Limit of the ratio of risks of James-Stein estimators with unknown variance

Djamel Benmansour, Abdennour Hamdaoui
Department of Mathematics, University of -Abou Bekr Belkaid-Tlemcen 13000 Algeria

Abstract. We study the estimate of the mean $\theta$ of a Gaussian random variable $X \sim N_p(\theta, \sigma^2 I_p)$ in $\mathbb{R}^p$, $\sigma^2$ unknown and estimated by the chi-square variable $S^2 (S^2 \sim \sigma^2 \chi^2_n)$. We particularly study the bounds and limits of the ratios of the risks, of the James-Stein estimator $\delta_{JS}(X)$ and of its positive-part $\delta_{JS}^+(X)$, with that of the maximum likelihood estimator $X$ when $p \to \infty$. If $\lim_{p \to +\infty} \frac{\|\theta\|_2}{p\sigma^2} = c$, we show that the ratios of the risks of the James-Stein estimator $\delta_{JS}(X)$ and its positive-part $\delta_{JS}^+(X)$, with that of the maximum likelihood estimator $X$ tend to the same value $\frac{2}{1+c}$ when $p \to \infty$. If $n$ and $p$ tend to infinity we show that the ratios of the risks tend to $\frac{2}{1+c}$. We graphically illustrate the ratios of the risks corresponding to the James-Stein estimators $\delta_{JS}(X)$ and its positive-part $\delta_{JS}^+(X)$, with that of the maximum likelihood estimator $X$ for diverse values of $n$ and $p$.

1. Introduction

The estimate of the average $\theta$ of a multidimensional Gaussian law $N_p(\theta, \sigma^2 I_p)$ in $\mathbb{R}^p$, $I_p$ means the matrix unit, has known many developments since the articles of C. Stein [7], [8] and W. James and C. Stein [5].

We cite also others works and generalizations papers [2–4]. In these works one has estimated the average of a Multidimensional Gaussian law $N_p(\theta, \sigma^2 I_p)$ in $\mathbb{R}^p$ by estimators with retrecisor deduced from the empirical average those are proved better in quadratic cost than the empirical average. These studies for the majority were made when $\sigma^2$ is known.

More precisely, if $X$ represents an observation or a sample of multidimensional Gaussian law $N_p(\theta, \sigma^2 I_p)$, the aim is to estimate $\theta$ by an estimator $\delta(X)$ relatively at the quadratic cost:

$$L(\delta, \theta) = \|\delta(X) - \theta\|^p_p,$$

where $\| \cdot \|_p$ is the usual norm in $\mathbb{R}^p$. We associate his function of risk:

$$R(\delta, \theta) = E_{\theta}(L(\delta, \theta)).$$

W. James and C. Stein [8], have introduced a class of James-Stein estimators improving $\delta(X) = X$, when the dimension of the space of the observations $p$ is $\geq 3$, noted

$$\delta_{JS}(X) = \left(1 - \frac{p - 2}{\|X\|^2}ight) X.$$
A. J. Baranchik [1] proposes the positive-part of the James-Stein estimator dominating the James-Stein estimator when \( p \geq 3 \):

\[
\delta_{JS}^0(X) = \max \left( 0, 1 - \frac{p-2}{\|X\|^2} \right) X.
\]

G. Casella and J. T. Hwang [3] studied the case where \( \sigma^2 \) is known and shown that if the limit of the ratio \( \frac{\hat{R}}{\hat{R}_{\hat{S}}} \), when \( p \) tends to infinity is a constant \( c > 0 \), then

\[
\lim_{p \to +\infty} \frac{R(\theta, \delta_{JS}^0(X))}{R(\theta, X)} = \lim_{p \to +\infty} \frac{R(\theta, \delta_{JS}^0(X))}{R(\theta, X)} = \frac{c}{1+c}, \quad c > 0.
\]

Li Sun [6] has considered the following model: \( (y_{ij}\|\theta_i, \sigma^2) \sim N(\theta_i, \sigma^2) \) \( i = 1, \ldots, n, \ j = 1, \ldots, m \) where \( E(y_{ij}) = \theta_j \) for the group \( j \) and \( Var(y_{ij}) = \sigma^2 \) is unknown. The James-Stein estimators is written in this case

\[
\delta^{JS} = (\delta_i^{JS}, \ldots, \delta_m^{JS})^t \text{ avec } \left( 1 - \frac{(m-3)S^2}{(N+2)T^2} \right) (\bar{y}_j - \bar{y}) + \bar{y}, \ j = 1, \ldots, m,
\]

where

\[
S^2 = \sum_{i=1}^{n} \sum_{j=1}^{m} (y_{ij} - \bar{y}_{ij}), \quad T^2 = n \sum_{j=1}^{m} (\bar{y}_j - \bar{y})^2, \quad \bar{y}_{ij} = \frac{\sum_{i=1}^{n} y_{ij}}{n}, \quad \bar{y}_j = \frac{\sum_{j=1}^{m} y_j}{m}.
\]

\( N = (n-1)m \), he gives a lower bound for the ratio \( \frac{R(\theta, \delta_{JS}^0(X))}{R(\theta, X)} \), that we find in (10). We give in (14) an upper bound of the same ratio.

