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Characterization of Probability Distributions Through Order Statistics

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Abstract.
A family of continuous probability distribution has been characterized through the difference of two conditional expectations, conditioned on a non-adjacent order statistics. Further, some of its deductions are also discussed.
Keywords. Characterization, continuous distributions, conditional expectation, order statistics.

1 Introduction

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a continuous population having the probability density function (pdf) $f(x)$ and the distribution function (df) $F(x)$ and let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be the corresponding order statistics. Then the conditional pdf of $X_{s:n}$ given $X_{r:n} = x$, $1 \leq r < s \leq n$, is ([2])

$$\Big[\frac{(n-r)!}{(s-r-1)!(n-s)!}\Big] \frac{[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}}{[1-F(x)]^{n-r}} f(y) \quad x \leq y \tag{1.1}$$

Conditional moments of order statistics are extensively used in characterizing the probability distributions. Various approaches are available in the literature. For a detailed survey one may refer to [3, 5, 6, 8] amongst others. In this paper, a general class of distribution

$$F(x) = 1 - e^{-ah(x)}, \quad a \neq 0 \tag{1.2}$$

has been characterized through expectation of function of order statistics.

It may be noted that the *df* $F(x)$ in (1.2) and the corresponding pdf $f(x)$ are linked by the following equation

$$1 - F(x) = \frac{f(x)}{ah'(x)} \tag{1.3}$$

For the applications of characterization of distributions, one may refer to [4] and [1].

2 Characterization theorem

Theorem 2.1. *Let X be an absolutely continuous random variable with the df $F(x)$ and the pdf $f(x)$ in the interval (α, β), where α and β may be finite or infinite, then for $1 \leq m < r < s \leq n$,*

$$E[h(X_{s:n}) - h(X_{r:n})|X_{m:n} = x] = \frac{1}{a} \sum_{j=r}^{s-1} \frac{1}{(n-j)} \tag{2.1}$$

if and only if

$$F(x) = 1 - e^{-ah(x)}, \quad a \neq 0 \tag{2.2}$$

where $h(x)$ is a monotonic and differentiable function of x such that $h(x) \to 0$ as $x \to \alpha$ and $h(x)\{1 - F(x)\} \to 0$ as $x \to \beta$.

Proof. First we will prove (2.2) implies (2.1). It can be seen ([7]) that for $1 \leq r < s \leq n$,

$$\begin{aligned} &E[h(X_{s:n}) - h(X_{s-1:n})|X_{r:n} = x] \\ &= \binom{n-r}{s-r-1} \frac{1}{[1-F(x)]^{n-r}} \int_x^\beta h'(y)[F(y) - F(x)]^{s-r-1}[1-F(y)]^{n-s+1} dy \end{aligned}$$

Therefore, for $1 \leq m < r < s \leq n$,

$$\begin{aligned} &E[h(X_{s:n}) - h(X_{r:n})|X_{m:n} = x] \\ &\quad = \sum_{i=0}^{s-r-1} E[h(X_{s-i:n}) - h(X_{s-i-1:n})|X_{m:n} = x] \\ &\quad = \sum_{j=r}^{s-1} \binom{n-m}{j-m} \frac{1}{[1-F(x)]^{n-m}} \int_x^\beta h'(y)[F(y) - F(x)]^{j-m}[1-F(y)]^{n-j} dy \\ &\quad = \frac{1}{a} \sum_{j=r}^{s-1} \frac{1}{(n-j)}, \text{ in view of (1.3)} \end{aligned}$$

This proves the necessary part.

To prove the sufficiency part, let

$$c = \frac{1}{a} \sum_{j=r}^{s-1} \frac{1}{(n-j)}$$

Then

$$E[h(X_{s:n}) - h(X_{r:n}) | X_{m:n} = x] = c$$

implies

$$\begin{aligned} &\frac{(n-m)!}{(s-m-1)!(n-s)!} \int_x^\beta h(y)[F(y)-F(x)]^{s-m-1}[1-F(y)]^{n-s} f(y)dy \\ &\quad - \frac{(n-m)!}{(r-m-1)!(n-r)!} \int_x^\beta h(y)[F(y)-F(x)]^{r-m-1}[1-F(y)]^{n-r} f(y)dy \\ &\quad = c[1-F(x)]^{n-m} \end{aligned} \tag{2.3}$$

