

Limit of the ratio of risks of James-Stein estimators with unknown variance

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Abstract. We study the estimate of the mean θ of a Gaussian random variable $X \sim N_p(\theta, \sigma^2 I_p)$ in IR^p , σ^2 unknown and estimated by the chi-square variable S^2 ($S^2 \sim \sigma^2 \chi_n^2$). We particularly study the bounds and limits of the ratios of the risks, of the James-Stein estimator $\delta_{JS}(X)$ and of its positive-part $\delta_{JS}^+(X)$, with that of the maximum likelihood estimator X when $p \rightarrow \infty$. If $\lim_{p \rightarrow +\infty} \frac{\|\theta\|^2}{p\sigma^2} = c$, we show that the ratios of the risks of the James-Stein estimator $\delta_{JS}(X)$ and its positive-part $\delta_{JS}^+(X)$, with that of the maximum likelihood estimator X tend to the same value $\frac{2}{1+c}$ when $p \rightarrow \infty$. If n and p tend to infinity we show that the ratios of the risks tend to $\frac{c}{1+c}$. We graphically illustrate the ratios of the risks corresponding to the James-Stein estimators $\delta_{JS}(X)$ and its positive-part $\delta_{JS}^+(X)$, with that of the maximum likelihood estimator X for diverse values of n and p .

1. Introduction

The estimate of the average θ of a multidimensional Gaussian law $N_p(\theta, \sigma^2 I_p)$ in IR^p , I_p means the matrix unit, has known many developments since the articles of C. Stein [7], [8] and W. James and C. Stein [5].

We cite also others works and generalizations papers [2–4]. In these works one has estimated the average of a Multidimensional Gaussian law $N_p(\theta, \sigma^2 I_p)$ in IR^p by estimators with retrecissor deduced from the empirical average those are proved better in quadratic cost than the empirical average. These studies for the majority were made when σ^2 is known.

More precisely, if X represents an observation or a sample of multidimensional Gaussian law $N_p(\theta, \sigma^2 I_p)$, the aim is to estimate θ by an estimator $\delta(X)$ relatively at the quadratic cost:

$$L(\delta, \theta) = \|\delta(X) - \theta\|_p^2,$$

where $\|\cdot\|_p$ is the usual norm in IR^p . We associate his function of risk:

$$R(\delta, \theta) = E_\theta(L(\delta, \theta)).$$

W. James and C. Stein [8], have introduced a class of James-Stein estimators improving $\delta(X) = X$, when the dimension of the space of the observations p is ≥ 3 , noted

$$\delta_{JS}(X) = \left(1 - \frac{p-2}{\|X\|^2}\right)X.$$

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A. J. Baranchik [1] proposes the positive-part of the James-Stein estimator dominating the James-Stein estimator when $p \geq 3$:

$$\delta_{JS}^+(X) = \max\left(0, 1 - \frac{p-2}{\|X\|^2}\right)X.$$

G. Casella and J. T. Hwang [3] studied the case where σ^2 is known and shown that if the limit of the ratio $\frac{\|\theta\|^2}{p}$, when p tends to infinity is a constant $c > 0$, then

$$\lim_{p \rightarrow +\infty} \frac{R(\theta, \delta_{JS}(X))}{R(\theta, X)} = \lim_{p \rightarrow +\infty} \frac{R(\theta, \delta_{JS}^+(X))}{R(\theta, X)} = \frac{c}{1+c}, \quad c > 0.$$

Li Sun [6] has considered the following model: $(y_{ij}|\theta_j, \sigma^2) \sim N(\theta_j, \sigma^2)$ $i = 1, \dots, n$, $j = 1, \dots, m$ where $E(y_{ij}) = \theta_j$ for the group j and $Var(y_{ij}) = \sigma^2$ is unknown. The James-Stein estimators is written in this case

$$\delta^{JS} = (\delta_1^{JS}, \dots, \delta_m^{JS})^t \quad \text{avec} \quad \left(1 - \frac{(m-3)S^2}{(N+2)T^2}\right)(\bar{y}_i - \bar{y}) + \bar{y}, \quad j = 1, \dots, m,$$

where

$$S^2 = \sum_{i=1}^n \sum_{j=1}^m (y_{ij} - \bar{y}_j), \quad T^2 = n \sum_{j=1}^m (\bar{y}_j - \bar{y}), \quad \bar{y}_j = \frac{\sum_{i=1}^n y_{ij}}{n}, \quad \bar{y} = \frac{\sum_{j=1}^m \bar{y}_j}{m},$$

$N = (n-1)m$, he gives a lower bound for the ratio $\frac{R(\theta, \delta_{JS}(X))}{R(\theta, X)}$, that we find in (10). We give in (14) an upper bound of the same ratio.