It shows after that if: \( q = \lim_{m \to +\infty} \frac{\sum_{i=1}^{m} (\theta_i - \bar{\theta})^2}{m} \) exists, then \( \lim_{m \to +\infty} \frac{R(\theta, \delta_{JS}^0(X))}{R(\theta, X)} = \lim_{m \to +\infty} \frac{R(\theta, \delta_{JS}^0(X))}{R(\theta, X)} = \frac{q}{q+\frac{2}{\pi}} \).

In Section 2 we recall a lemma of G. Casella and J. T. Hwang [3] which we generalize if \( \sigma^2 \) is unknown and a technical lemma to calculate a lower bound for the ratio \( \frac{R(\theta, \delta_{JS}^0(X))}{R(\theta, X)} \). We give, indeed, another demonstration of the limit of the ratio \( \frac{R(\theta, \delta_{JS}^0(X))}{R(\theta, X)} \) when \( p \to \infty \), because this last enabled us to more easily deduce the limit from the ratio \( \frac{R(\theta, \delta_{JS}^0(X))}{R(\theta, X)} \) when \( n \) and \( p \) tend simultaneously to infinity.

We generalize the results of G. Casella and J. T. Hwang [3] in Section 3 (case where \( \sigma^2 \) is unknown), by giving a lower bound and an upper bound of ratio \( \frac{R(\theta, \delta_{JS}^0(X))}{R(\theta, X)} \), and its limit when \( p \) tends to infinity and \( n \) fixes on the one hand, and on the other hand, when \( n \) and \( p \) tend simultaneously to infinity, and this by supposing that \( \lim_{p \to +\infty} \frac{\|y\|^2}{p^2} = c > 0 \).

In Section 4, we give lower and upper bounds of the ratio \( \frac{R(\theta, \delta_{JS}^0(X))}{R(\theta, X)} \) and its limit when \( p \) tends to infinity and \( n \) fixes on the one hand, and on the other hand, when \( n \) and \( p \) tend simultaneously to infinity, and this in all the cases where \( \sigma^2 \) is unknown.

We take as estimator of \( \sigma^2 \) the statistics \( S^2 \) independent of \( X \) and of law \( \sigma^2 \chi^2_n \) in \( IR^+ \). In this case the James-Stein estimator and his positive-part are written respectively

\[
\delta_{JS}(X) = \left( 1 - \frac{(p-2)S^2}{(n+2)||X||^2} \right) X
\]

\[
\delta_{JS}^+(X) = \max \left( 0, 1 - \frac{(p-2)S^2}{(n+2)||X||^2} \right) X.
\]

In Section 5, we give a graphic illustration of different ratios of risks and the lower and upper bounds associated for various values of \( n \) and \( p \).
2. Preliminary

Let us recall that the risk of the maximum likelihood estimator $X$ is $p\sigma^2$ the risk of the James Stein estimator (given in (1) of $\theta$ is

$$R(\theta, \delta_{JS}(X)) = \sigma^2 \left\{ p - \frac{h}{n + 2} (p - 2)E \left( \frac{1}{p - 2 + 2K} \right) \right\},$$

where $K \sim P \left( \frac{||\theta||^2}{2\sigma^2} \right)$ being the law of Poisson of parameter $\frac{||\theta||^2}{2\sigma^2}$.

When $X \sim N_p(\theta, I_p)$ G. Casella and J. T. Hwang [3] have given the lemma 1 which expresses the following inequalities:

$$\frac{1}{(p - 2 + ||\theta||^2)} \leq E \left( \frac{1}{||X||^2} \right) \leq \frac{p}{(p - 2)(p + ||\theta||^2)} p \geq 3.$$

In the following lemma, we generalise this result when $X \sim N_p(\theta, \sigma^2 I_p)$ and $\sigma^2$ is unknown.

**Lemma 2.1.** Let $X \sim N_p(\theta, \sigma^2 I_p)$. If $p \geq 3$ then

$$\frac{1}{\sigma^2 (p - 2 + ||\theta||^2)} \leq E \left( \frac{1}{||X||^2} \right) \leq \frac{p}{\sigma^2 (p - 2 + ||\theta||^2)}.$$

**Proof.** We have

$$X \sim N_p(\theta, \sigma^2 I_p) \Rightarrow \frac{X}{\sigma} \sim N_p \left( \frac{\theta}{\sigma}, I_p \right) \Rightarrow \frac{1}{\sigma^2} ||X||^2 \sim \chi^2_p \left( \frac{||\theta||^2}{\sigma^2} \right)$$

$$E \left( \frac{1}{||X||^2} \right) = \frac{1}{\sigma^2} E \left( \frac{1}{\chi^2_p} \right) = \frac{1}{\sigma^2} \sum_{k \geq 0} \int_0^{+\infty} \frac{1}{\omega} \chi^2_{p+2k}(d\omega) \pi \left( \frac{||\theta||^2}{2\sigma^2} \right)$$

$$= \frac{1}{\sigma^2} E \left( \frac{1}{p - 2 + 2K} \right),$$

where $K \sim P \left( \frac{||\theta||^2}{2\sigma^2} \right)$ and (3) being the definition of the law of $\chi^2$ no centred. According to the inequality of Jensen we have

$$E \left( \frac{1}{||X||^2} \right) = \frac{1}{\sigma^2} E \left( \frac{1}{p - 2 + 2K} \right) \geq \frac{1}{\sigma^2 (p - 2 + ||\theta||^2)}.$$

