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Random Infinite Divisibility on $\mathbf{Z}_+$ and Generalized INAR(1) Models

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Abstract. Satheesh *et al.* (2010) had discussed two notions of random infinite divisibility of $\mathbf{Z}_+$-valued random variables (*r.v.*). Here we present two more results on random sums of $\mathbf{Z}_+$-valued *r.v.*s. Further, we discuss solutions to generalizations of a first order $\mathbf{Z}_+$-valued auto-regressive model by fruitfully applying these two notions of random infinite divisibility.

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1 Introduction

Discussion of the notion of infinitely divisible (ID) laws and their subclasses *viz.* stable, semi-stable, semi-selfdecomposable, α-decomposable, geometrically ID and compound distributions *etc.* on $\mathbf{Z}_+ = \{0, 1, 2, \dots\}$ has received a good amount of attention in recent years, *see e.g.* Aly and Bouzar (2000), Bouzar (2004, 2008) and Satheesh and Nair (2002). A whole chapter and many sections are devoted exclusively to the discussion of these in Steutel and van Harn (2004). Role of these distributions and their $\mathbf{Z}_+$ analogues in first-order auto-regressive (AR(1)) models were also investigated in many works *see, e.g.* the expository article by McKenzie (2003), Bouzar and Jayakumar (2006), Satheesh and Sandhya (2005, *corrections in* 2007), Bouzar and Satheesh (2008), Ristic *et al.* (2009) and Bakouch and Ristic (2009), and the references therein. Naturally, the discussion is to be based on probability generating functions (PGF) and many of the results applicable to the distributions on $\mathbf{R}$ have to be reformulated to those on $\mathbf{Z}_+$ and this poses a certain challenge too. Reformulating the first transfer theorem of Gnedenko, two approaches to $\mathbf{Z}_+$valued randomly ID laws were discussed in Satheesh *et al.* (2010)- the first one being the $\mathbf{Z}_+$valued analogue of N-ID laws of Gnedenko and Korolev (1996, p.145) and the second one that of the φ-ID laws of Satheesh (2004). Such limit distributions have been fruitfully used in discussing the relation between the marginals and the innovations of a generalized AR(1) model in Satheesh *et al.* (2008).

Here we need the definitions of N-ID laws and φ-ID laws for $\mathbf{Z}_+$-valued *r.v.*s. The notion of N-ID laws is based on a Laplce transform (LT) φ that is also the standard solution to the Poincare equation, $\varphi(t) = P(\varphi(\theta t))$ (see Gnedenko and Korolev, 1996, eqn. (6.7) on p.140). It may be noted that Aly and Bouzar (2000) has briefly touched upon this notion for $\mathbf{Z}_+$-valued *r.v.*s.

Definition 1.1 Let φ be the standard solution to the Poincare equation and N_θ a positive integer-valued *r.v.* having finite mean with PGF $P_\theta(s) = \varphi(\frac{1}{\theta}\varphi^{-1}(s))$, $\theta \in \Theta \subseteq (0, 1)$. A PGF $P(s)$ is N-ID if for each $\theta \in \Theta$ there exists a PGF $Q_\theta(s)$ that is independent of N_θ such that $P(s) = P_\theta(Q_\theta(s))$, for all $s \in [0, 1)$.

The notion of φ-ID laws is based on the following two lemmas, *see* Satheesh (2004) that are ramifications of Feller's proof of Bernstein's theorem (Feller, 1971, p.440).

Lemma 1.1. Given any LT φ, $\wp_\varphi = \{P_\theta(s) = s^j\varphi[(1 - s^k)/\theta]\}$, $s \in [0, 1]$, $j \geq 0$ & $k \geq 1$ integers and $\theta > 0$, describes a class of PGFs.