It shows after that if: $q = \frac{\lim_{m \rightarrow +\infty} \sum_{j=1}^m (\theta_j - \bar{\theta})^2}{m}$ exists, then $\lim_{m \rightarrow +\infty} \frac{R(\theta, \delta^{JS}(X))}{R(\theta, X)} = \lim_{m \rightarrow +\infty} \frac{R(\theta, \delta^{JS}(X))}{R(\theta, X)} = \frac{q}{q + \frac{\sigma^2}{n}}$.

In Section 2 we recall a lemma of G. Casella and J. T. Hwang [3] which we generalize if σ^2 is unknown and a technical lemma to calculate a lower bound for the ratio $\frac{R(\theta, \delta^{JS}(X))}{R(\theta, X)}$. We give, indeed, another demonstration of the limit of the ratio $\frac{R(\theta, \delta^{JS}(X))}{R(\theta, X)}$ when $p \rightarrow \infty$, because this last enabled us to more easily deduce the limit from the ratio $\frac{R(\theta, \delta^{JS}(X))}{R(\theta, X)}$ when n and p tend simultaneously to infinity.

We generalize the results of G. Casella and J. T. Hwang [3] in Section 3 (case where σ^2 is unknown), by giving a lower bound and an upper bound of ratio $\frac{R(\theta, \delta^{JS}(X))}{R(\theta, X)}$, and its limit when p tends to infinity and n fixes on the one hand, and on the other hand, when n and p tend simultaneously to infinity and this by supposing that $\lim_{p \rightarrow +\infty} \frac{\|\theta\|^2}{p\sigma^2} = c > 0$.

In Section 4, we give lower and upper bounds of the ratio $\frac{R(\theta, \delta_{JS}^+(X))}{R(\theta, X)}$ and its limit when p tends to infinity and n fixes on the one hand, and on the other hand, when n and p tend simultaneously to infinity, and this in all the cases where σ^2 is unknown.

We take as estimator of σ^2 the statistics S^2 independent of X and of law $\sigma^2 \chi_n^2$ in IR^+ . In this case the James-Stein estimator and his positive-part are written respectively

$$\delta_{JS}(X) = \left(1 - \frac{(p-2)S^2}{(n+2)\|X\|^2}\right)X \tag{1}$$

$$\delta_{JS}^+(X) = \max\left(0, 1 - \frac{(p-2)S^2}{(n+2)\|X\|^2}\right)X. \tag{2}$$

In Section 5, we give a graphic illustration of different ratios of risks and the lower and upper bounds associated for various values of n and p .

2. Preliminary

Let us recall that the risk of the maximum likelihood estimator X is $p\sigma^2$ the risk of the James Stein estimator (given in (1) of θ is

$$R(\theta, \delta_{J.S}(X)) = \sigma^2 \left\{ p - \frac{n}{n+2}(p-2)^2 E \left(\frac{1}{p-2+2K} \right) \right\},$$

where $K \sim P\left(\frac{\|\theta\|^2}{2\sigma^2}\right)$ being the law of Poisson of parameter $\frac{\|\theta\|^2}{2\sigma^2}$.

When $X \sim N_p(\theta, I_p)$ G. Casella and J. T. Hwang [3] have given the lemma 1 which expresses the following inequalities:

$$\frac{1}{(p-2+\|\theta\|^2)} \leq E \left(\frac{1}{\|X\|^2} \right) \leq \frac{p}{(p-2)(p+\|\theta\|^2)}, p \geq 3.$$

In the following lemma, we generalise this result when $X \sim N_p(\theta, \sigma^2 I_p)$ and σ^2 is unknown.