In the other hand, for any real function $h$ such that $E(h(\chi^2_q(\lambda)))\chi^2_q(\lambda))$ exists (see G. Casella [2]), we have

$$E(h(\chi^2_q(\lambda)))\chi^2_q(\lambda)) = qE(h(\chi^2_{q+2}(\lambda))) + 2AE(h(\chi^2_{q+4}(\lambda)))$$

for $q = p - 2, h(\omega) = \frac{1}{\lambda}, \lambda = \frac{||\theta||^2}{2\sigma^2}$ we obtain

$$\int_0^{+\infty} \chi^2_{p-2}(\lambda, d\omega) = (p - 2) \int_0^{+\infty} \frac{1}{\omega} \chi^2_p(\lambda, d\omega) + \frac{||\theta||^2}{\sigma^2} \int_0^{+\infty} \frac{1}{\omega} \chi^2_{p+2}(\lambda, d\omega).$$

Then

$$1 = (p - 2)E \left( \frac{1}{||X||^2} \right) + \frac{||\theta||^2}{\sigma^2} E \left( \frac{1}{p + 2K} \right) \text{ with } K \sim P \left( \frac{||\theta||^2}{2\sigma^2} \right).$$

(5)
Thus
\[ \sigma^2 E \left( \frac{1}{\|X\|^2} \right) = \frac{1}{p-2} \left[ 1 - \frac{\|\theta\|^2}{\sigma^2} E \left( \frac{1}{p+2K} \right) \right]. \]

Hence
\[ E \left( \frac{1}{\|X\|^2} \right) \leq \frac{1}{\sigma^2} \frac{1}{p-2} \left[ 1 - \frac{\|\theta\|^2}{\sigma^2} \frac{1}{p+\|\theta\|} \right]. \quad (6) \]

Thus
\[ E \left( \frac{1}{\|X\|^2} \right) \leq \frac{1}{\sigma^2(p-2)} \left[ \frac{p}{p+\|\theta\|} \right]. \]

The equality (5) came from \( E \left( \chi^2_{p-2}(\lambda) \right) = 1 \), with \( K \sim \mathcal{P} \left( \frac{\|\theta\|^2}{\sigma^2} \right) \) and the inequality (6) came from Jensen’s inequality. Hence the result.

We recall that if \( X \) is of a random variable of multidimensional Gaussian law \( X \sim N_p(\theta, \sigma^2 I_p) \) in \( \mathbb{R}^p \), then \( U = \|X\|^2 \sim \chi^2_p(\lambda) \) where \( \chi^2_p(\lambda) \) designates a noncentral \( \chi^2 \) distribution with \( p \) degrees of freedom and noncentrality parameter \( \lambda = \frac{\|\theta\|^2}{\sigma^2} \).

**Lemma 2.2.** Let \( f \) be a real function defined on \( \mathbb{R} \) and \( X \) a random variable of multidimensional Gaussian law \( X \sim N_p(\theta, \sigma^2 I_p) \) in \( \mathbb{R}^p \). If for \( p \geq 1 \), \( E[f(U)\chi^2_p(\lambda)] \) exists, then:

a) If \( f \) nonincreasing we have
\[ E[f(U)\chi^2_{p+2}(\lambda)] \leq E[f(U)\chi^2_p(\lambda)]. \quad (7) \]

b) If \( f \) nondecreasing we have:
\[ E[f(U)\chi^2_{p+2}(\lambda)] \geq E[f(U)\chi^2_p(\lambda)]. \quad (8) \]

**Proof.** a) We have
\[ E[(f(U)\chi^2_{p+2}(\lambda))] - E[(f(U)\chi^2_p(\lambda))] = E\left[f(U)\left( \frac{\frac{u}{p+2K}}{p} - 1 \right)\chi^2_p(\lambda)\right] E[f(U)\chi^2_p(\lambda)] E\left[\frac{\frac{u}{p+2K}}{p} - 1\right] \chi^2_p(\lambda) \]

because the covariance of two functions one increasing and the other decreasing is negative or null, with \( K \sim \mathcal{P}(\frac{\|\theta\|^2}{\sigma^2}) \). However
\[ E\left[\left( \frac{\frac{u}{p+2K}}{p} - 1 \right)\chi^2_p(\lambda)\right] = 0, \]
then
\[ E[(f(U)\chi^2_{p+2}(\lambda))] \leq E[(f(U)\chi^2_p(\lambda))]. \]

Hence the result a), (in the same manner we get b). Thus the result.
3. Bounds and limit of the ratio of the risks of James-Stein estimator to the maximum likelihood estimator

**Theorem 3.1.** If \( \lim_{p \to +\infty} \frac{\|\theta\|_2^2}{p \sigma^2} = c > 0 \), then

\[
\lim_{p \to +\infty} \frac{R(\theta, \delta_{JS}(X))}{R(\theta, X)} = \frac{2}{n+2} + c.
\]

**Proof.** We have

\[
R(\theta, \delta_{JS}(X)) = \sigma^2 \left\{ p - \frac{n}{n+2} \left( p - 2 \right)^2 E \left( \frac{1}{p - 2 + 2K} \right) \right\}
\]

where \( K \sim \mathcal{P}(\|\theta\|_2^2) \). Then

\[
\frac{R(\theta, \delta_{JS}(X))}{R(\theta, X)} = 1 - \frac{n}{n+2} \left( p - 2 \right)^2 \frac{1}{p - 2 + 2K} \]

\[
\leq 1 - \frac{n}{n+2} \left( p - 2 \right)^2 \frac{1}{p - 2 + \frac{\|\theta\|_2^2}{\sigma^2}} \leq \frac{n}{n+2} \left( p - 2 \right)^2 \frac{1}{p - 2 + \frac{\|\theta\|_2^2}{\sigma^2}}.
\]