Lemma 1.2. Given a *r.v.* U with LT φ, the $\mathbf{Z}_+$-valued *r.v*s N_θ with PGF $P_\theta \in \wp_\varphi$ satisfy

$$\theta N_\theta \stackrel{d}{\to} kU \text{ as } \theta \downarrow 0,\ k \geq 1 \text{ integer.}$$

The transfer theorem in Satheesh *et al.* (2010) is given below. An underlying assumption in this development is that the $\mathbf{Z}_+$-valued *r.v.*s $\{X_{n,i}\}$ considered in the partial sums are assumed to have a positive mass at the origin, which is a necessary condition for their PGFs to be ID, and further ensures that their PGFs do not vanish.

Theorem 1.1 Let $X_{n,1}, X_{n,2}, \ldots$ be a sequence of independent and identically distributed (*i.i.d.*) $\mathbf{Z}_+$-valued *r.v.*s with PGF $P_n(s)$. Let N_n be another $\mathbf{Z}_+$-valued *r.v.* with distribution $p_{n,k} = P\{N_n = k\}$, distribution function $F_n(x) = P\{N_n < x\}$ and for each n, let N_n be independent of $X_{n,1}, X_{n,2}, \ldots$. Let there exists a sequence $\{k_n\} \subset Z_+$ such that as $n \to \infty$, $k_n \to \infty$ and

(i) $\sum_{i=1}^{k_n} X_{n,i} \xrightarrow{d} X$, a *r.v.* with PGF $R(s)$,

(ii) $\frac{N_n}{k_n} \xrightarrow{d} N$, a *r.v.* with LT φ.

Then as $n \to \infty$, $S_{Nn} = \sum_{i=1}^{N_n} X_{n,i} \xrightarrow{d} Z$, and the PGF of Z is

$$\varphi(-\log R(s)) = \varphi\{\lambda(1 - G(s))\}$$

where $\lambda > 0$ and $G(s)$ is a PGF with $G(0) = 0$.

Note 1.1 The limit X in (i) is ID and hence $R(s) = \exp(-\lambda(1-G(s)))$, $\lambda > 0$ and $G(s)$ is a PGF with $G(0) = 0$, (Theorem 3.2 in Steutel and van Harn, 2004, p.30).

Definition 1.2 (Satheesh *et al.*, 2010) Let φ be a LT. A PGF $P(s)$ is φ-ID if there exists a sequence of positive numbers $\theta_n \downarrow 0$ as $n \to \infty$ and a sequence of PGF's $Q_n(s)$ such that

$$P(s) = \lim_{n\to\infty} \varphi(\frac{1 - Q_n(s)}{\theta_n}).$$

The purpose of this note is to discuss two more results on random sums of $\mathbf{Z}_+$-valued *r.v.*s in Section 2 and then demonstrate the use of the two notions in finding solutions to a $\mathbf{Z}_+$-valued AR(1) (INAR(1)) model in Section 3.

2 More on random sums of $\mathbf{Z}_+$-valued random variables

The results in this section give some necessary and sufficient conditions for the existence of φ-ID laws.

Theorem 2.1 Let $\{Q_\theta(s), \theta \in \Theta\}$ be a family of PGFs. Then

$$\lim_{\theta\downarrow 0} \varphi((1 - Q_\theta(s))/\theta)$$

exists and is φ-ID *iff* there exists an ID PGF $R(s)$ such that,

$$\lim_{\theta\downarrow 0}\frac{1-Q_\theta(s)}{\theta}=-\log R(s). \tag{2.1}$$

Proof. Suppose (2.1) holds. Then since φ is continuous

$$\lim_{\theta\downarrow 0}\varphi(\frac{1-Q_\theta(s)}{\theta})=\varphi(\lim_{\theta\downarrow 0}\frac{1-Q_\theta(s)}{\theta})=\varphi(-\log R(s))$$

