

## Marshall-Olkin Extended Uniform Distribution

K.K. Jose<sup>a</sup>, E. Krishna<sup>b</sup>

<sup>a</sup>Department of Statistics, St. Thomas College, Pala, Kerala, India

<sup>b</sup>Department of Statistics, St. Joseph's College, Alappuzha, Kerala, India

**Abstract.** The Marshall-Olkin Extended Uniform (MOEU) distribution is introduced. MOEU distribution is expressed as a mixture distribution with exponential distribution as mixing density. Limiting distributions of sample maxima and sample minima are derived. Record value properties of the new distribution are investigated. We develop minification processes and the sample path with MOEU stationary marginal distribution. Estimates of unknown parameters are obtained and the results are validated using simulation studies.

### 1. Introduction

Marshall and Olkin [6] introduced a new family of distributions in an attempt to add a parameter to a family of distributions. Let  $\bar{F}(x) = P(X > x)$  be the survival function of a random variable  $X$  and  $\alpha > 0$  be a parameter. Then

$$\bar{G}(x, \alpha) = \frac{\alpha \bar{F}(x)}{1 - (1 - \alpha)\bar{F}(x)}, \quad -\infty < x < \infty, \quad \alpha > 0, \quad (1)$$

is a proper survival function.  $\bar{G}(x, \alpha)$  is called Marshall-Olkin family of distributions.

The probability density function (p.d.f.) corresponding to (1) is given by

$$g(x, \alpha) = \frac{\alpha f(x)}{[1 - (1 - \alpha)\bar{F}(x)]^2}, \quad -\infty < x < \infty, \quad \alpha > 0,$$

where  $f(x)$  is the p.d.f. corresponding to  $\bar{F}(x)$ . The hazard (failure) rate function is given by

$$h(x, \alpha) = \frac{r(x)}{1 - (1 - \alpha)\bar{F}(x)}, \quad \text{where } r(x) = \frac{f(x)}{\bar{F}(x)}.$$

Similar models were considered, for example by Alice and Jose [1, 2]. Ristić and Popović [7] discussed a new uniform AR(1) time series model.

In this paper, we introduce the Marshall-Olkin extended uniform (MOEU) distribution in Section 2 and various properties are studied. In Section 3, we derive the p.d.f. of  $n^{\text{th}}$  record value statistics and its

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2010 *Mathematics Subject Classification.* Primary 62M10

*Keywords.* Autoregressive processes, Marshall-Olkin family, Minification processes, Mixture distribution, Record values, Survival function, Time series modeling

Received: 31 March 2011; Revised: 03 October 2011; Accepted: 23 October 2011

*Email addresses:* [kkjstc@gmail.com](mailto:kkjstc@gmail.com) (K.K. Jose), [ekrishna48@gmail.com](mailto:ekrishna48@gmail.com) (E. Krishna)

shape properties are discussed. A recurrence relation for the moments of record values is also developed. The limiting distribution of sample extremes are derived. In Section 4, we introduce first order stationary autoregressive processes with uniformly distributed marginals and develop the sample paths for various values of the parameters  $p$  and  $\theta$ . The estimate of the parameters are obtained and the results are verified using simulation studies. The extension to  $k^{th}$  order is also discussed.

## 2. Marshall-Olkin Extended Uniform (MOEU) Distribution and Properties

Let  $X$  follows  $U(0, \theta)$  distribution, where  $\theta > 0$ . Then  $\bar{F}(x) = 1 - (x/\theta)$ . Substituting in (1) we get a new distribution denoted by MOEU  $(\alpha, \theta)$  with survival function

$$\bar{G}(x, \alpha, \theta) = \frac{\alpha(\theta - x)}{\alpha\theta + (1 - \alpha)x}, \quad 0 < x < \theta, \quad \alpha > 0.$$

The corresponding p.d.f. is obtained as

$$g(x, \alpha, \theta) = \frac{\alpha\theta}{[\alpha\theta + (1 - \alpha)x]^2}, \quad 0 < x < \theta, \quad \alpha > 0.$$

The graphs of p.d.f. and distribution function (d.f.) are drawn in Figures 1 and 2. The shape of the p.d.f.  $g(x, \alpha, \theta)$  (1) depends on parameter  $\alpha$ . Namely, if  $\alpha \in (0, 1)$ , then the p.d.f. is a decreasing function on  $(0, \theta)$  with  $g(0, \alpha, \theta) = 1/(\alpha\theta)$  and  $g(\theta, \alpha, \theta) = \alpha/\theta$ . Otherwise, if  $\alpha > 1$ , then the p.d.f. is an increasing function on  $(0, \theta)$  with  $g(0, \alpha, \theta) = 1/(\alpha\theta)$  and  $g(\theta, \alpha, \theta) = \alpha/\theta$ .