Lemma 2.1. *Let $X \sim N_p(\theta, \sigma^2 I_p)$. If $p \geq 3$ then*

$$\frac{1}{\sigma^2(p-2+\frac{\|\theta\|^2}{\sigma^2})} \leq E \left(\frac{1}{\|X\|^2} \right) \leq \frac{p}{\sigma^2(p-2)(p+\frac{\|\theta\|^2}{\sigma^2})}.$$

Proof. We have

$$\begin{aligned} X \sim N_p(\theta, \sigma^2 I_p) &\Rightarrow \frac{X}{\sigma} \sim N_p\left(\frac{\theta}{\sigma}, I_p\right) \Rightarrow \frac{1}{\sigma^2} \|X\|^2 \sim \chi_p^2\left(\frac{\|\theta\|^2}{\sigma^2}\right) \\ E \left(\frac{1}{\|X\|^2} \right) &= \frac{1}{\sigma^2} E \left(\frac{1}{\|\frac{X}{\sigma}\|^2} \right) = \frac{1}{\sigma^2} \sum_{k \geq 0} \int_0^{+\infty} \frac{1}{\omega} \chi_{p+2k}^2(d\omega) \pi \left(\frac{\|\theta\|^2}{2\sigma^2} \right) \\ &= \frac{1}{\sigma^2} E \left(\frac{1}{p-2+2K} \right) \end{aligned} \tag{3}$$

where $K \sim P\left(\frac{\|\theta\|^2}{2\sigma^2}\right)$ and (3) being the definition of the law of χ^2 no centred. According to the inequality of Jensen we have

$$E \left(\frac{1}{\|X\|^2} \right) = \frac{1}{\sigma^2} E \left(\frac{1}{p-2+2K} \right) \geq \frac{1}{\sigma^2(p-2+\frac{\|\theta\|^2}{\sigma^2})}.$$

In the other hand, for any real function h such that $E(h)\chi_q^2((\lambda))\chi_q^2(\lambda)$ exists (see G. Casella [2]), we have

$$E(h(\chi_q^2(\lambda))\chi_q^2(\lambda)) = qE(h(\chi_{q+2}^2(\lambda))) + 2\lambda E(h(\chi_{q+4}^2(\lambda))) \tag{4}$$

for $q = p-2$, $h(\omega) = \frac{1}{\omega}$, $\lambda = \frac{\|\theta\|^2}{2\sigma^2}$ we obtain

$$\int_0^{+\infty} \chi_{p-2}^2(\lambda, d\omega) = (p-2) \int_0^{+\infty} \frac{1}{\omega} \chi_p^2(\lambda, d\omega) + \frac{\|\theta\|^2}{\sigma^2} \int_0^{+\infty} \frac{1}{\omega} \chi_{p+2}^2(\lambda, d\omega).$$

Then

$$1 = (p-2) E \left(\frac{1}{\|\frac{X}{\sigma}\|^2} \right) + \frac{\|\theta\|^2}{\sigma^2} E \left(\frac{1}{p+2K} \right) \text{ with } K \sim P \left(\frac{\|\theta\|^2}{2\sigma^2} \right). \tag{5}$$

Thus

$$\sigma^2 E\left(\frac{1}{\|X\|^2}\right) = \frac{1}{p-2} \left[1 - \frac{\|\theta\|^2}{\sigma^2} E\left(\frac{1}{p+2K}\right)\right].$$

Hence

$$E\left(\frac{1}{\|X\|^2}\right) \leq \frac{1}{\sigma^2} \frac{1}{p-2} \left[1 - \frac{\|\theta\|^2}{\sigma^2} \frac{1}{p + \frac{\|\theta\|^2}{\sigma^2}}\right]. \tag{6}$$

Thus

$$E\left(\frac{1}{\|X\|^2}\right) \leq \frac{1}{\sigma^2(p-2)} \left[\frac{p}{p + \frac{\|\theta\|^2}{\sigma^2}}\right].$$

The equality (5) came from $E(\chi_{p-2}^2(\lambda)) = 1$, with $K \sim P\left(\frac{\|\theta\|^2}{2\sigma^2}\right)$ and the inequality (6) came from Jensen's inequality. Hence the result. \square

We recall that if X is of a random variable of multidimensional Gaussian law $X \sim N_p(\theta, \sigma^2 I_p)$ in IR^p , then $U = \|X\|^2 \sim \chi_p^2(\lambda)$ where $\chi_p^2(\lambda)$ designates a noncentral χ^2 distribution with p degrees of freedom and noncentrality parameter $\lambda (= \frac{\|\theta\|^2}{\sigma^2})$.