exists. The result follows from definition 2.1 if we prove that the last term in the above equation is a PGF. Recall that since $R(s)$ is an ID PGF, for each $n\geq 1$, $[R(s)]^{1/n}=E(s^{Y_n})$ is the PGF of some *r.v.* Y_n. Consider, for each fixed $n\geq 1$, *i.i.d.* *r.v.*s $X_{n,1}$, $X_{n,2}$, ..., $X_{n,n}$ with PGF $[R(s)]^{1/n}$. Then for each n, $\sum_{j=1}^{n}X_{n,j}$ has PGF $R(s)$ and hence converges to some *r.v.* Y in distribution. Let N_n be a positive integer-valued *r.v.* independent of $\{X_{n,j}\}$ for each n, such that the LT of $\frac{N_n}{n}$ is φ. Then by Theorem 1.1, $\sum_{j=1}^{N_n}X_{n,j}\xrightarrow{d}Z$, where Z is a *r.v.* with PGF $\varphi(-\log R(s))$. Conversely we have

$$\lim_{\theta\downarrow 0}\varphi(\frac{1-Q_\theta(s)}{\theta})$$

is φ-ID. But

$$\lim_{\theta\downarrow 0}\varphi(\frac{1-Q_\theta(s)}{\theta})=\varphi(\lim_{\theta\downarrow 0}\frac{1-Q_\theta(s)}{\theta})=\varphi(-\log\lim_{\theta\downarrow 0}\exp(-\frac{1-Q_\theta(s)}{\theta})).$$

Here again $\lim_{\theta\downarrow 0}\exp(-\frac{1-Q_\theta(s)}{\theta}))$ is the weak limit of compound Poisson LTs and by assumption it follows that this limit (which is ID) exists. That is, there exists a PGF $R(s)$, that is ID (hence no zeroes) such that (2.1) holds.

Now we consider the $\mathbf{Z}_+$-valued analogue of Theorem 4.9 of Satheesh (2004) which in turn is the φ-ID analogue of Theorem 4.6.5 of Gnedenko and Korolev (1996) and we closely follow the proof therein. Let $\{X_{\theta,i}\}$ be $\mathbf{Z}_+$-valued, *i.i.d.* *r.v.*s for every $\theta\in\Theta$ with PGF Q_θ and N_θ be a positive integer-valued *r.v.* independent of $\{X_{\theta,i}\}$ for all $\theta\in\Theta$ with PGF $P_\theta(s)=\varphi(\frac{1-s}{\theta})$, where φ is a LT. Let $\left[\frac{1}{\theta}\right]$ denote the integer part of $\frac{1}{\theta}$. Then

Theorem 2.2 Let $R(s)$ be a PGF and $P(s)=\varphi(-\log R(s))$ be another PGF that is φ-ID. Then,

$$\lim_{\theta\downarrow 0}P_\theta(Q_\theta(s))=\lim_{\theta\downarrow 0}\varphi(\frac{1-Q_\theta(s)}{\theta})=P(s) \tag{2.2}$$

iff $R(s)$ is ID and

$$\lim_{\theta\downarrow 0}Q_\theta{}^{\left[\frac{1}{\theta}\right]}(s)=R(s). \tag{2.3}$$

Proof. The sufficiency of the condition (2.3) follows from the (transfer) Theorem 1.1 for $\mathbf{Z}_+$-valued *r.v.*s by invoking the relation $\theta\left[\frac{1}{\theta}\right] \to 1$ as $\theta \downarrow 0$ and since $\theta N_\theta \xrightarrow{d} U$. Conversely (2.2) implies

$$\lim_{\theta\downarrow 0} \varphi(\frac{1-Q_\theta(s)}{\theta}) = \varphi(-\log R(s))$$

and by (2.1) we have

$$\lim_{\theta\downarrow 0} Q_\theta(s) = 1. \tag{2.4}$$

Since $\varphi(\frac{1-Q_\theta(s)}{\theta})$ is a PGF that is φ-ID for every $\theta \in \Theta$, by remark 2.1 in Satheesh *et al.* (2010) $\lim_{\theta\downarrow 0}\varphi(\frac{1-Q_\theta(s)}{\theta})$ is also φ-ID. Hence there exists a PGF $H(s)$ that is ID such that

$$\lim_{\theta\downarrow 0} \frac{1-Q_\theta(s)}{\theta} = -\log H(s), \forall s \in (0,1). \tag{2.5}$$