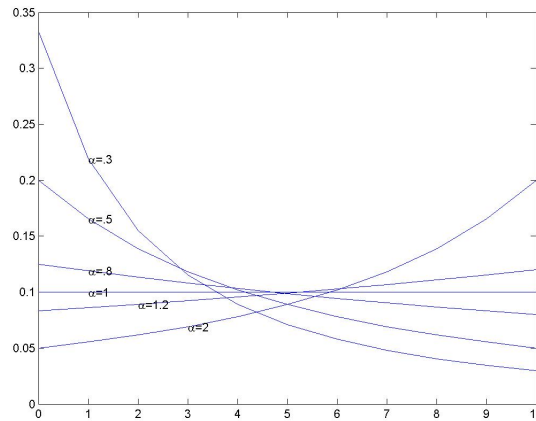


Figure 1: Graph of  $g(x)$  for  $\theta = 10$  and for various values of  $\alpha$ .

The hazard rate function of a random variable  $X$  with MOEU $(\alpha, \theta)$  distribution is

$$h(x, \alpha, \theta) = \frac{\theta}{[\alpha\theta + (1 - \alpha)x](\theta - x)}.$$

For  $\alpha \leq 0.3$  the hazard rate is initially decreasing and there exists an interval where it changes to be IFR. For  $\alpha > 0.3$  the hazard function is evidently IFR. The graph of hazard rate function is drawn in Figure 3.

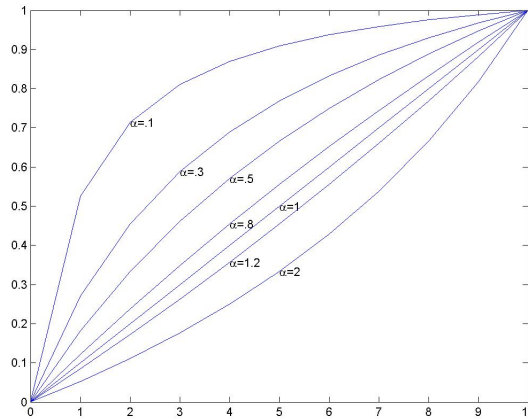


Figure 2: Graph of  $G(x)$  for  $\theta = 10$  and for various values of  $\alpha$ .

2.1. Mean, variance, quantiles

In this section we consider a random variable  $X$  with MOEU( $\alpha, \theta$ ) distribution. Let us first consider the higher-order moments. We have

$$E(X^r) = \int_0^\theta x^r \frac{\alpha\theta}{[\alpha\theta + (1-\alpha)x]^2} dx = \frac{\theta^r}{\alpha(r+1)} \left( -r \cdot {}_2F_1 \left( 1, r+1; r+2; \frac{\alpha-1}{\alpha} \right) + \alpha(r+1) \right).$$

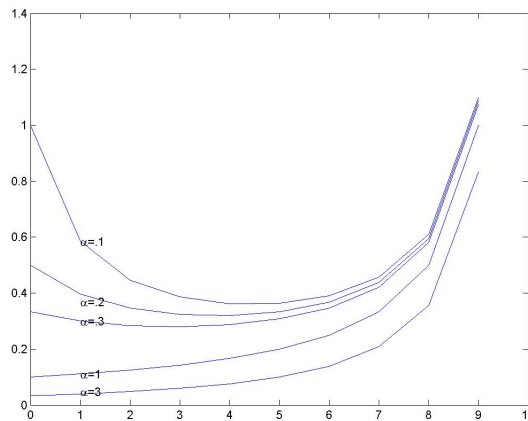


Figure 3: Graph of  $h(x)$  for  $\theta = 10$  and for various values of  $\alpha$ .

Specially, the mean and the variance of a random variable  $X$  with MOEU( $\alpha, \theta$ ) distribution are, respectively,

$$\begin{aligned} \mu'_1 &= \frac{\alpha\theta}{(1-\alpha)^2}(\alpha - \log \alpha - 1), \\ \mu_2 &= \frac{\alpha\theta^2}{(1-\alpha)^4}[(1-\alpha)^2 - \alpha(\log \alpha)^2]. \end{aligned}$$

The coefficient of variation is

$$C_V = \frac{\sqrt{(1-\alpha)^2 - \alpha(\log \alpha)^2}}{\sqrt{\alpha}(\alpha - \log \alpha - 1)}, \alpha > 0,$$

and it depends only on parameter  $\alpha$ .