Lemma 2.2. *Let f be a real function defined on IR and X a random variable of multidimensional Gaussian law $X \sim N_p(\theta, \sigma^2 I_p)$ in IR^p . If for $p \geq 1$, $E[(f(U)\chi_p^2(\lambda))]$ exists, then:*

a) *If f nonincreasing we have*

$$E[(f(U)\chi_{p+2}^2(\lambda))] \leq E[(f(U)\chi_p^2(\lambda))]. \tag{7}$$

b) *If f nondecreasing we have:*

$$E[(f(U)\chi_{p+2}^2(\lambda))] \geq E[(f(U)\chi_p^2(\lambda))]. \tag{8}$$

Proof. a) We have

$$E[(f(U)\chi_{p+2}^2(\lambda))] - E[(f(U)\chi_p^2(\lambda))] = E\left[f(U)\left(\frac{u}{(p+2K)} - 1\right)\chi_p^2(\lambda)\right] E[f(U)\chi_p^2(\lambda)] E\left[\left(\frac{u}{(p+2K)} - 1\right)\chi_p^2(\lambda)\right]$$

because the covariance of two functions one increasing and the other decreasing is negative or null, with $K \sim P\left(\frac{\|\theta\|^2}{2\sigma^2}\right)$. However

$$E\left[\left(\frac{u}{(p+2K)} - 1\right)\chi_p^2(\lambda)\right] = 0,$$

then

$$E[(f(U)\chi_{p+2}^2(\lambda))] \leq E[(f(U)\chi_p^2(\lambda))].$$

Hence the result a), (in the same manner we get b). Thus the result. \square

3. Bounds and limit of the ratio of the risks of James-Stein estimator to the maximum likelihood estimator

Theorem 3.1. *If $\lim_{p \rightarrow +\infty} \frac{\|\theta\|^2}{p\sigma^2} = c > 0$, then*

$$\lim_{p \rightarrow +\infty} \frac{R(\theta, \delta_{J.S}(X))}{R(\theta, X)} = \frac{\frac{2}{n+2} + c}{1 + c}.$$

Proof. We have

$$R(\theta, \delta_{J.S}(X)) = \sigma^2 \left\{ p - \frac{n}{n+2} (p-2)^2 E \left(\frac{1}{p-2+2K} \right) \right\} \tag{9}$$

where $K \sim P\left(\frac{\|\theta\|^2}{2\sigma^2}\right)$. Then

$$\begin{aligned} \frac{R(\theta, \delta_{J.S}(X))}{R(\theta, X)} &= 1 - \frac{n}{n+2} \frac{(p-2)^2}{p} E \left(\frac{1}{p-2+2K} \right) \\ &= 1 - \frac{n}{n+2} \frac{\sigma^2 (p-2)^2}{p} E \left(\frac{1}{\|X\|^2} \right) \\ &\leq 1 - \frac{n}{n+2} \frac{(p-2)^2}{p} \frac{1}{\left(p-2 + \frac{\|\theta\|^2}{\sigma^2}\right)} \end{aligned} \tag{10}$$

$$\leq 1 - \frac{n}{n+2} \frac{(p-2)^2}{p^2} \frac{1}{\frac{p-2}{p} + \frac{\|\theta\|^2}{p\sigma^2}}. \tag{11}$$

The inequality (10) is obtained from lemma 2.1. Hence

$$\begin{aligned} \lim_{p \rightarrow +\infty} \frac{R(\theta, \delta_{J.S}(X))}{R(\theta, X)} &\leq \lim_{p \rightarrow +\infty} \left\{ 1 - \frac{n}{n+2} \frac{(p-2)^2}{p^2} \frac{1}{\frac{p-2}{p} + \frac{\|\theta\|^2}{p\sigma^2}} \right\} \\ &\leq 1 - \frac{n}{(n+2)(1+c)} \\ &\leq \frac{(n+2)(1+c) - n}{(n+2)(1+c)} \end{aligned} \tag{12}$$

$$\leq \frac{\frac{2}{n+2} + c}{1 + c}. \tag{13}$$

Where $\lim_{p \rightarrow +\infty} \frac{\|\theta\|^2}{p\sigma^2} = c$. In the other hand

$$\begin{aligned} \frac{R(\theta, \delta_{J.S}(X))}{R(\theta, X)} &= 1 - \frac{n}{n+2} \frac{(p-2)^2}{p} E \left(\frac{1}{p-2+2K} \right) \\ &= 1 - \frac{n}{n+2} \frac{(p-2)^2}{p} \sigma^2 E \left(\frac{1}{\|X\|^2} \right). \end{aligned}$$

