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Moments of order statistics from extended type-I generalized logistic distribution

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Abstract. The concept of extended type-I generalized logistic distribution is introduced by Olapade [2]. In this paper explicit expression for moments of order statistics for the given distribution is obtained and some computational work is also carried out. Further, recurrence relations for marginal and joint moment generating functions of order statistics are derived.

1. Introduction

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a continuous population having probability density function *(pdf)* $f(x)$ and distribution function *(df)* $F(x)$. Let $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ be the corresponding order statistics. Then the pdf of $X_{r:n}$, the r–th order statistic is given by David and Nagaraja [1]

$$f_{r:n}(x) = C_{r:n}[F(x)]^{r-1}[1 - F(x)]^{n-r} f(x), \ -\infty < x < \infty.$$

Let us denote $\alpha_{r:n}^{(k)} = E(X_{r:n}^k)$, k–th moment of r–th order statistic. Then

$$\alpha_{r:n}^{(k)} = C_{r:n} \int_{-\infty}^{\infty} x^k [F(x)]^{r-1}[1 - F(x)]^{n-r} f(x) dx,$$

where

$$C_{r:n} = \frac{n!}{(r-1)!(n-r)!} = [B(r, n-r+1)]^{-1}$$

and $B(m, n)$ is a complete Beta function.

A random variable X is said to have extended type-I generalized logistic distribution (Olapade [2]) if its pdf is given by

$$f(x) = \frac{ape^{-x}}{(1 + ae^{-x})^{p+1}} \ , \ -\infty < x < \infty, \ a, p > 0, \tag{1}$$

and the df is given by

$$F(x) = \frac{1}{(1 + a \ e^{-x})^p} \ .$$

When $p = a = 1$, we have the ordinary logistic distribution and when $a = 1$, we have the type-I generalized logistic distribution (Balakrishnan and Leung [3]).

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2. Exact Moments

For extended type-I generalized logistic distribution, the pdf of $X_{r:n}$, $1 \leq r \leq n$, may be written as

$$\begin{aligned} f_{r:n}(x) &= [B(r, n-r+1)]^{-1}[1+ae^{-x}]^{-p(r-1)}[1-(1+ae^{-x})^{-p}]^{n-r}ape^{-x}(1+ae^{-x})^{-p-1} \\ &= ap[B(r, n-r+1)]^{-1}\sum_{i=0}^{n-r}(-1)^i\binom{n-r}{i}e^{-x}[1+ae^{-x}]^{-[p(r+i)+1]}. \end{aligned} \tag{2}$$

Theorem 2.1. *For distribution as given in* (1), $1 \leq r \leq n$ *and* $k = 1, 2, \ldots,$ *we have that*

$$\alpha_{r:n}^{(k)} = \frac{ap}{B(r, n-r+1)}\sum_{i=0}^{n-r}(-1)^{i+k}\binom{n-r}{i}\frac{\partial^k}{\partial[-p(r+i)]^k}[a^{-1}B(p(r+i), 1)]. \tag{3}$$

Proof. We have

$$\begin{aligned} \alpha_{r:n}^{(k)} &= \frac{ap}{B(r, n-r+1)}\int_{-\infty}^{\infty}x^k\sum_{i=0}^{n-r}(-1)^i\binom{n-r}{i}e^{-x}[1+ae^{-x}]^{-[p(r+i)+1]}dx \\ &= \frac{ap}{B(r, n-r+1)}\sum_{i=0}^{n-r}(-1)^i\binom{n-r}{i}\int_{-\infty}^{\infty}x^k\frac{e^{-(-p(r+i))x}}{[e^x+a]^{p(r+i)+1}}\,dx. \end{aligned}$$

Now, using the following result from Prudnikov et al. [6]

$$\int_{-\infty}^{\infty}\frac{x^n\ e^{-px}}{(e^x+z)^\rho}dx = (-1)^n\frac{\partial^n}{\partial p^n}[z^{-p-\rho}B(-p, (p+\rho))],$$

we obtain the required result. □

Remark 2.2. *When* $p = a = 1$, *the expression* (3) *reduces to ordinary logistic distribution and*

$$\alpha_{r:n}^{(k)} = \frac{1}{B(r, n-r+1)}\sum_{i=0}^{n-r}(-1)^{i+k}\binom{n-r}{i}\frac{\partial^k}{\partial[-(r+i)]^k}\ B(r+i, 1).$$