The inequality (10) is obtained from lemma 2.1. Hence

\[
\lim_{p \to +\infty} \frac{R(\theta, \delta_{JS}(X))}{R(\theta, X)} \leq \lim_{p \to +\infty} \left\{ 1 - \frac{n}{n+2} \left( p - 2 \right)^2 \frac{1}{p - 2 + \frac{\|\theta\|_2^2}{\sigma^2}} \right\}
\]

\[
\leq 1 - \frac{n}{(n+2)(1+c)} \leq \frac{(n+2)(1+c) - n}{(n+2)(1+c)} \leq \frac{2}{n+2} + c.
\]

Where \( \lim_{p \to +\infty} \frac{\|\theta\|_2^2}{p \sigma^2} = c \). In the other hand

\[
\frac{R(\theta, \delta_{JS}(X))}{R(\theta, X)} = 1 - \frac{n}{n+2} \left( p - 2 \right)^2 \frac{1}{p - 2 + 2K} \]

\[
\leq 1 - \frac{n}{n+2} \left( p - 2 \right)^2 \frac{1}{\|X\|^2} \]

According to lemma 2.1 we have

\[
\frac{R(\theta, \delta_{JS}(X))}{R(\theta, X)} \geq 1 - \frac{n}{n+2} \left( p - 2 \right)^2 \frac{p}{p + \|X\|^2} \]

\[
\geq 1 - \frac{n}{n+2} \left( p - 2 \right)^2 \frac{1}{p \left( 1 + \frac{\|X\|^2}{\sigma^2} \right)}.
\]
Thus we obtain lower and upper bounds of the ratio \( \frac{R(\theta, \delta_{JS}(X))}{R(\theta, X)} \) deduced from (10) and (14):

\[
1 - \frac{n}{n + 2} \left( \frac{p - 2}{p} \right) \frac{1}{1 + \frac{\|\theta\|}{p \sigma^2}} \leq \frac{R(\theta, \delta_{JS}(X))}{R(\theta, X)} \leq 1 - \frac{n}{n + 2} \left( \frac{p - 2}{p} \right)^2 \frac{1}{1 + \frac{\|\theta\|}{p \sigma^2}}. \tag{15}
\]

Passaging to the limit when \( p \to +\infty \) we obtain

\[
\lim_{p \to +\infty} \frac{R(\theta, \delta_{JS}(X))}{R(\theta, X)} \geq \lim_{p \to +\infty} \left( 1 - \frac{n}{n + 2} \left( \frac{p - 2}{p} \right) \frac{1}{1 + \frac{\|\theta\|}{p \sigma^2}} \right) \geq 1 - \frac{n}{n + 2} \frac{1}{1 + \frac{\|\theta\|}{\sigma^2}} = \frac{n + 2 - 1}{n + 2} = \frac{n + 1}{n + 2} \geq 1 - \frac{n}{n + 2} \frac{1}{1 + \frac{\|\theta\|}{\sigma^2}} \geq 1 - \frac{n}{n + 2} \frac{1}{1 + \frac{\|\theta\|}{\sigma^2}}.
\]

where \( \lim_{p \to +\infty} \frac{\|\theta\|}{\sigma^2} = c \). Combining (13) and (16) we obtain

\[
\lim_{p \to +\infty} \frac{R(\theta, \delta_{JS}(X))}{R(\theta, X)} = \frac{1}{\frac{n + 2 - 1}{n + 2} + \frac{1}{1 + \frac{\|\theta\|}{\sigma^2}}} = \frac{1 + c}{n + 2}.
\]

Hence the result.

**Corollary 3.2.** If \( \lim_{p \to +\infty} \frac{\|\theta\|}{\sigma^2} = c > 0 \), we have

\[
\lim_{n, p \to +\infty} \frac{R(\theta, \delta_{JS}(X))}{R(\theta, X)} = \frac{c}{1 + c}.
\]

**Proof.** It is obtained immediately from (11) and (14). \( \square \)

4. Bounds and limit of the ratio of the risks of the positive-part James-Stein estimator to the maximum likelihood estimator

The results of the positive-part James-Stein estimator \( \delta_{JS}^+(X) \) are similar to the results obtained on the James-Stein estimator \( \delta_{JS}(X) \).

Indeed, we denote \( \alpha = \frac{p - 2}{n + 2} \), and recall that:

\[
\delta_{JS}^+(X) = \left( 1 - \alpha \frac{S^2}{\|X\|^2} \right)^+ X = \phi_{JS}^+(X) X \left( 1 - \alpha \frac{S^2}{\|X\|^2} \right) X I_{\{\alpha \frac{S^2}{\|X\|^2} \leq 1\}},
\]

\[
\delta_{JS}^-(X) = \left( 1 - \alpha \frac{S^2}{\|X\|^2} \right)^- X = \phi_{JS}^-(X) X \left( \alpha \frac{S^2}{\|X\|^2} - 1 \right) X I_{\{\alpha \frac{S^2}{\|X\|^2} \geq 1\}}.
\]

\( I_{\{\alpha \frac{S^2}{\|X\|^2} \geq 1\}} \) showing the indicating function of the set \( \{\alpha \frac{S^2}{\|X\|^2} \geq 1\} \).