According to lemma 2.1 we have

$$\begin{aligned} \frac{R(\theta, \delta_{J.S}(X))}{R(\theta, X)} &\geq 1 - \frac{n}{n+2} \frac{(p-2)^2}{(p-2)p} \frac{p}{\left(p + \frac{\|X\|^2}{\sigma^2}\right)} \\ &\geq 1 - \frac{n}{n+2} \frac{(p-2)^2}{p} \frac{1}{\left(1 + \frac{\|X\|^2}{p\sigma^2}\right)}. \end{aligned} \tag{14}$$

Thus we obtain lower and upper bounds of the ratio $\frac{R(\theta, \delta_{JS}(X))}{R(\theta, X)}$ deduced from (10) and (14)

$$1 - \frac{n}{n+2} \frac{(p-2)}{p} \frac{1}{\left(1 + \frac{\|\theta\|^2}{p\sigma^2}\right)} \leq \frac{R(\theta, \delta_{JS}(X))}{R(\theta, X)} \leq 1 - \frac{n}{n+2} \frac{(p-2)^2}{p} \frac{1}{\left(p-2 + \frac{\|\theta\|^2}{\sigma^2}\right)}. \tag{15}$$

Passaging to the limit when $p \rightarrow +\infty$ we obtain

$$\begin{aligned} \lim_{p \rightarrow +\infty} \frac{R(\theta, \delta_{JS}(X))}{R(\theta, X)} &\geq \lim_{p \rightarrow +\infty} \left\{ 1 - \frac{n}{n+2} \frac{(p-2)}{p} \frac{1}{\left(1 + \frac{\|\theta\|^2}{p\sigma^2}\right)} \right\} \\ &\geq 1 - \frac{n}{(n+2)(1+c)} \\ &\geq \frac{\frac{2}{n+2} + c}{1+c} \end{aligned} \tag{16}$$

where $\lim_{p \rightarrow +\infty} \frac{\|\theta\|^2}{p\sigma^2} = c$. Combining (13) and (16) we obtain

$$\lim_{p \rightarrow +\infty} \frac{R(\theta, \delta_{JS}(X))}{R(\theta, X)} = \frac{\frac{2}{n+2} + c}{1+c}.$$

Hence the result. \square

Corollary 3.2. *If $\lim_{p \rightarrow +\infty} \frac{\|\theta\|^2}{p\sigma^2} = c > 0$, we have*

$$\lim_{n, p \rightarrow +\infty} \frac{R(\theta, \delta_{JS}(X))}{R(\theta, X)} = \frac{c}{1+c}.$$

Proof. It is obtained immediately from (11) and (14). \square

4. Bounds and limit of the ratio of the risks of the positive-part James-Stein estimator to the maximum likelihood estimator

The results of the positive -part James-Stein estimator $\delta_{JS}^+(X)$ are similar to the results obtained on the James-Stein estimator $\delta_{JS}(X)$.

Indeed, we denote $\alpha = \frac{p-2}{n+2}$, and recall that:

$$\begin{aligned} \delta_{JS}^+(X) &= \left(1 - \alpha \frac{S^2}{\|X\|^2}\right)^+ X = \phi_{JS}^+(X) X \left(1 - \alpha \frac{S^2}{\|X\|^2}\right) XI_{\left(\alpha \frac{S^2}{\|X\|^2} \leq 1\right)} \\ \delta_{JS}^-(X) &= \left(1 - \alpha \frac{S^2}{\|X\|^2}\right)^- X = \phi_{JS}^-(X) X \left(\alpha \frac{S^2}{\|X\|^2} - 1\right) XI_{\left(\alpha \frac{S^2}{\|X\|^2} \geq 1\right)} \end{aligned}$$

$I_{\left(\alpha \frac{S^2}{\|X\|^2} \geq 1\right)}$ showing the indicating function of the set $\left(\alpha \frac{S^2}{\|X\|^2} \geq 1\right)$.

We will denote for the needs for demonstration and each time that it will be necessary, for the sequel, by: $Y = \frac{X}{\sigma}, \beta = \frac{\theta}{\sigma}, T^2 = \frac{S^2}{\sigma^2}$ and $U = \|Y\|^2$.