Remark 2.3. *When* $a = 1$, *the expression* (3) *reduces to type-I generalized logistic distribution and*

$$\alpha_{r:n}^{(k)} = \frac{p}{B(r, n-r+1)}\sum_{i=0}^{n-r}(-1)^{i+k}\binom{n-r}{i}\frac{\partial^k}{\partial[-p(r+i)]^k}\ B(p(r+i), 1).$$

When $k = 1$ and $k = 2$, we have

$$\alpha_{r:n}^{(1)} = \frac{p}{B(r, n-r+1)}\sum_{i=0}^{n-r}(-1)^i\binom{n-r}{i}\big[B(p(r+i), 1)\{\log(a)+\psi(p(r+i))-\psi(1)\}\big] \tag{4}$$

and

$$\alpha_{r:n}^{(2)} = \frac{p}{B(r, n-r+1)}\sum_{i=0}^{n-r}(-1)^i\binom{n-r}{i}B(p(r+i), 1)\big[\{\psi(p(r+i))-\psi(1)+[\log(a)+\psi(p(r+i))-\psi(1)]^2\}\big] \tag{5}$$

where $\psi(x)$ is a digamma function defined by $\psi(x) = \frac{d}{dx}\big(\ln\Gamma(x)\big) = \frac{\Gamma'(x)}{\Gamma(x)}$, $x \neq 0, -1, -2, \ldots$ as obtained in Balakrishnan and Leung [3].

Thus, making use of the explicit expression in (4) and (5), we can obtain the mean and variance of $X_{r:n}$.

In Table 1, it may be noted that the well known property of order statistics $\sum_{i=1}^{n}E(X_{i:n}) = nE(X)$ (David and Nagaraja [1]) is satistied.

Table 1: Means of order statistics of type-I generalized logistic distribution

n	r	$p = 1$	$p = 1.5$	$p = 2$	$p = 2.5$
1	1	0.0000	0.6137	1.0000	1.2804
2	1	−1.0000	−0.2726	0.1667	0.4774
	2	1.0000	1.5000	1.8333	2.0833
3	1	−1.5000	−0.6928	−0.2167	0.1151
	2	0.0000	0.5678	0.9333	1.2021
	3	1.5000	1.9661	2.2833	2.5239
4	1	−1.8333	−0.9642	−0.4595	−0.1116
	2	−0.5000	0.1213	0.5119	0.7951
	3	0.5000	1.0143	1.3547	1.6089
	4	1.8333	2.2833	2.5928	2.8289
5	1	−2.0833	−1.1632	−0.6353	−0.2742
	2	−0.8333	−0.1676	0.2436	0.5387
	3	0.0000	0.5547	0.9143	1.1797
	4	0.8333	1.3208	1.6484	1.8952
	5	2.0833	2.5239	2.8289	3.0624

Table 2: Variances of order statistics of type-I generalized logistic distribution

n	r	$p = 1$	$p = 1.5$	$p = 2$	$p = 2.5$
1	1	3.2898	2.5797	2.2898	2.1352
2	1	2.2898	1.5485	1.2621	1.1148
	2	2.2898	2.0398	1.9287	1.8662
3	1	2.0398	1.2786	0.9929	0.8490
	2	1.2898	1.0288	0.9187	0.8586
	3	2.0398	1.8936	1.8262	1.7875
4	1	1.9287	1.1534	0.8676	0.7257
	2	1.0398	0.7708	0.6611	0.6025
	3	1.0398	0.8881	0.8211	0.7836
	4	1.9287	1.8262	1.7780	1.7501
5	1	1.8662	1.0801	0.7939	0.6533
	2	0.9287	0.6535	0.5439	0.4863
	3	0.7898	0.6337	0.5670	0.5303
	4	0.9287	0.8229	0.7749	0.7477
	5	1.8662	1.7875	1.7501	1.7282

3. Recurrence relations for moments generating function

For extended type-I generalized logistic distribution defined in (1), we have the relation

$$F(x) = \frac{(a+e^x)}{ap} f(x) \,. \tag{6}$$

Using the relation (6), we shall derive the recurrence relations for moments generating function *(mgf)* of order statistics from extended type-I generalized logistic distribution.