We will denote for the needs of demonstration and each time that it will be necessary, for the sequel, by: \( Y = \frac{X}{\sigma} \), \( \beta = \frac{\phi}{\sigma} \), \( T^2 = \frac{S^2}{\sigma^2} \), and \( U = \|Y\|^2 \).

Then we have \( Y \sim N(\beta, I_p), T^2 \sim \chi^2_n \) and \( U \sim \chi^2_p(\lambda) \), where \( \chi^2_n \) designates the law of the chi square centered with \( n \) degrees of freedom and \( \chi^2_p(\lambda) \) designates a noncentral \( \chi^2 \) distribution with \( p \) degrees of freedom and noncentrality parameter \( \lambda = \frac{\|\theta\|}{\sigma^2} \).

The following lemma is a recall of the expression of the risk of \( \delta_{JS}^+(X) \).
Lemma 4.1. We have that
\[
R(\theta, \delta_{JS}(X)) = R(\theta, \delta_{JS}(X)) + E \left\{ \left[ ||X||^2 + 2(p - 2)\sigma^2 \theta \frac{S^2}{||X||^2} - \alpha^2 \frac{(S^2)^2}{||X||^2} - 2p\sigma^2 \right] I_{\frac{\phi}{\phi} \geq 1} \right\}.
\] (17)

Proof. We have
\[
R(\theta, \delta_{JS}(X)) = E \left( ||\phi_{JS}(X)X - \theta||^2 \right)
= E \left( ||\phi_{JS}^+(X)X - \phi_{JS}(X)||^2 \right)
= E \left( (||\phi_{JS}^+(X)X - \theta||^2) + E \left( \left( \phi_{JS}^-(X) \right)^2 ||X||^2 \right) - 2E \left( \langle \phi_{JS}^+(X)X - \phi_{JS}(X) \rangle \right) \right)
= R(\theta, \delta_{JS}(X)) + E \left( \left( \phi_{JS}^-(X) \right)^2 ||X||^2 \right) - 2E \left( \langle X - \theta, \phi_{JS}^-(X) \rangle \right) + 2E \left( \langle X, \delta_{JS}^-(X) \rangle \right).
\]

Then
\[
R(\theta, \delta_{JS}(X)) = R(\theta, \delta_{JS}^+(X)) + E \left\{ \left( \frac{S^2}{||X||^2} - 1 \right) I_{\frac{\phi}{\phi} \geq 1} \right\}
- 2E \left\{ t(X - \theta)X \left( \alpha \frac{S^2}{||X||^2} - 1 \right) I_{\frac{\phi}{\phi} \geq 1} \right\} + 2E \left\{ \left( \frac{S^2}{||X||^2} - 1 \right) I_{\frac{\phi}{\phi} \geq 1} \right\} ||X||^2.
\]

According to the lemma of P. Shao and W. E. Strawderman ([7, Lemma 2.1]) we have,
\[
E \left\{ t(X - \theta)X \left( \alpha \frac{S^2}{||X||^2} - 1 \right) I_{\frac{\phi}{\phi} \geq 1} \right\} = \sigma^2 E \left\{ -2\alpha \frac{S^2}{\sigma^2||Y||^2} + p \left( \frac{S^2}{\sigma^2||Y||^2} - 1 \right) I_{\frac{\phi}{\phi} \geq 1} \right\}
= \sigma^2 E \left\{ (p - 2)\alpha \frac{S^2}{\sigma^2||Y||^2} - p I_{\frac{\phi}{\phi} \geq 1} \right\}
= \sigma^2 E \left\{ (p - 2)\alpha \frac{S^2}{\sigma^2||Y||^2} - p I_{\frac{\phi}{\phi} \geq 1} \right\}
= \sigma^2 E \left\{ (p - 2)\alpha \frac{S^2}{||X||^2} - p I_{\frac{\phi}{\phi} \geq 1} \right\}
\]

thus
\[
R(\theta, \delta_{JS}(X)) = R(\theta, \delta_{JS}^+(X)) + E \left\{ \frac{\alpha^2 (S^2)^2}{||X||^2} - 2\alpha S^2 + ||X||^2 \right\} I_{\frac{\phi}{\phi} \geq 1}
- 2\sigma^2 E \left\{ (p - 2)\alpha \frac{S^2}{||X||^2} - p I_{\frac{\phi}{\phi} \geq 1} \right\} + 2E \left\{ \left( \frac{S^2}{||X||^2} - 1 \right) I_{\frac{\phi}{\phi} \geq 1} \right\} ||X||^2.
\]

Therefore
\[
R(\theta, \delta_{JS}(X)) = R(\theta, \delta_{JS}^+(X)) + E \left\{ ||X||^2 + 2\sigma^2 (p - 2)\alpha \frac{S^2}{||X||^2} - \alpha^2 \frac{(S^2)^2}{||X||^2} - 2p\sigma^2 \right\} I_{\frac{\phi}{\phi} \geq 1}.
\]

Hence the result. \(\square\)
The fact that \( R(\theta, \delta_{1S}(X)) \leq R(\theta, \delta_{J}(X)) \) (Baranchick [1]), an upper bound of the ratio \( \frac{R(\theta, \delta_{1S}(X))}{R(\theta, \delta_{J}(X))} \) would be for example the upper bound of the ratio \( \frac{R(\theta, \delta_{1S}(X))}{R(\theta, \delta_{J}(X))} \) that we know the asymptotic character. Thus we will interesting ourselves in what follows on a lower bound of the ratio \( \frac{R(\theta, \delta_{1S}(X))}{R(\theta, \delta_{J}(X))} \).