The  $q^{th}$  quantile of a random variable  $X$  with MOEU( $\alpha, \theta$ ) distribution is given by

$$x_q = G^{-1}(q) = \frac{q\alpha\theta}{1 - q(1 - \alpha)}, 0 \leq q \leq 1,$$

where  $G^{-1}(\cdot)$  is the inverse distribution function. In particular, the median of a random variable with the MOEU( $\alpha, \theta$ ) distribution is given by  $\text{median}(X) = \alpha\theta/(1 + \alpha)$ .

### 2.2. Mixtures

Let  $\bar{F}(x/\lambda)$ ,  $-\infty < x < \infty$ ,  $-\infty < \lambda < \infty$ , be the conditional survival function of a continuous random variable  $\Lambda$ . Let  $\Lambda$  follows a distribution with probability density function  $m(\lambda)$ . A distribution with survival function

$$\bar{F}(x) = \int_{-\infty}^{\infty} \bar{F}(x/\lambda)m(\lambda)d\lambda, -\infty < x < \infty,$$

is called a mixture distribution with mixing density  $m(\lambda)$ . The following result shows that MOEU distribution can be expressed as a mixture.

**Theorem 2.1.** *Let  $X$  be a continuous random variable with conditional survival function*

$$\bar{F}(x/\lambda) = \frac{\theta - x}{\theta} e^{-(1-\alpha)\lambda x}, 0 < x < \theta,$$

*with probability density function  $m(\lambda) = \alpha\theta e^{-\alpha\theta\lambda}$ ,  $\lambda > 0$ . Then the random variable  $X$  has the MOEU( $\alpha, \theta$ ) distribution.*

*Proof.* Under these assumptions, we obtain that the survival function of the random variable  $X$  is

$$\bar{F}(x) = \alpha(\theta - x) \int_0^{\infty} e^{-[\alpha\theta + (1-\alpha)x]\lambda} d\lambda = \frac{\alpha(\theta - x)}{\alpha\theta + (1 - \alpha)x}.$$

Thus the random variable  $X$  has MOEU( $\alpha, \theta$ ) distribution.  $\square$

**Remark 2.2.** *We can obtain new parameter families of distributions in terms of existing ones with the help of mixture distribution.*

## 3. Distributions of order statistics and record values

### 3.1. Limiting distributions of sample extremes

Let  $X_1, X_2, \dots, X_n$  be a random sample of size  $n$  from MOEU( $\alpha, \theta$ ) distribution. Then the sample minima and sample maxima are respectively  $X_{1:n} = \min(X_1, X_2, \dots, X_n)$  and  $X_{n:n} = \max(X_1, X_2, \dots, X_n)$ .

**Theorem 3.1.** *Let  $X_1, X_2, \dots, X_n$  be a random sample of size  $n$  from MOEU( $\alpha, \theta$ ) distribution. Then*

- (i)  $\lim_{n \rightarrow \infty} P(X_{1:n} \leq b_n^* t) = 1 - e^{-t}$ ,  $t > 0$ , where  $b_n^* = \alpha\theta/(n - 1 + \alpha)$ .
- (ii)  $\lim_{n \rightarrow \infty} P(X_{n:n} \leq a_n + b_n t) = e^{-t}$ ,  $t > 0$ , where  $a_n = \theta$  and  $b_n = \alpha\theta(n - 1)/(1 + \alpha(n - 1))$ .

*Proof.* (i) We use the asymptotic result for  $X_{1:n}$  (Arnold, Balakrishnan and Nagaraja [3, pp. 210–214]) by which

$$\lim_{n \rightarrow \infty} P(X_{1:n} \leq a_n^* + b_n^* t) = 1 - e^{-t^c}, \quad t > 0, \quad c > 0,$$

where  $a_n^* = F^{-1}(0)$  and  $b_n = F^{-1}(1/n) - F^{-1}(0)$  if and only if  $F^{-1}(0)$  is finite and for all  $t > 0$  and  $c > 0$

$$\lim_{\epsilon \rightarrow 0^+} \frac{F(F^{-1}(0) + \epsilon t)}{F(F^{-1}(0) + \epsilon)} = t^c.$$

For a random variable  $X$  with the MOEU( $\alpha, \theta$ ) distribution we have that  $G^{-1}(0) = 0$  is finite and

$$\lim_{\epsilon \rightarrow 0^+} \frac{G(\epsilon t)}{G(\epsilon)} = t \lim_{\epsilon \rightarrow 0^+} \frac{\alpha\theta + (1 - \alpha)\epsilon}{\alpha\theta + (1 - \alpha)\epsilon t} = t.$$

Thus we obtain that  $c = 1$ ,  $a_n^* = 0$  and  $b_n^* = \alpha\theta/(n - 1 + \alpha)$ .