Let us denote the mgf of $X_{r:n}$ by $M_{r:n}(t)$,

$$M_{r:n}(t) = C_{r:n} \int_{-\infty}^{\infty} e^{tx} \big[F(x)\big]^{r-1} \big[1-F(x)\big]^{n-r} f(x)dx$$

and the joint mgf of $X_{r:n}$ and $X_{s:n}$ $(1 \le r < s \le n)$ by $M_{r,s:n}(t_1,t_2)$,

$$M_{r,s:n}(t_1,t_2) = C_{r,s:n} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1 x + t_2 y} \big[F(x)\big]^{r-1} \big[F(y)-F(x)\big]^{s-r-1} \big[1-F(y)\big]^{n-s} f(x)f(y)dxdy,$$

where

$$C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}.$$

Lemma 3.1. *For the distribution as given in* (1) *and* $1 \le r \le n$, $M_{r:n}(t)$ *exists.*

Proof. In view of (2), we have

$$M_{r:n}(t) = \frac{ap}{B(r,n-r+1)} \sum_{i=0}^{n-r} (-1)^i \binom{n-r}{i} \int_{-\infty}^{\infty} e^{-x(1-t)} [1 + a\ e^{-x}]^{-p(r+i)-1} dx.$$

Since, $\int_0^\infty \frac{x^{\alpha}-1}{(x+z)^{\beta}}\ dx = Z^{\alpha-\beta}\ B(\alpha,\beta-\alpha)$ (Prudnikov et al. [6]), we have that

$$M_{r:n}(t) = \frac{a^t p}{B(r,n-r+1)} \sum_{i=0}^{n-r} (-1)^i \binom{n-r}{i}\ B[1-t, p(r+i)+t],\ t<1.$$

Hence the lemma. □

Lemma 3.2. *For* $2 \le r \le n$ *and* $n \ge 2$,

$$(i)\ \ M_{r:n}(t) - M_{r-1:n}(t) = \binom{n}{r-1} t \int_{-\infty}^{\infty} e^{tx} [F(x)]^{r-1} [1-F(x)]^{n-r+1} dx \tag{7}$$

$$(ii)\ \ M_{r:n}(t) - M_{r-1:n-1}(t) = \binom{n-1}{r-1} t \int_{-\infty}^{\infty} e^{tx} [F(x)]^{r-1} [1-F(x)]^{n-r+1} dx \tag{8}$$

$$(iii)\ \ M_{r-1:n-1}(t) - M_{r-1:n}(t) = \binom{n-1}{r-2} t \int_{-\infty}^{\infty} e^{tx} [F(x)]^{r-1} [1-F(x)]^{n-r+1} dx \tag{9}$$

Proof. We have from Ali and Khan [4] that

$$E\big[g(X_{r:n})\big] - E\big[g(X_{r-1:n})\big] = \binom{n}{r-1} \int_{\alpha}^{\beta} g'(x) [F(x)]^{r-1} [1-F(x)]^{n-r+1} dx, \tag{10}$$

where $g(x)$ is a Borel measurable function of x and $x \in (\alpha,\beta)$.

Let $g(X) = e^{tX}$, then (7) can be proved in view of (10). The equations (8) and (9) can be seen on the lines of (7). □

Theorem 3.3. *For extended type-I distribution as given in* (1) *and* $2 \leq r \leq n$, $n \geq 2$, $j = 1, 2, \ldots$

$$(i)\ \ M_{r:n}^{(j)}(t) = \Big(1+\frac{t}{p(r-1)}\Big)M_{r-1:n}^{(j)}(t)+\frac{1}{p(r-1)}\left(jM_{r-1:n}^{(j-1)}(t)+\frac{t}{a}M_{r-1:n}^{(j)}(t+1)+\frac{j}{a}M_{r-1:n}^{(j-1)}(t+1)\right), \quad (11)$$