We have the following result.

**Theorem 4.2.** For all \( p \geq 3 \), we have the following minoration of the ratio of the risks \( \frac{R(\theta, \delta_{1S}(X))}{R(\theta, \delta_{J}(X))} \):

\[
\frac{R(\theta, \delta_{1S}(X))}{R(\theta, \delta_{J}(X))} \geq \frac{R(\theta, \delta_{1S}(X))}{p \sigma^2} + \frac{(p + \lambda)}{p} \int_0^{+\infty} I_P \left( \chi_n^2 \geq \frac{u}{\alpha}, \chi_{p+4}(\lambda, du) \right) \frac{(an + 4)}{p} \int_0^{+\infty} I_P \left( \chi_n^2 \geq \frac{u}{\alpha}, \chi_{p-2}(\lambda, du) \right)
\]

where \( F_{(p, n, \lambda)}(\alpha) \) is a function to distribution of Fisher with \( p \) and \( n \) degrees of freedom and parameter of noncentrality \( \lambda = \frac{||q||^2}{\sigma^2} \).

**Proof.** Taking again the equality (17) we have

\[
R(\theta, \delta_{1S}(X)) = R(\theta, \delta_{J}(X)) + E \left\{ \left[ ||X||^2 + 2(p - 2) \alpha \sigma^2 \frac{S^2}{||X||^2} - \alpha^2 \frac{(S^2)^2}{||X||^2} - 2p \sigma^2 \right] I_{\frac{\alpha S^2}{||X||^2} \geq 1} \right\}
\]

Then

\[
I_1 = E \left( ||X||^2 I_{\frac{\alpha S^2}{||X||^2} \geq 1} \right)
\]

\[
= \alpha^2 E \left( \frac{||X||^2}{\sigma^2} I_{\frac{\alpha S^2}{||X||^2} \geq 1} \right)
\]

\[
= \alpha^2 \int_0^{+\infty} \left( \int_0^{+\infty} \chi_n^2(0, ds) \right) u \chi_p^2(\lambda, du).
\]

Taking \( h(u) = \int_0^{+\infty} \chi_n^2(0, ds) \) and \( q = p \), and applying (4) we obtain

\[
I_1 = \sigma^2 p \int_0^{+\infty} I_P \left( \chi_n^2 \geq \frac{u}{\alpha}, \chi_{p+4}(\lambda, du) + \sigma^2 \lambda \int_0^{+\infty} I_P \left( \chi_n^2 \geq \frac{u}{\alpha}, \chi_{p+4}(\lambda, du) \right)
\]

\[
\geq \sigma^2 (p + \lambda) \int_0^{+\infty} I_P \left( \chi_n^2 \geq \frac{u}{\alpha}, \chi_{p+4}(\lambda, du) \right)
\]

The inequality (20) comes from inequality (7) of lemma 2.2. On the other hand

\[
I_2 = \sigma^2 E \left( \left[ 2(p - 2) \alpha \frac{S^2}{||X||^2} - 2p \right] I_{\frac{\alpha S^2}{||X||^2} \geq 1} \right)
\]

\[
\geq -4 \sigma^2 E \left( I_{\frac{\alpha S^2}{||X||^2} \geq 1} \right)
\]

\[
\geq -4 \sigma^2 \int_0^{+\infty} \left( \int_0^{+\infty} \chi_n^2(0, ds) \right) \chi_p^2(\lambda, du)
\]

\[
\geq -4 \sigma^2 \int_0^{+\infty} I_P \left( \chi_n^2 \geq \frac{u}{\alpha}, \chi_p^2(\lambda, du) \right)
\]

\[
\geq -4 \sigma^2 \int_0^{+\infty} I_P \left( \chi_n^2 \geq \frac{u}{\alpha}, \chi_{p+4}(\lambda, du) \right)
\]
The inequality (21) comes from inequality (7) of lemma 2.2 and by observing that $\text{IP}(\chi_n^2 \geq \frac{u}{\alpha}) \geq \text{IP}(\chi_n^2 \geq \frac{u}{\alpha})$ for all $n \geq 1$. On the other hand

$$I_3 = -\alpha^2 \sigma^2 \int_0^{+\infty} \left( \int_0^{+\infty} (\chi_n^2(0,dt^2)) \frac{1}{u} \chi_n^2(\lambda,du) \right) \geq \frac{\sigma^2}{(n+2)} \int_0^{+\infty} \left( \int_0^{+\infty} (\chi_n^2(0,dt^2)) \chi_{n+2}(\lambda,du) \right).$$

