_1), \widehat{D}_{\Gamma_2}(\mathcal{R}_2)),$$

$$\widehat{D}_{\Gamma_1||\Gamma_2}(\mathcal{R}_{\Gamma_1||\Gamma_2}) = \min(\widehat{D}_{\Gamma_1}(\mathcal{R}_1), \widehat{D}_{\Gamma_2}(\mathcal{R}_2)),$$

$$\widehat{C}(\mathcal{R}_{\Gamma_1\to\Gamma_2}) = \begin{cases} \widehat{C}(\mathcal{R}_1), \widehat{D}_{\Gamma_1}(\mathcal{R}_1) > \widehat{D}_{\Gamma_2}(\mathcal{R}_2), \\ \widehat{C}(\mathcal{R}_2), \widehat{D}_{\Gamma_1}(\mathcal{R}_1) < \widehat{D}_{\Gamma_2}(\mathcal{R}_2), \\ \widehat{C}(\mathcal{R}_1) + \widehat{C}(\mathcal{R}_2),\ else, \end{cases}$$

$$\widehat{C}(\mathcal{R}_{\Gamma_1||\Gamma_2}) = \begin{cases} \widehat{C}(\mathcal{R}_1), \widehat{D}_{\Gamma_1}(\mathcal{R}_1) < \widehat{D}_{\Gamma_2}(\mathcal{R}_2), \\ \widehat{C}(\mathcal{R}_2), \widehat{D}_{\Gamma_1}(\mathcal{R}_1) > \widehat{D}_{\Gamma_2}(\mathcal{R}_2), \\ \min(\widehat{C}(\mathcal{R}_1), \widehat{C}(\mathcal{R}_2)),\ else, \end{cases}$$

If $\widehat{P}_{\Gamma_i} \sim \exp(-\widehat{C}(\mathcal{L}_i)h^{-\widehat{D}_{\Gamma_i}(\mathcal{L}_i)}),\ i = 1, 2,\ h \to 0,$ *then*

$$\overline{P}_{\Gamma_1 \to \Gamma_2} \sim \exp(-\widehat{C}(\mathcal{L}_{\Gamma_1 \to \Gamma_2})h^{-\widehat{D}_{\Gamma_1 \to \Gamma_2}(\mathcal{L}_{\Gamma_1 \to \Gamma_2})}),$$

$$\overline{P}_{\Gamma_1 \| \Gamma_2} \sim \exp(-\widehat{C}(\mathcal{L}_{\Gamma_1 \| \Gamma_2})h^{-\widehat{D}_{\Gamma_1 \| \Gamma_2}(\mathcal{L}_{\Gamma_1 \| \Gamma_2})}),$$

To calculate $\widehat{D}_{\Gamma_1 \to \Gamma_2}(\mathcal{L}_{\Gamma_1 \to \Gamma_2}),\ \widehat{D}_{\Gamma_1 \| \Gamma_2}(\mathcal{L}_{\Gamma_1 \| \Gamma_2}),\ \widehat{C}(\mathcal{L}_{\Gamma_1 \to \Gamma_2}),\ \widehat{C}(\mathcal{L}_{\Gamma_1 \| \Gamma_2})$ *it is necessary to replace in previous four equalities* $\min,\ \max,\ \mathcal{R},\ \Gamma_1 \to \Gamma_2,\ \Gamma_1 \| \Gamma_2$ *by* $\max,\ \min,\ \mathcal{L},\ \Gamma_1 \| \Gamma_2,\ \Gamma_1 \to \Gamma_2$. *For the pseudo minimal ways* $\widehat{I}$ *and cross sections* $\widehat{J}$ *the formulas (7.1), (7.2) may be used if to replace* D, I, J *by* $\widehat{D}, \widehat{I}, \widehat{J}$ *respectively.*

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