(ii) For the maximal order statistics  $X_{n:n}$  we have

$$\lim_{n \rightarrow \infty} P(X_{n:n} \leq a_n + b_n t) = e^{-(t)^k}, \quad t > 0, \quad k > 0,$$

which is Weibull type where  $a_n = F^{-1}(1)$ ,  $b_n = F^{-1}(1) - F^{-1}(1 - 1/n)$  if and only if  $F^{-1}(1)$  is finite and there exists a constant  $k > 0$  such that

$$\lim_{\epsilon \rightarrow 0^+} \frac{\bar{F}(F^{-1}(1) - \epsilon t)}{\bar{F}(F^{-1}(1) - \epsilon)} = t^k.$$

For a random variable  $X$  with the MOEU( $\alpha, \theta$ ) distribution  $G^{-1}(1) = \theta$  is finite and we have

$$\lim_{\epsilon \rightarrow 0^+} \frac{\bar{G}(G^{-1}(1) - \epsilon t)}{\bar{G}(G^{-1}(1) - \epsilon)} = t \lim_{\epsilon \rightarrow 0^+} \frac{\theta - (1 - \alpha)\epsilon}{\theta - (1 - \alpha)\epsilon t} = t.$$

Thus we obtain that  $k = 1$ ,  $a_n = \theta$  and  $b_n = \alpha\theta(n - 1)/(1 + \alpha(n - 1))$ .  $\square$

**Remark 3.2.** (i) If  $\alpha = 1$ , i.e. for uniform distribution the norming constants  $b_n^*$ ,  $b_n$  are respectively  $b_n^* = \theta/n$  and  $b_n = (n - 1)/(n\theta)$ .

(ii) If the limiting distributions of the random variables  $(X_{1:n} - a_n^*)/b_n^*$  and  $(X_{n:n} - a_n)/b_n$  are denoted by  $G^*(t)$  and  $G(t)$ , then from (Arnold, Balakrishnan and Nagaraja [3, pp. 210–214]) for any finite  $i > 1$ , the limiting distributions of the random variables  $(X_{i:n} - a_n^*)/b_n^*$  and  $(X_{n-i+1} - a_n)/b_n$  are, respectively, given by

$$\begin{aligned} \lim_{n \rightarrow \infty} P(X_{i:n} \leq a_n^* + b_n^* t) &= 1 - \sum_{j=0}^{i-1} (1 - G^*(t))^j \frac{[-\log(1 - G^*(t))]^j}{j!} \\ \lim_{n \rightarrow \infty} P(X_{n-i+1:n} \leq a_n + b_n t) &= \sum_{j=0}^{i-1} G(t)^j \frac{(-\log G(t))^j}{j!}. \end{aligned}$$

From Theorem 3.1 it follows that for any finite  $i > 1$  the limiting distributions of the  $i^{\text{th}}$  and  $(n - i + 1)^{\text{th}}$  order statistics from the MOEU distribution respectively are given by

$$\begin{aligned} \lim_{n \rightarrow \infty} P\left(X_{1:n} \leq \frac{\alpha\theta t}{n - 1 + \alpha}\right) &= 1 - \sum_{j=0}^{i-1} e^{-t} \frac{t^j}{j!} = 1 - P(Z < i) \\ \lim_{n \rightarrow \infty} P\left(X_{n-i+1:n} \leq \theta + \frac{\alpha\theta(n - 1)t}{1 + \alpha(n - 1)}\right) &= \sum_{j=0}^{i-1} e^{-t} \frac{t^j}{j!} = P(Z < i) \end{aligned}$$

where  $Z$  follows the Poisson distribution with mean  $t$ .

3.2. Record values

Record values and associated statistics are of greater importance in many real life situations involving data relating to sports, weather, economics, life testing etc. Balakrishnan and Ahsanullah [5] and Arnold, Balakrishnan and Nagaraja [4] provide excellent discussions on the theory of record values. Let  $X_i, i \geq 1$  be a sequence of i.i.d. random variables having an absolutely continuous c.d.f.  $F(x)$  and p.d.f.  $f(x)$ . An observation  $X_j$  will be called an upper record value if its value exceeds that of all previous observations. Thus  $X_j$  is an upper record if  $X_j \geq X_i$  for every  $i < j$ . The p.d.f. of  $n^{th}$  record value say  $R_n$  is given by

$$f_{R_n}(x) = \frac{f(x)[- \log(1 - F(x))]^n}{n!}.$$

If  $g_{R_n}(x)$  denote the density function of  $n^{th}$  record value from MOEU( $\alpha, \theta$ ) we have

$$g_{R_n}(x) = \frac{\alpha\theta}{n![\alpha\theta + (1 - \alpha)x]^2} \left( - \log \frac{\alpha(\theta - x)}{\alpha\theta + (1 - \alpha)x} \right)^n, \quad 0 < x < \theta, \alpha > 0. \tag{2}$$