Then we have $Y \sim N(\beta, I_p), T^2 \sim \chi_n^2$ and $U \sim \chi_p^2(\lambda)$ where χ_n^2 designates the law of the chi square centered with n degrees of freedom and $\chi_p^2(\lambda)$ designates a noncentral χ^2 distribution with p degrees of freedom and noncentrality parameter $\lambda \left(= \frac{\|\theta\|^2}{\sigma^2}\right)$.

The following lemma is a recall of the expression of the risk of $\delta_{JS}^+(X)$.

Lemma 4.1. We have that

$$R(\theta, \delta_{JS}^+(X)) = R(\theta, \delta_{JS}(X)) + E \left\{ \left[\|X\|^2 + 2(p-2)\sigma^2 \alpha \frac{S^2}{\|X\|^2} - \alpha^2 \frac{(S^2)^2}{\|X\|^2} - 2p\sigma^2 \right] I_{\alpha \frac{S^2}{\|X\|^2} \geq 1} \right\}. \quad (17)$$

Proof. We have

$$\begin{aligned} R(\theta, \delta_{JS}(X)) &= E \left(\|\delta_{JS}(X) - \theta\|^2 \right) \\ &= E \left(\|\phi_{JS}^+(X)X - \theta - \phi_{JS}^-(X)X\|^2 \right) \\ &= E \left(\|\phi_{JS}^+(X)X - \theta\|^2 \right) + E \left(\left[\phi_{JS}^-(X) \right]^2 \|X\|^2 \right) - 2E \left\{ \langle \phi_{JS}^+(X)X - \theta, \phi_{JS}^-(X)X \rangle \right\} \\ &= R(\theta, \delta_{JS}^+(X)) + E \left(\left[\phi_{JS}^-(X) \right]^2 \|X\|^2 \right) - 2E \left\{ \langle -\theta, \phi_{JS}^-(X)X \rangle \right\} \\ &= R(\theta, \delta_{JS}^+(X)) + E \left(\left[\phi_{JS}^-(X) \right]^2 \|X\|^2 \right) - 2E \left\{ \langle X - \theta, \phi_{JS}^-(X)X \rangle \right\} + 2E \left\{ \langle X, \delta_{JS}^-(X)X \rangle \right\}. \end{aligned}$$

Then

$$\begin{aligned} R(\theta, \delta_{JS}(X)) &= R(\theta, \delta_{JS}^+(X)) + E \left\{ \left(\left(\alpha \frac{S^2}{\|X\|^2} - 1 \right) I_{\alpha \frac{S^2}{\|X\|^2} \geq 1} \right)^2 \|X\|^2 \right\} \\ &\quad - 2E \left\{ {}^t(X - \theta)X \left(\left(\alpha \frac{S^2}{\|X\|^2} - 1 \right) I_{\alpha \frac{S^2}{\|X\|^2} \geq 1} \right) \right\} + 2E \left\{ \left(\left(\alpha \frac{S^2}{\|X\|^2} - 1 \right) I_{\alpha \frac{S^2}{\|X\|^2} \geq 1} \right) \|X\|^2 \right\}. \end{aligned}$$

According to the lemma of P. Shao and W. E. Strawderman ([7, Lemma 2.1]) we have,

$$\begin{aligned} &E \left\{ {}^t(X - \theta)X \left(\left(\alpha \frac{S^2}{\|X\|^2} - 1 \right) I_{\alpha \frac{S^2}{\|X\|^2} \geq 1} \right) \right\} \\ &= \sigma^2 E \left\{ \left(-2\alpha \frac{S^2}{\sigma^2 \|Y\|^2} + p \left(\alpha \frac{S^2}{\sigma^2 \|Y\|^2} - 1 \right) \right) I_{\alpha \frac{S^2}{\sigma^2 \|Y\|^2} \geq 1} \right\} \\ &= \sigma^2 E \left\{ \left((p-2)\alpha \frac{S^2}{\sigma^2 \|Y\|^2} - p \right) I_{\alpha \frac{S^2}{\sigma^2 \|Y\|^2} \geq 1} \right\} \\ &= \sigma^2 E \left\{ \left((p-2)\alpha \frac{S^2}{\sigma^2 \|Y\|^2} - p \right) I_{\alpha \frac{S^2}{\sigma^2 \|Y\|^2} \geq 1} \right\} \\ &= \sigma^2 E \left\{ \left((p-2)\alpha \frac{S^2}{\|X\|^2} - p \right) I_{\alpha \frac{S^2}{\|X\|^2} \geq 1} \right\} \end{aligned}$$