$$(ii)\ \ M_{r:n}^{(j)}(t) = M_{r-1:n-1}^{(j)}(t)+\frac{n-r+1}{np(r-1)}\left(tM_{r-1:n}^{(j)}(t)+tM_{r-1:n}^{(j-1)}(t)+\frac{t}{a}M_{r-1:n}^{(j)}(t+1)+\frac{1}{a}M_{r-1:n}^{(j-1)}(t+1)\right), \quad (12)$$

$$(iii)\qquad M_{r-1:n-1}^{(j)}(t) = \Big(1+\frac{t}{np}\Big)M_{r-1:n}^{(j)}(t)+\frac{j}{np}M_{r-1:n}^{(j-1)}(t)+\frac{t}{nap}M_{r-1:n}^{(j)}(t+1)+\frac{j}{nap}M_{r-1:n}^{(j-1)}(t+1), \quad (13)$$

where $M_{r:n}^{(j)}(t)$ *is the* $j-th$ *derivative of* $M_{r:n}(t)$.

Proof. In view of equations (6) and (7), we have

$$M_{r:n}(t) - M_{r-1:n}(t) = \frac{t}{ap}\binom{n}{r-1}\int_{-\infty}^{\infty} e^{tx}[F(x]^{r-2}[1-F(x)]^{n-r+1}(a+e^x)f(x)dx$$

$$= \frac{C_{r-1:n}t}{(r-1)p}\int_{-\infty}^{\infty} e^{tx}[F(x)]^{r-2}[1-F(x)]^{n-r+1}f(x)dx + \frac{C_{r-1:n}t}{(r-1)ap}\int_{-\infty}^{\infty} e^{(t+1)x}[F(x)]^{r-2}[1-F(x)]^{n-r+1}f(x)dx$$

$$= \frac{t}{p\ (r-1)}M_{r-1:n}(t) + \frac{t}{ap(r-1)}M_{r-1:n}(t+1).$$

After rearranging the terms, we get

$$M_{r:n}(t) = \Big(1 + \frac{t}{p(r-1)}\Big)M_{r-1:n}(t) + \frac{t}{ap(r-1)}M_{r-1:n}(t+1). \quad (14)$$

Differentiating (14) *w.r.t.* t j times, we get (11). The equations (12) and (13) can be established on the lines of (11). □

Lemma 3.4. *For* $2 \leq r < s \leq n$ *and* $n \geq 2$

$$M_{r,s:n}(t_1,t_2) - M_{r-1,s:n}(t_1,t_2) = \frac{n!\ t_1}{(r-1)!(s-r)!(n-s)!}\int_{-\infty}^{\infty}\int_{y}^{\infty} e^{t_1x+t_2y} \times[F(y)-F(x)]^{s-r}[1-F(y)]^{n-s}f(y)dydx. \quad (15)$$

Proof. The Lemma can be prooved in view of Lemma 2.2 of Khan et al. [5]. □

Theorem 3.5. *For* $2 \leq r < s \leq n$ *and* $j, k = 0, 1, 2, \ldots$

$$M_{r,s:n}^{(j,k)}(t_1,t_2) = \Big(1+\frac{t_1}{p(r-1)}\Big)M_{r-1,s:n}^{(j,k)}(t_1,t_2) + \frac{j}{p(r-1)}M_{r-1,s:n}^{(j-1,k)}(t_1,t_2) + \frac{t_1}{ap(r-1)}M_{r-1,s:n}^{(j,k)}(t_1+1,t_2) + \frac{j}{ap(r-1)}M_{r-1,s:n}^{(j-1,k)}(t_1+1,t_2),$$

where, $M_{r,s:n}^{(j,k)}(t_1,t_2)$ *is the* $j,k-th$ *partial derivative of* $M_{r,s:n}(t_1,t_2)$.

Proof. In view of (6) and (15), we get

$$M_{r,s:n}(t_1,t_2) = \Big(1+\frac{t_1}{p(r-1)}\Big)M_{r-1,s:n}(t_1,t_2) + \frac{t_1}{ap(r-1)}M_{r-1:s:n}(t_1+1,t_2) \quad (16)$$

Differentiate (16) j times *w.r.t.* t_1 and k times *w.r.t* t_2, we have the required result. □

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