The last inequality comes from (4) while taking $h(u) = \frac{1}{u} \int_0^{+\infty} (\chi_n^2(0,ds), q = p - 2$ and the fact that $\alpha = \frac{p-2}{n+2}$. However

$$\int_0^{+\infty} (\chi_n^2(0,dt^2)) = n \int_0^{+\infty} (\chi_n^2(0,dt^2)$$

$$= n(n+2) \int_0^{+\infty} \chi_n^2(0,dt^2)$$

$$= n(n+2) \text{IP}(\chi_n^2 \geq \frac{u}{\alpha}).$$

Then we have

$$I_3 \geq -\sigma^2 \alpha n \int_0^{+\infty} \text{IP}(\chi_n^2 \geq \frac{u}{\alpha}) \chi_{n+2}(\lambda,du).$$

According to (14), (20), (21) and (23) we have

$$\frac{R(\theta, \delta_{15}(X))}{p \sigma^2} \geq \frac{R(\theta, \delta_{15}(X))}{p \sigma^2} + \frac{I_1}{p \sigma^2} + \frac{I_2}{p \sigma^2} + \frac{I_3}{p \sigma^2}$$

$$\geq \frac{R(\theta, \delta_{15}(X))}{p \sigma^2} + \frac{\sigma^2 (p+\lambda)}{p \sigma^2} \int_0^{+\infty} \left( \text{IP}(\chi_n^2 \geq \frac{u}{\alpha}) \right) \chi_{n+2}(\lambda,du)$$

$$- \frac{4\sigma^2}{p \sigma^2} \int_0^{+\infty} \left( \text{IP}(\chi_n^2 \geq \frac{u}{\alpha}) \right) \chi_{n+2}(\lambda,du)$$

$$\geq \frac{R(\theta, \delta_{15}(X))}{p \sigma^2} + \frac{(p+\lambda)}{p} \int_0^{+\infty} \text{IP}(\chi_n^2 \geq \frac{u}{\alpha}) \chi_{n+2}(\lambda,du) - \frac{(an+4)}{p} \int_0^{+\infty} \text{IP}(\chi_n^2 \geq \frac{u}{\alpha}) \chi_{n+2}(\lambda,du).$$

Thus the result. □

**Remark 4.3.** The lower bound given by the inequality (18) is a bound enough “fine” and close to the ratio $\frac{R(\theta, \delta_{15}(X))}{p \sigma^2}$ as soon as $\lambda$ moves away from zero. The simulations in Section 5 illustrates it rather well. On the other hand for the passage to limit of the ratio $\frac{R(\theta, \delta_{15}(X))}{p \sigma^2}$ when $p$ tends to infinity or when $p$ and $n$ tend simultaneously to infinity, a less fine bound min would be enough. Then we have the next result on the limit of the ratio of the risks $\frac{R(\theta, \delta_{15}(X))}{R(\theta, X)}$.

**Proposition 4.4.** If $\lim_{p \to +\infty} \frac{R(\theta, \delta_{15}(X))}{R(\theta, X)} = c$ and $c > 0$, we have:

$$\lim_{p \to +\infty} \frac{R(\theta, \delta_{15}(X))}{R(\theta, X)} = \frac{2\pi + c}{1+c}.$$
Proof. Baranchick [1] showed that \( R(\theta, \delta_{I_S}(X)) \leq R(\theta, \delta_{I}(X)) \) for \( p \geq 3 \) and all \( \theta, \sigma \in IR^d IR^+ \). Thus the upper bound in (11) plays the role of the upper bound of \( \frac{R(\theta, \delta_{I_S}(X))}{R(\theta, X)} \). It is enough to show that the limit of the lower bound is higher or equal to \( \frac{n^2 + \xi}{1 + c} \). Indeed according to (24) we have

\[
\frac{R(\theta, \delta_{I_S}(X))}{R(\theta, X)} \geq \frac{R(\theta, \delta_{I}(X))}{p \alpha^2} + \frac{(p + \lambda)}{p} \int_0^{+\infty} IP \left( \chi_n^2 \geq \frac{u}{\alpha} \right) \chi_{p+4}^2(\lambda, du)
\]

because

\[
\frac{(p + \lambda)}{p} \int_0^{+\infty} IP \left( \chi_n^2 \geq \frac{u}{\alpha} \right) \chi_{p+4}^2(\lambda, du) \geq \int_0^{+\infty} IP \left( \chi_n^2 \geq \frac{u}{\alpha} \right) \chi_{p+4}^2(\lambda, du)
\]

\[
= \int_0^{+\infty} IP \left( \chi_n^2 \geq \frac{u}{\alpha} \right) \chi_{p+4}^2(\lambda, du) \geq -1 \quad \text{and} \quad - \frac{n(p - 2)}{\lambda(p + 2)} \geq -1.
\]

Let us denote that like \( \alpha = \frac{n^2 - \xi}{\sigma^2} \) and thus tends to \(+\infty\) when \( p \to +\infty \), we have according to the theorem of Lebesgue by taking for example, the increasing sequel with \( p \left( \int_0^{+\infty} \chi_n^2(0, ds) = IP \left( \chi_n^2 \geq \frac{\xi}{2} \right) \right) \)

\[
\lim_{p \to +\infty} IP \left( \chi_n^2 \geq \frac{u}{\alpha} \right) = IP \left( \chi_n^2 \geq 0 \right) = 1, \quad \forall \ n \geq 1
\]

thus

\[
\lim_{p \to +\infty} \int_0^{+\infty} IP \left( \chi_n^2 \geq \frac{u}{\alpha} \right) \chi_{p+4}^2(\lambda, du) = 1
\]

where \( \lambda = \frac{n^2 + \xi}{\sigma^2} \). Finally we obtains

\[
\lim_{p \to +\infty} \frac{R(\theta, \delta_{I_S}(X))}{R(\theta, X)} \geq \frac{\xi}{n^2 + \xi} + c \quad \frac{1}{1 + c}.
\]

Thus the result. \( \square \)

The case where \( n \) and \( p \) tend simultaneously to \(+\infty\) is given in the following theorem.