thus

$$\begin{aligned} R(\theta, \delta_{JS}(X)) &= R(\theta, \delta_{JS}^+(X)) + E \left\{ \left[\alpha^2 \frac{(S^2)^2}{\|X\|^2} - 2\alpha S^2 + \|X\|^2 \right] I_{\alpha \frac{S^2}{\|X\|^2} \geq 1} \right\} \\ &\quad - 2\sigma^2 E \left\{ \left((p-2)\alpha \frac{S^2}{\|X\|^2} - p \right) I_{\alpha \frac{S^2}{\|X\|^2} \geq 1} \right\} + 2E \left\{ \left(\left(\alpha \frac{S^2}{\|X\|^2} - 1 \right) I_{\alpha \frac{S^2}{\|X\|^2} \geq 1} \right) \|X\|^2 \right\}. \end{aligned}$$

Therefore

$$R(\theta, \delta_{JS}^+(X)) = R(\theta, \delta_{JS}(X)) + E \left\{ \left[\|X\|^2 + 2\sigma^2(p-2)\alpha \frac{S^2}{\|X\|^2} - \alpha^2 \frac{(S^2)^2}{\|X\|^2} - 2p\sigma^2 \right] I_{\alpha \frac{S^2}{\|X\|^2} \geq 1} \right\}.$$

Hence the result. \square

The fact that $R(\theta, \delta_{JS}^+(X)) \leq R(\theta, \delta_{JS}(X))$ (Baranchick [1]), an upper bound of the ratio $\frac{R(\theta, \delta_{JS}^+(X))}{R(\theta, X)}$ would be for example the upper bound of the ratio $\frac{R(\theta, \delta_{JS}(X))}{R(\theta, X)}$ that we know the asymptotic character. Thus we will interesting ourselves in what follows on a lower bound of the ratio $\frac{R(\theta, \delta_{JS}^+(X))}{R(\theta, X)}$.

We have the following result.

Theorem 4.2. For all $p \geq 3$, we have the following minoration of the ratio of the risks $\frac{R(\theta, \delta_{JS}^+(X))}{R(\theta, X)}$:

$$\begin{aligned} \frac{R(\theta, \delta_{JS}^+(X))}{p\sigma^2} &\geq \frac{R(\theta, \delta_{JS}(X))}{p\sigma^2} + \frac{(p + \lambda)}{p} \int_0^{+\infty} IP\left(\chi_n^2 \geq \frac{u}{\alpha}\right) \chi_{p+4}^2(\lambda, du) \frac{(\alpha n + 4)}{p} \int_0^{+\infty} IP\left(\chi_{n+4}^2 \geq \frac{u}{\alpha}\right) \chi_{p-2}^2(\lambda, du) \\ &= \frac{R(\theta, \delta_{JS}^+(X))}{p\sigma^2} + \frac{(p + \lambda)}{p} F_{(p+4, n, \lambda)}(\alpha) - \frac{(\alpha n + 4)}{p} F_{(p-2, n+4, \lambda)}(\alpha) \end{aligned} \tag{18}$$

where $F_{(p, n, \lambda)}(\alpha)$ is a function to distribution of Fisher with p and n degrees of freedom and parameter of noncentrality $\lambda (= \frac{\|\theta\|^2}{\sigma^2})$.

Proof. Taking again the equality (17) we have

$$R(\theta, \delta_{JS}^+(X)) = R(\theta, \delta_{JS}(X)) + E \left\{ \left[\|\|X\|^2 + 2(p - 2)\sigma^2 \alpha \frac{S^2}{\|\|X\|^2} - \alpha^2 \frac{(S^2)^2}{\|\|X\|^2} - 2p\sigma^2 \right] I_{\alpha \frac{S^2}{\|\|X\|^2} \geq 1} \right\}.$$

Then

$$\begin{aligned} I_1 &= E \left(\|\|X\|^2 I_{\alpha \frac{S^2}{\|\|X\|^2} \geq 1} \right) \\ &= \sigma^2 E \left(\frac{\|\|X\|^2}{\sigma^2} I_{\alpha \frac{S^2}{\|\|X\|^2} \geq 1} \right) \\ &= \sigma^2 \int_0^{+\infty} \left(\int_0^{+\infty} \chi_n^2(0, ds) \right) u \chi_p^2(\lambda, du). \end{aligned} \tag{19}$$