On the other hand, for $|\kappa| \leq 1$ we have

$$\log Q_\theta^{\left[\frac{1}{\theta}\right]}(s) = \left[\frac{1}{\theta}\right]\log(1-(1-Q_\theta(s))) = \left[\frac{1}{\theta}\right](Q_\theta(s)-1)+\kappa\left[\frac{1}{\theta}\right]|Q_\theta(s)-1|^2. \tag{2.6}$$

Hence by (2.4) and (2.5) we get from (2.6),

$$\lim_{\theta\downarrow 0} Q_\theta{}^{\left[\frac{1}{\theta}\right]}(s) = H(s), \forall s \in (0,1). \tag{2.7}$$

Again applying the transfer theorem for $\mathbf{Z}_+$-valued *r.v.*s it follows that

$$\lim_{\theta\downarrow 0} \varphi(\frac{1-Q_\theta(s)}{\theta}) = \varphi(-\log H(s)).$$

Hence by (2.2), $H(s) \equiv R(s)$. That is, by (2.7), (2.3) is true with $R(s)$ being ID, completing the proof.

We can now see that by virtue of Lemma 1.2 and Theorem 1.1, limit distributions of random sums of *i.i.d.* $\mathbf{Z}_+$-valued *r.v.*s is φ-ID when the PGF of the indexing *r.v.* is in $\wp_\varphi$ described in Lemma 1.1. The parameter k here stands for the gaps in the support of the distribution specified by the PGF $P_\theta(s)$. That is, if there is a probability mass at $x = n$, then the next integer that carries a probability mass is $n+k$. Lemma 1.2 shows that this k appears in the limit distribution of θN_θ as well. Such distributions enable us to model situations where we need combine k *i.i.d.* observations. In the next section we will see such situations in the $\mathbf{Z}_+$-valued setup.

3 An INAR(1) model and its generalization

Consider the $\mathbf{Z}_+$-valued analogue of the AR(1) model in Lawrence and Lewis (1981) described by the $\mathbf{Z}_+$-valued *r.v.*s $\{X_n, n \in Z\}$ with innovations (*i.i.d. r.v.*s) $\{\epsilon_n\}$ such that for each n, ϵ_n is independent of X_{n-1}, as below.

$$X_n = \begin{cases} \epsilon_n, & \text{with probability } p, \\ X_{n-1} + \epsilon_n, & \text{with probability } (1-p). \end{cases} \tag{3.1}$$

In terms of PGFs, assuming stationarity this is equivalent to

$$\begin{aligned} P(s) &= pP_\epsilon(s) + (1-p)P(s)P_\epsilon(s) \\ &= \frac{pP_\epsilon(s)}{1-(1-p)P_\epsilon(s)}, \quad \text{which proves} \end{aligned} \tag{3.2}$$

Theorem 3.1*a* If $\{X_n\}$ describes the model (3.1) that is stationary for some $p \in (0,1)$ *then* X_n is compound geometric(p) and the innovations are the components in the compound.

Conversely, we have

Theorem 3.1*b* If $P(s)$ is the PGF of a compound geometric(p) distribution with components having PGF $P_\epsilon(s)$, then there exists a stationary AR(1) process $\{X_n, n \in Z\}$ with structure given in (3.1) such that $P(s)$ is the PGF of X_n and $P_\epsilon(s)$ that of ϵ_n.