**Theorem 4.5.** If \( \lim_{p \to +\infty} \frac{\sigma^d}{\sigma^d} = c > 0 \), we have

\[
\lim_{n,p \to +\infty} \frac{R(\theta, \delta_{I_S}(X))}{R(\theta, X)} = \frac{c}{1 + c}.
\]

**Proof.** On the one hand, we have

\[
\lim_{n,p \to +\infty} \frac{R(\theta, \delta_{I_S}(X))}{R(\theta, X)} \leq \lim_{n,p \to +\infty} \frac{R(\theta, \delta_{I}(X))}{R(\theta, X)} = \frac{c}{1 + c}\]

(27)
because $R(\theta, \delta^*_J(X)) \leq R(\theta, \delta_{IS}(X))$ for $p \geq 3$ and all $\theta, \sigma \in IR^pIR^+$. In the other hand, by beginning again (25) we have

$$\lim_{n,p \to +\infty} \frac{R(\theta, \delta^*_J(X))}{R(\theta, X)} \geq \frac{c}{1 + c} + \lim_{n,p \to +\infty} \int_0^{+\infty} \left( IP\left( \chi^2_n \geq \frac{u}{\alpha} \right) \right) \chi^2_{p+4}(\lambda, du) - 1.$$  (28)

However

$$IP\left( \chi^2_n \geq \frac{u}{\alpha} \right) = IP\left( \sum_{i=1}^n y_i^2 \geq \frac{u(n + 2)}{(p - 2)} \right) = IP\left( \frac{\sum_{i=1}^n y_i^2}{n} \geq \frac{u}{(p - 2)} + \frac{2u}{n(p - 2)} \right)$$

where $y_1, y_2, \ldots, y_n$ are independent, Gaussian random variables centered reduced. Thus by the strong law of the great numbers we have

$$\lim_{n,p \to +\infty} IP\left( \chi^2_n \geq \frac{u}{\alpha} \right) = \lim_{n,p \to +\infty} IP\left( \frac{\sum_{i=1}^n y_i^2}{n} \geq \frac{u}{(p - 2)} + \frac{2u}{n(p - 2)} \right) = \lim_{n,p \to +\infty} IP\left( \sum_{i=1}^n y_i^2 \geq 0 \right) = IP(1 \geq 0) = 1$$

ainsi

$$\lim_{n,p \to +\infty} \int_0^{+\infty} \left( IP\left( \chi^2_n \geq \frac{u}{\alpha} \right) \right) \chi^2_{p+4}(\lambda, du) = \int_0^{+\infty} \chi^2_{p+4}(\lambda, du) = 1.$$  (29)

Combining (27), (28) and (29), there is finally the result. \(\Box\)

5. Simulations

We illustrate graphically in what follows the ratios of the risks $R(\theta, \delta^*_J(X)) / R(\theta, X)$, as well as the evolution of the bound min and max associated, given respectively by the expressions (15) and (18), for $\sigma^2 = 1$ and different values of $n$ and $p$. 
Figure 1: Graph of the relative risks and their minima and maxima for $n = 50$ and $p = 4$, according to $\lambda = \frac{|\theta|}{\sigma^2}$ and $\sigma^2 = 1$.

Figure 2: Graph of the relative risks and their minima and maxima for $n = 100$ and $p = 10$, according to $\lambda = \frac{|\theta|}{\sigma^2}$ and $\sigma^2 = 1$. 
Conclusion

In the case of the estimate of the average $\theta$ of a multidimensional Gaussian law $N_p(\theta, I_p)$ in $\mathbb{R}^p$, Casella, G and J, T, Tzon Hwang [3] showed that if $\lim_{p \to +\infty} \frac{||\theta||^2}{p} = c_1 > 0$ then the ratio $\frac{R(\theta, \delta J, S(X))}{R(\theta, X)}$ tends to $\frac{c_1}{1+c_1}$. By taking the same model, namely $X \sim N_p(\theta, \sigma^2 I_p)$ with this time $\sigma^2$ unknown, and estimated by the statistic $S^2$ independent of $X$ and of law $\sigma^2 \chi^2_n$ in $\mathbb{R}^+$. We have for our part, showed that we obtain a similar ratio depend on the size $n$ of the sample, with that found by the latter, as soon as $\lim_{p \to +\infty} \sigma^2 \frac{\chi^2_n}{p^2} = c > 0$. Moreover we obtain a ratio constant and independent of $n$, when $n$ and $p$ tend simultaneously to $+\infty$ and this without taking account of an unspecified relation of order or functional calculus between $n$ and $p$. Li Sun. [6] is him also interested if $\sigma^2$ is unknown, but studied the behavior of the ratio $\frac{R(\theta, \delta J, S(X))}{R(\theta, X)}$ and $\frac{R(\theta, \delta J, S(X))}{R(\theta, X)}$, when only $p$ tends to infinity. An idea would be to see whether we obtain similar ratios in the general case of the symmetrical spherical models.

References