Differentiating (2.3) $(r-m)$ times w.r.t. x, we have,

$$\begin{aligned} \frac{(n-r)!}{(s-r-1)!(n-s)!} \int_x^\beta h(y)[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s} f(y)dy \\ = \{h(x)+c\}[1-F(x)]^{n-r} \end{aligned} \tag{2.4}$$

Integrating LHS of (2.4) by parts and simplifying, we get

$$\begin{aligned} &\frac{(n-r)!}{(s-r-2)!(n-s+1)!} \int_x^\beta h(y)[F(y)-F(x)]^{s-r-2}[1-F(y)]^{n-s+1} f(y)dy \\ &\quad + \frac{(n-r)!}{(s-r-1)!(n-s+1)!} \int_x^\beta h'(y)[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s+1} dy \\ &\quad = \{h(x)+c\}[1-F(x)]^{n-r} \end{aligned} \tag{2.5}$$

From (2.4) and (2.5) it follows that

$$\begin{aligned} &\frac{(n-r)!}{(s-r-1)!(n-s+1)!} \int_x^\beta h'(y)[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s+1} dy \\ &\quad + \{h(x)+c_1\}[1-F(x)]^{n-r} = \{h(x)+c\}[1-F(x)]^{n-r} \end{aligned}$$

where $c_1 = \frac{1}{a} \sum_{j=r}^{s-2} \frac{1}{(n-j)}$.
That is,

$$\frac{[1-F(x)]^{n-r}}{a(n-s+1)} = \frac{(n-r)!}{(s-r-1)!(n-s+1)!} \int_x^\beta h'(y)[F(y)-F(x)]^{s-r-1} [1-F(y)]^{n-s+1} dy$$

Differentiating $(s-r)$ times both sides w.r.t x, we get

$$h'(x)[1-F(x)] = \frac{f(x)}{a}$$

and hence the Theorem. □

Remark 2.1. At $r = m$, we have

$$E[h(X_{s:n})|X_{r:n} = x] = h(x) + \frac{1}{a}\sum_{j=r}^{s-1}\frac{1}{(n-j)}$$

if and only if $1 - F(x) = e^{-ah(x)}$ as obtained in [5].

Corollary 2.1.1. *Under the conditions given in Theorem 2.1 and for* $1 \leq r < s \leq n$,

$$E[h(X_{s:n}) - h(X_{r:n})] + h(x) = E[h(X_{s:n})|X_{r:n} = x] \tag{2.6}$$

if and only if

$$F(x) = 1 - e^{-ah(x)}, \quad a \neq 0. \tag{2.7}$$

Proof. Follows simply from Theorem 2.1 and Remark 2.1. $\square$

Table 1: Examples based on the *d.f* $F(x) = 1 - e^{-ah(x)}$

Distribution	$F(x)$	a	$h(x)$
Exponential	$1 - e^{-\theta x}$ $0 < x < \infty$	θ	x
Weibull	$1 - e^{-\theta x^p}$ $0 < x < \infty$	θ	x^p
Pareto	$1 - (\frac{x}{\alpha})^{-\theta}$ $\alpha < x < \infty$	θ	$\log(\frac{x}{\alpha})$
Lomax	$1 - [1 + (\frac{x}{\alpha})]^{-p}$ $0 < x < \infty$	p	$\log[1 + (\frac{x}{\alpha})]$
Gompertz	$1 - \exp[-\frac{\lambda}{\mu}(e^{\mu x} - 1)]$ $0 < x < \infty$	$\frac{\lambda}{\mu}$	$e^{\mu x} - 1$
Beta of II	$1 - (1 + x)^{-1}$ $0 < x < \infty$	1	$\log(1 + x)$
Beta of I	$1 - (1 - x)^{\theta}$ $0 < x < 1$	$-\theta$	$\log(1 - x)$
Extreme value I	$1 - \exp[-e^x]$ $-\infty < x < \infty$	1	e^x
Log logistic	$1 - (1 + \theta x^p)^{-1}$ $0 < x < \infty$	1	$\log(1 + \theta x^p)$
Burr Type IX	$1 - \left[\frac{c\{(1+e^x)^k - 1\}}{2} + 1\right]^{-1}$ $-\infty < x < \infty$	1	$\log\left[\frac{c\{(1+e^x)^k - 1\}}{2} + 1\right]$
Burr Type XII	$1 - (1 + \theta x^p)^{-m}$ $0 < x < \infty$	m	$\log(1 + \theta x^p)$

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