Proof. We proceed as in the proof of Theorem 2.2 in Bouzar and Satheesh (2008). We have $P(s)$ and $P_\epsilon(s)$ as defined by (3.2). By the Kolmogorov extension theorem (Breiman, 1968), there exists a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ on which one can define a random variable X_0 with PGF $P(s)$ and a sequence of *i.i.d. r.v.*s $\{\epsilon_n\}$ with a common PGF $P_\epsilon(s)$, with the further property that X_0 and $\{\epsilon_n\}$ are independent. We then construct a single-ended AR(1) process $\{X_n, n \geq 0\}$ via equation (3.1). This implies that for every $n \geq 1$, the PGF $P_n(s)$ of X_n satisfies

$$P_n(s) = pP_\epsilon(s) + (1-p)P_{n-1}(s)P_\epsilon(s) \tag{3.3}$$

with $P_0(s) = P(s)$. It follows by (3.2) and (3.3) that $P_n(s) = P(s)$ for every $n \geq 1$. Therefore, the X_ns are identically distributed. Since $\{X_n, n \geq 0\}$ is a Markov chain, its stationarity ensues. The existence of the doubly infinite extension $\{X_n, n \in Z\}$ follows from Proposition 6.5, p. 105, in Breiman (1968).

Suppose we demand (3.1) to be satisfied for each $p \in (0,1)$, then (3.2) is true for each $p \in (0,1)$ with $P_\epsilon(s)$ replaced by $P_{\epsilon,p}(s)$. Hence by Aly and Bouzar (2000) X_n must be $\mathbf{Z}_+$-valued geometrically ID, that is

Theorem 3.2*a* If $\{X_n\}$ describes the model (3.1) that is stationary for each $p \in (0,1)$ *then* X_n is $\mathbf{Z}_+$-valued geometrically ID.

Now proceeding as in the proof of Theorem 3.1b, we have

Theorem 3.2*b* If $P(s)$ is the PGF of a geometrically ID distribution with components having PGF $P_{\epsilon,p}(s)$ for each $p \in (0,1)$, then there exists a stationary AR(1) process $\{X_n, n \in Z\}$ with structure given in (3.1) such that $P(s)$ is the PGF of X_n and $P_{\epsilon,p}(s)$ that of ϵ_n.

Next we demonstrate the advantage of φ-ID approach over the N-ID approach in the following INAR(1) model which is a generalization of (3.1). Here we also need Harris(a,k) distributions described by its PGF

$$P(s) = \frac{s}{\{a-(a-1)s^k\}^{1/k}}$$

for $a > 1$ and $k \geq 1$ integer (see Sandhya *et al.* (2008) for more on this distribution). It has been shown in Satheesh *et al.* (2008) that the PGF of Harris(a,k) distribution can be derived from the LT of gamma$(1/k)$ distribution as the standard solution to the Poincare equation and also by using Lemma 1.1. In this generalization, the INAR(1) sequence $\{X_n\}$ is composed of k independent INAR(1) sequences $\{Y_{n,i}\}, i = 1,2,\ldots\ k$ and where for each n, $\{Y_{n,i}\}$ are independent. That is, for each n, $X_n = \sum_{i=1}^k Y_{n,i}$ and $\epsilon_n = \sum_{i=1}^k \epsilon_{n,i}$ where $\{Y_{n,i}\}$ is an $i.i.d$ sequence and similarly $\{\epsilon_{n,i}\}$ is also an $i.i.d$ sequence, k being a fixed positive integer. Further, it is also assumed that for each n, $\epsilon_{n,i}$ is independent of $Y_{n-1,i}$ for all $i = 1,2,\ldots\ k$. Situations where such a model can be useful have been discussed in Satheesh *et al.* (2006, 2008).

$$\sum_{i=1}^{k} Y_{n,i} = \begin{cases} \sum_{i=1}^k \epsilon_{n,i}, & \text{with probability } p, \\ \sum_{i=1}^k Y_{n-1,i} + \sum_{i=1}^k \epsilon_{n,i}, & \text{with probability } (1-p). \end{cases} \tag{3.4}$$

In terms of PGFs, assuming stationarity (3.4) is equivalent to

$$\begin{aligned} P_Y^k(s) &= pP_\epsilon^k(s) + (1-p)P_Y^k(s)P_\epsilon^k(s) \\ &= \frac{pP_\epsilon^k(s)}{1-(1-p)P_\epsilon^k(s)}. \\ \text{That is,} P_Y(s) &= \frac{P_\epsilon(s)}{\{a-(a-1)P_\epsilon^k(s)\}^{1/k}},\ a = \frac{1}{p}. \end{aligned} \tag{3.5}$$

Hence we have

Theorem 3.3*a* If $\{Y_{n,i}\}$ describes the model (3.4) that is stationary for some $p \in (0,1)$ *then* $Y_{n,i}$ is a Harris(a,k)-sum and the innovations are the components in the Harris(a,k)-sum, $a = \frac{1}{p}$.