Let us consider the shapes of the p.d.f.  $g_{R_n}$ . The first derivative of the function  $\log g_{R_n}$  is

$$(\log g_{R_n})' = \frac{H(x)}{(\alpha\theta + (1 - \alpha)x) \log \frac{\alpha\theta + (1 - \alpha)x}{\alpha(\theta - x)}}$$

where

$$H(x) = -2(1 - \alpha)(\theta - x) \log \frac{\alpha\theta + (1 - \alpha)x}{\alpha(\theta - x)} + n\theta.$$

If  $\alpha \in (0, 1)$  and  $H(x_0) < 0$ , where  $x_0$  is the solution of the equation

$$\log \frac{\alpha\theta + (1 - \alpha)x}{\alpha(\theta - x)} = \frac{\theta}{\alpha\theta + (1 - \alpha)x},$$

then the function  $H(x)$  has two roots  $0 < x_1 < x_2 < \theta$ . In this case, the p.d.f.  $g_{R_n}$  is an increasing function on  $(0, x_1) \cup (x_2, \theta)$  and a decreasing function on  $(x_1, x_2)$ . Otherwise, the p.d.f.  $g_{R_n}$  is an increasing function on  $(0, \theta)$ . In both cases,  $g_{R_n}(0) = 0$  and  $g_{R_n}(\theta) = \infty$ .

3.2.1. Recurrence relation for moments of record values

Let us derive now the recurrence relation for the moments of records (see Balakrishnan and Ahsanullah [5]). The recurrence relation can be used to compute all the single moments of record values which is useful for the inference. For the MOEU( $\alpha, \theta$ ) distribution with p.d.f.  $g(x)$  and d.f.  $G(x)$  it is easy to see that

$$(\alpha\theta + (1 - \alpha)x) \left( 1 - \frac{x}{\theta} \right) g(x) = 1 - G(x),$$

which implies that for a record  $R_n$  we obtain

$$(\alpha\theta + (1 - \alpha)x) \left( 1 - \frac{x}{\theta} \right) g_{R_n}(x) = \frac{(1 - G(x))(- \log(1 - G(x)))^n}{n!}. \tag{3}$$

This relation will be used to derive a recurrence relation for the moments of record values.

**Theorem 3.3.** For  $n \geq 2$  and  $r = 0, 1, 2, \dots$ , we have

$$E(R_n^{r+2}) = \frac{\theta}{(1 - \alpha)(r + 1)} [2\alpha - r(1 - 2\alpha)] E(R_n^{r+1}) + \frac{\alpha\theta^2}{1 - \alpha} E(R_n^r) + \frac{\theta}{(1 - \alpha)(r + 1)} E(R_{n-1}^{r+1}).$$

*Proof.* From (3) we obtain that

$$\begin{aligned} \alpha\theta^2 E(R_n^r) + \theta(1 - 2\alpha)E(R_n^{r+1}) - (1 - \alpha)E(R_n^{r+2}) &= \frac{\theta}{n!} \int_0^\theta x^r (1 - G(x)) \{-\log(1 - G(x))\}^n dx \\ &= \frac{\theta}{r + 1} [E(R_n^{r+1}) - E(R_{n-1}^{r+1})]. \end{aligned}$$

Finally, the result follows after some simplifications.  $\square$

#### 4. Applications in Autoregressive Time Series Modeling

Now we discuss some applications of MOEU distribution in autoregressive time series modeling. We construct a first order autoregressive minification process with structure as follows. Consider an AR(1) structure

$$X_n = \begin{cases} \varepsilon_n, & w.p. \quad p \\ \min(X_{n-1}, \varepsilon_n), & w.p. \quad 1 - p, \quad 0 \leq p \leq 1, \quad n \geq 1, \end{cases} \tag{4}$$

where  $\{\varepsilon_n\}$  is a sequence of i.i.d. random variables with uniform distribution in  $(0, \theta)$  and is independent of  $\{X_n\}$ .

**Theorem 4.1.** *Consider the AR(1) structure given by (4) with  $X_0$  distributed as MOEU( $p, \theta$ ) distribution. Then  $\{X_n, n \geq 0\}$  is a stationary Markovian autoregressive model with MOEU( $p, \theta$ ) marginals if  $\{\varepsilon_n\}$  is distributed as uniform in  $(0, \theta)$ .*

*Proof.* From (4) it follows that

$$\bar{F}_{X_n}(x) = p\bar{F}_{\varepsilon_n}(x) + (1 - p)\bar{F}_{X_{n-1}}(x)\bar{F}_{\varepsilon_n}(x). \tag{5}$$

Using the fact that  $X_0$  has MOEU( $p, \theta$ ) distribution and  $\varepsilon_1$  has  $U(0, \theta)$  distribution, we obtain that for  $n = 1$  that

$$\bar{F}_{X_1}(x) = [p + (1 - p)\bar{F}_{X_0}(x)]\bar{F}_{\varepsilon_1}(x) = \frac{p(\theta - x)}{p\theta + (1 - p)x},$$

which means that  $X_1$  has MOEU( $p, \theta$ ) distribution.