Further, proceeding as in the proof of Theorem 3.1*b*, we have

Theorem 3.3*b* If $P(s)$ is the PGF of a Harris(a,k)-sum distribution with components having PGF $P_\epsilon(s)$, then there exists a stationary AR(1) process $\{Y_{n,i}, n \in Z\}$ with structure given in (3.4) such that $P(s)$ is the PGF of $Y_{n,i}$ and $P_\epsilon(s)$ that of $\epsilon_{n,i}$.

Again suppose we demand (3.4) to be satisfied for each $p \in (0,1)$. Then (3.5) is true for each $p \in (0,1)$ with $P_\epsilon(s)$ replaced by $P_{\epsilon,p}(s)$. Hence $Y_{n,i}$ must be $\mathbf{Z}_+$-valued Harris(a,k)-ID. Thus we have

Theorem 3.4*a* If $\{Y_{n,i}\}$ describes the model (3.4) that is stationary for each $p \in (0,1)$ *then* $Y_{n,i}$ is Harris(a,k)-ID with PGF (see also Theorem 2.6 in Satheesh *et al.* (2010))

$$P(s) = \frac{1}{\{1 - \log Q(s)\}^{1/k}}.$$

Conversely, proceeding as in the proof of Theorem 3.1*b* we have

Theorem 3.4*b* If $P(s)$ is the PGF of a Harris(a,k)-ID distribution with components having PGF $P_{\epsilon,p}(s)$ for each $p \in (0,1)$, then there exists a stationary AR(1) process $\{Y_{n,i}, n \in Z\}$ with structure given in (3.4) such that $P(s)$ is the PGF of $Y_{n,i}$ and $P_{\epsilon,p}(s)$ that of $\epsilon_{n,i}$.

A problem in the above development is how to identify the PGF $Q(s)$. However, consider the φ-ID approach and let us demand only that (3.4) is satisfied for some $p \downarrow 0$ through some $\{p_n\}$. Notice that here the requirement is apparently weaker than the one in Theorem 3.4. That is, we are considering

$$P(s) = \lim_{p \downarrow 0} \frac{pP^k_{\epsilon,p}(s)}{\{1 - (1-p)P^k_{\epsilon,p}(s)\}}.$$

Hence by Lemma 1.2 above and the (transfer) Theorem 1.1 the RHS here converges weakly to the PGF $\dfrac{1}{\{1 - k\log Q(s)\}^{1/k}}$ where $Q(s)$ is the weak limit (assuming its existence) of $P_{\epsilon,p}(s)$'s as $p \downarrow 0$ through $\{p_n\}$. As $P_{\epsilon,p}(s)$'s are *i.i.d* for each p, the weak limit $Q(s)$ must be ID and we have

Theorem 3.5 If $\{Y_{n,i}\}$ describes the model (3.4) that is stationary for a null sequence $\{p_n\}$ *then* $Y_{n,i}$ is gamma$(\frac{1}{k})$-ID with PGF $\dfrac{1}{\{1 - k\log Q(s)\}^{1/k}}$ where $Q(s)$ is the weak limit of $P_{\epsilon,p}(s)$'s as $p \downarrow 0$ through $\{p_n\}$ (*c.f* Theorem 2.7 in Satheesh *et al.* (2010)).