Assume that  $X_{n-1} \stackrel{d}{=} \text{MOEU}(p, \theta)$ . Then by induction method we can establish that  $\{X_n\}$  is distributed as MOEU( $p, \theta$ ). Even if  $X_0$  is arbitrary, it is easy to establish that  $\{X_n\}$  is stationary and is asymptotically marginally distributed as MOEU( $p, \theta$ ).

Under stationarity equilibrium, from (5) it follows that

$$\bar{F}_X(x) = \frac{p\bar{F}_\varepsilon(x)}{1 - (1 - p)\bar{F}_\varepsilon(x)},$$

and

$$\bar{F}_\varepsilon(x) = \frac{\bar{F}_X(x)}{p + (1 - p)\bar{F}_X(x)}.$$

If  $\{\varepsilon_n\}$  has  $U(0, \theta)$  distribution, then  $\{X_n\}$  is distributed as MOEU( $p, \theta$ ). The converse is also true.  $\square$

Figures 4, 5 and 6 shows sample paths of MOEU AR(1) process for different values of  $p$  and  $\theta$ .

We consider now the joint survival function of the random variables  $X_n$  and  $X_{n-1}$ . We have that

$$\begin{aligned} \bar{F}(x, y) &= P(X_n > x, X_{n-1} > y) = (p\bar{F}_X(y) + (1 - p)\bar{F}_X(\max(x, y))) \cdot \bar{F}_\varepsilon(x) \\ &= \begin{cases} (p\bar{F}_X(y) + (1 - p)\bar{F}_X(x)) \cdot \bar{F}_\varepsilon(x), & 0 < y < x < \theta, \\ \bar{F}_X(y) \cdot \bar{F}_\varepsilon(x), & 0 < x < y < \theta. \end{cases} \end{aligned}$$

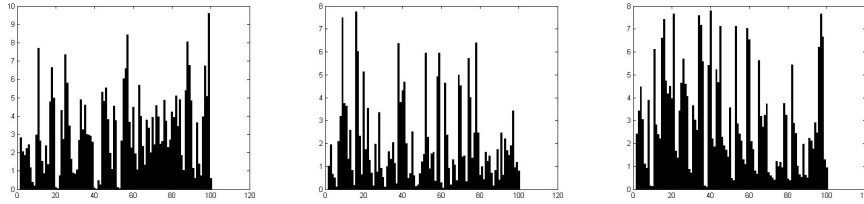


Figure 4: Sample path for  $\theta = 10$  and for various values of  $p = 0.5, 0.2, 0.8$ .

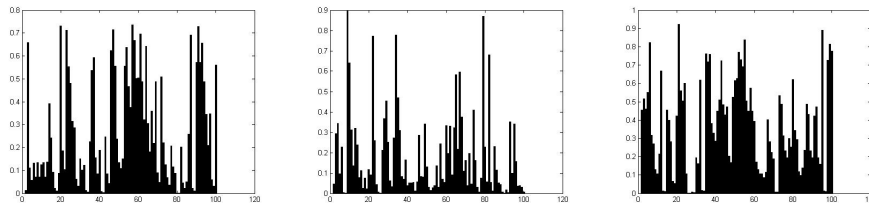


Figure 5: Sample path for  $\theta = 1$  and for various values of  $p = 0.5, 0.2, 0.8$ .

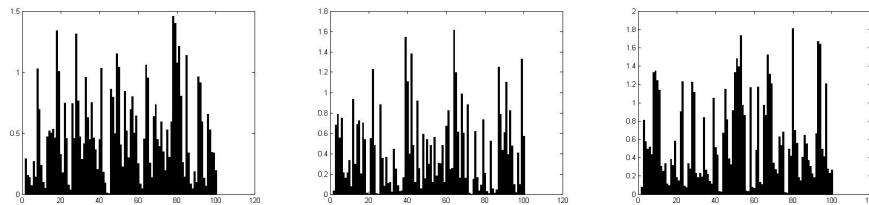


Figure 6: Sample path for  $\theta = 2$  and for various values of  $p = 0.5, 0.2, 0.8$ .



Thus the joint p.d.f. of the random variables  $X_n$  and  $X_{n-1}$  is

$$g(x, y) = \begin{cases} \frac{p^2}{(p\theta + (1-p)y)^2}, & 0 < y < x < \theta, \\ \frac{p}{(p\theta + (1-p)y)^2}, & 0 < x < y < \theta. \end{cases} \quad (6)$$

The joint distribution of the random variables  $X_n$  and  $X_{n-1}$  is not a continuous since we have

$$P(X_n = X_{n-1}) = (1-p)P(\varepsilon_n \geq X_{n-1}) = p(1-p) \int_0^\theta \frac{\theta - x}{(p\theta + (1-p)x)^2} dx = \frac{1-p+p \log p}{1-p} > 0. \quad (7)$$

Now consider the conditional probability  $P(X_n \leq x | X_{n-1} = y)$ . We have

$$P(X_n \leq x | X_{n-1} = y) = 1 - p \frac{\theta - x}{\theta} - (1-p) \frac{\theta - x}{\theta} I(y > x)$$

where  $I(\cdot)$  is indicator function. Thus the conditional p.d.f. of  $X_n$  on  $X_{n-1} = y$  is

$$g(x|y) = \begin{cases} \frac{1}{\theta}, & y > x \\ \frac{p}{\theta}, & y < x. \end{cases}$$

Also, we have  $P(X_n = X_{n-1} | X_{n-1} = y) = (1-p)(\theta - y)/\theta$ . From (6) and (7), we get the conditional moment as

$$E(X_n | X_{n-1}) = \frac{p\theta}{2} + (1-p)X_{n-1} - \frac{1-p}{2\theta} X_{n-1}^2.$$

Now, the product moment of the random variables  $X_n$  and  $X_{n-1}$  is

$$E(X_n X_{n-1}) = E(X_{n-1} E(X_n | X_{n-1})) = \frac{3p\theta^2(1-p^2 + 2p \log p)}{4(1-p)^3}. \quad (8)$$

From (8) and (2) we obtain that the autocovariance function and the autocorrelation function at lag 1 are respectively

$$\begin{aligned} Cov(X_n, X_{n-1}) &= \frac{p\theta^2 (3 - 7p + 5p^2 - p^3 - 2(1-p)p \log p - 4p \log^2 p)}{4(1-p)^4}, \\ Corr(X_n, X_{n-1}) &= \frac{3 - 7p + 5p^2 - p^3 - 2(1-p)p \log p - 4p \log^2 p}{4(1 - 2p + p^2 - p \log^2 p)}. \end{aligned}$$

The autocorrelation function at lag 1 is a decreasing function on  $p$  and lies in  $(0, 3/4)$ . Thus the autocorrelation between the random variables  $X_n$  and  $X_{n-1}$  is positive.

#### 4.1. Estimation of Parameters

Let us consider now the estimation of the unknown parameters  $p$  and  $\theta$ . These estimators can be easily found by using the equations (2) and (7), i.e. the estimators  $\hat{p}$  and  $\hat{\theta}$  are the solutions of the equations

$$\begin{aligned} \frac{\hat{p}\hat{\theta}(\hat{p} - \log \hat{p} - 1)}{(1 - \hat{p})^2} &= \frac{1}{N} \sum_{i=0}^{N-1} X_i \\ \frac{1 - \hat{p} + \hat{p} \log \hat{p}}{1 - \hat{p}} &= \frac{1}{N-1} \sum_{i=1}^{N-1} I(X_i = X_{i-1}), \end{aligned}$$

where  $I(\cdot)$  as before an indicator function and  $X_0, X_1, \dots, X_{N-1}$  is a realization of AR(1) minification model with MOEU( $p, \theta$ ) marginals of size  $N$ . In Table 1 we present some results of the estimation. We

simulated up to 10000 realizations of a Marshall-Olkin extended uniform minification process for some true values of the parameters  $p$  and  $\theta$ . The simulations are repeated 100 times and for each data set the standard deviations (SD) of the estimates are computed and given in brackets. The table shows that the standard deviation of the estimate is lesser for large samples.

Now we develop a  $k^{th}$  order autoregressive model. Consider an autoregressive model of order  $k$  with structure as

$$X_n = \begin{cases} \varepsilon_n, & \text{w.p. } p_0 \\ \min(X_{n-1}, \varepsilon_n), & \text{w.p. } p_1 \\ \vdots \\ \min(X_{n-k}, \varepsilon_n), & \text{w.p. } p_k, \end{cases}$$

such that  $0 < p_i < 1$ ,  $p_1 + p_2 + \dots + p_k = 1 - p_0$ , where  $\{\varepsilon_n\}$  is a sequence of i.i.d. random variables following uniform distribution over  $(0, \theta)$ , independent of  $\{X_{n-1}, X_{n-2}, \dots\}$ . Then

$$\bar{F}_{X_n}(x) = p_0 \bar{F}_{\varepsilon_n}(x) + p_1 \bar{F}_{X_{n-1}}(x) \bar{F}_{\varepsilon_n}(x) + \dots + p_k \bar{F}_{X_{n-k}}(x) \bar{F}_{\varepsilon_n}(x).$$

Under stationary equilibrium, we have that  $F_X(x) = p_0 \bar{F}_\varepsilon(x) + p_1 \bar{F}_X(x) \bar{F}_\varepsilon(x) + \dots + p_k \bar{F}_X(x) \bar{F}_\varepsilon(x)$ , which on simplification leads to  $F_X(x) = p_0 \bar{F}_\varepsilon(x) / (1 - (1 - p_0) \bar{F}_\varepsilon(x))$ . Evidently we can extend Theorem 4.1 in this case also.