Remarks. Whether we approach the model (3.4) by N-ID or φ-ID laws we need consider weak limits of the distributions concerned. The advantage of using the approach of φ-ID laws is that the requirement is apparently weaker and the limit law is a function of $Q(s)$ which is the weak limit of the distribution of the innovations in the model. It may also be noticed that gamma($\frac{1}{k}$)-ID laws identified in Theorem 3.5 also satisfy the requirement of Harris(a, k)-sum for each $p \in (0, 1)$, $p = \frac{1}{a}$.

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References

[1] Aly, E E and Bouzar, N (2000), On geometric infinite divisibility and stability, *Ann. Inst. Statist. Math.* **52**, 790–799.

[2] Bakouch, H S and Ristic, M M (2009), Zero truncated Poisson integer-valued AR(1) model, *Metrika*, Published online at; DOI 10.1007/s00184-009-0252-5.

[3] Bouzar, N (2004), Discrete semi-stable distributions, *Ann. Inst. Statist. Math.* **56**, 497–510.

[4] Bouzar, N (2008), Semi-selfdecomposable distributions on $\mathbf{Z}_+$, *Ann. Inst. Statist. Math.*, **60**, 901–917.

[5] Bouzar, N and Jayakumar, K (2006), Time series with discrete semi-stable marginals, *Statistical Papers*, (online first), doi: 10.1007/s00362-006-0040-5.

[6] Bouzar, N and Satheesh, S (2008), Comments on α-decomposability, *Metron*, **LXVI**, 243–252.

[7] Breiman, L (1968), Probability, Addison-Wesley.

[8] Feller, W (1971), *An Introduction to Probability Theory and Its applications*, Volume 2, 2nd Edition, Wiley, New York.

[9] Gnedenko, B V and Korolev, V Yu. (1996), *Random Summation-Limit Theorems and Applications*, CRC Press, Boca Raton.

[10] Lawrence, A J and Lewis, P A W (1981), A new autoregressive time series model in exponential variables, (NEAR(1)), *Adv. Appl. Probab.* **13**, 826–845.

[11] McKenzie, E (2003), Discrete variate time series, In: Shanbhag D N(ed.) *et al.*, in *Stochastic Processes: Modelling and Simulation*, Amsterdam: North-Holland. Handb. Stat., **21**, 573606.

[12] Ristic, M N; Bakouch, H S and Nastic, A S (2009), A new geometric first-order integer-valued autoregressive (NGINAR(1)) process, *J. Statist. Plan. Infer.*, **139**, 2218–2226.

[13] Sandhya, E; Sherly, S and Raju, N (2008), Harris family of discrete distributions, In: *Some Recent Innovations in Statistics*, A special volume in honor of Professor T S K Moothathu, Published by the Department of Statistics, University of Kerala, India, 57–72.

[14] Satheesh, S (2004). Another look at random infinite divisibility, *Statist. Meth.*, **6**, 123–144.

[15] Satheesh, S and Nair, N U (2002), Some classes of distributions on the non-negative lattice, *J. Ind. Statist. Assoc.* **40**, 41–58.

[16] Satheesh, S and Sandhya, E (2005), Semi-selfdecomposable laws and related processes, *J. Ind. Statist. Assoc.*, **43**, 157-166, *see also corrections* by the authors in *J. Ind. Statist. Assoc.*, (2007) **45**, 123–127.

[17] Satheesh, S; Sandhya, E and Sherly, S (2006), A Generalization of Stationary AR(1) Schemes, *Statist. Meth.*, **8**, 213–225.

[18] Satheesh, S; Sandhya, E and Rajasekharan, K E (2008), A generalization and extension of an autoregressive model, *Statist. Probab. Lett.*, **78**, 1369–1374.

[19] Satheesh, S; Sandhya, E and Lovely, A T (2010), Limit distributions of random sums of $\mathbf{Z}_+$-valued random variables, *Commu. Statist. - Theor. Meth.*, **39**, 1979–1984.

[20] Steutel, F W and van Harn, K (2004), *Infinite Divisibility of Probability Distributions on the Real Line*, Mercel Dekker, Inc. New York-Basel.