Table 1: The results of the estimation of the parameters for some given values: a)  $p = 0.1, \theta = 0.5$ , b)  $p = 0.1, \theta = 1$ , c)  $p = 0.1, \theta = 5$ , d)  $p = 0.5, \theta = 0.5$ , e)  $p = 0.5, \theta = 1$ , f)  $p = 0.5, \theta = 5$ , g)  $p = 0.9, \theta = 0.5$ , h)  $p = 0.9, \theta = 1$ , i)  $p = 0.9, \theta = 5$ .

Sample size	$\hat{p}$	$\hat{\theta}$	$\hat{p}$	$\hat{\theta}$	$\hat{p}$	$\hat{\theta}$
100	0.1028 (0.0400)	0.5136 (0.0906)	0.1114 (0.0375)	0.9907 (0.1678)	0.1003 (0.0374)	5.1098 (0.8548)
500	0.1010 (0.0172)	0.4993 (0.0335)	0.1035 (0.0176)	0.9860 (0.0805)	0.1020 (0.0177)	5.0327 (0.3940)
1000	0.1008 (0.0117)	0.4999 (0.0220)	0.1030 (0.0111)	0.9983 (0.0574)	0.1003 (0.0118)	5.0117 (0.2639)
5000	0.1008 (0.0054)	0.4995 (0.0104)	0.1001 (0.0050)	1.0005 (0.0229)	0.1004 (0.0049)	4.9907 (0.1160)
10000	0.1008 (0.0042)	0.4996 (0.0071)	0.1000 (0.0036)	0.9985 (0.0141)	0.1004 (0.0036)	5.0008 (0.0792)
Sample size	$\hat{p}$	$\hat{\theta}$	$\hat{p}$	$\hat{\theta}$	$\hat{p}$	$\hat{\theta}$
100	0.4906 (0.0609)	0.5020 (0.0436)	0.4966 (0.0651)	1.0012 (0.0860)	0.5105 (0.0739)	5.0414 (0.4330)
500	0.4990 (0.0275)	0.4998 (0.0196)	0.4958 (0.0310)	0.9999 (0.0403)	0.5029 (0.0265)	5.0124 (0.1874)
1000	0.4976 (0.0204)	0.4989 (0.0130)	0.4975 (0.0214)	0.9978 (0.0270)	0.5056 (0.0194)	4.9985 (0.1202)
5000	0.4997 (0.0087)	0.4992 (0.0066)	0.5002 (0.0095)	1.0001 (0.0114)	0.5022 (0.0088)	5.0012 (0.0604)
10000	0.5006 (0.0059)	0.4998 (0.0041)	0.5007 (0.0067)	1.0012 (0.0082)	0.5009 (0.0063)	4.9998 (0.0386)
Sample size	$\hat{p}$	$\hat{\theta}$	$\hat{p}$	$\hat{\theta}$	$\hat{p}$	$\hat{\theta}$
100	0.9025 (0.0443)	0.5059 (0.0355)	0.9036 (0.0392)	0.9932 (0.0571)	0.8987 (0.0379)	5.0079 (0.2908)
500	0.8985 (0.0164)	0.4998 (0.0112)	0.8982 (0.0173)	1.0028 (0.0277)	0.8974 (0.0169)	4.9828 (0.1228)
1000	0.8989 (0.0124)	0.5004 (0.0090)	0.8986 (0.0123)	1.0011 (0.0200)	0.8996 (0.0119)	4.9828 (0.1058)
5000	0.9009 (0.0061)	0.5004 (0.0039)	0.9004 (0.0060)	1.0004 (0.0092)	0.8985 (0.0058)	4.9971 (0.0422)
10000	0.9006 (0.0042)	0.5004 (0.0031)	0.9006 (0.0045)	1.0005 (0.0064)	0.8992 (0.0045)	4.9987 (0.0254)

## **Acknowledgements**

The authors thank the Editor Miroslav M. Ristić, for the help extended in developing the ideas in Section 4. They are also very grateful to the referee for the suggestions and comments which significantly improved the quality of the manuscript.

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