Taking $h(u) = \int_0^{+\infty} \chi_n^2(0, ds)$ and $q = p$, and applying (4) we obtain

$$\begin{aligned} I_1 &= \sigma^2 p \int_0^{+\infty} IP\left(\chi_n^2 \geq \frac{u}{\alpha}\right) \chi_{p+2}^2(\lambda, du) + \sigma^2 \lambda \int_0^{+\infty} IP\left(\chi_n^2 \geq \frac{u}{\alpha}\right) \chi_{p+4}^2(\lambda, du) \\ &\geq \sigma^2 (p + \lambda) \int_0^{+\infty} IP\left(\chi_n^2 \geq \frac{u}{\alpha}\right) \chi_{p+4}^2(\lambda, du). \end{aligned} \tag{20}$$

The inequality (20) comes from inequality (7) of lemma 2.2. On the other hand

$$\begin{aligned} I_2 &= \sigma^2 E \left(\left[2(p - 2)\alpha \frac{S^2}{\|\|X\|^2} - 2p \right] I_{\alpha \frac{S^2}{\|\|X\|^2} \geq 1} \right) \\ &\geq -4\sigma^2 E \left(I_{\alpha \frac{S^2}{\|\|X\|^2} \geq 1} \right) \\ &\geq -4\sigma^2 \int_0^{+\infty} \left(\int_0^{+\infty} \chi_n^2(0, ds) \right) \chi_p^2(\lambda, du) \\ &\geq -4\sigma^2 \int_0^{+\infty} IP\left(\chi_n^2 \geq \frac{u}{\alpha}\right) \chi_p^2(\lambda, du) \\ &\geq -4\sigma^2 \int_0^{+\infty} IP\left(\chi_{n+4}^2 \geq \frac{u}{\alpha}\right) \chi_{p-2}^2(\lambda, du). \end{aligned} \tag{21}$$

The inequality (21) comes from inequality (7) of lemma 2.2 and by observing that $IP\left(\chi_{n+4}^2 \geq \frac{u}{\alpha}\right) \geq IP\left(\chi_n^2 \geq \frac{u}{\alpha}\right)$ for all $n \geq 1$. On the other hand

$$\begin{aligned} I_3 &= -E\left(\alpha^2 \frac{(S^2)^2}{\|X\|^2} I_{\left(\alpha \frac{\sigma^2}{\|X\|^2} \geq 1\right)}\right) \\ I_3 &= -\alpha^2 \sigma^2 E\left(\frac{(T^2)^2}{U} I_{(T^2 \geq \frac{u}{\alpha})}\right) \\ &= -\alpha^2 \sigma^2 \int_0^{+\infty} \left(\int_{\frac{u}{\alpha}}^{+\infty} (t^2)^2 \chi_n^2(0, dt^2)\right) \frac{1}{u} \chi_p^2(\lambda, du) \\ &\geq -\frac{\sigma^2 \alpha}{(n+2)} \int_0^{+\infty} \left(\int_{\frac{u}{\alpha}}^{+\infty} (t^2)^2 \chi_n^2(0, dt^2)\right) \chi_{p-2}^2(\lambda, du). \end{aligned}$$

The last inequality comes from (4) while taking $h(u) = \frac{1}{u} \int_{\frac{u}{\alpha}}^{+\infty} (t^2)^2 \chi_n^2(0, ds)$, $q = p - 2$ and the fact that $\alpha = \frac{p-2}{n+2}$.

However

$$\begin{aligned} \int_{\frac{u}{\alpha}}^{+\infty} (t^2)^2 \chi_n^2(0, dt^2) &= n \int_{\frac{u}{\alpha}}^{+\infty} t^2 \chi_{n+2}^2(0, dt^2) \\ &= n(n+2) \int_{\frac{u}{\alpha}}^{+\infty} \chi_{n+4}^2(0, dt^2) \\ &= n(n+2) IP\left(\chi_{n+4}^2 \geq \frac{u}{\alpha}\right). \end{aligned} \tag{22}$$

Then we have

$$I_3 \geq -\sigma^2 \alpha n \int_0^{+\infty} IP\left(\chi_{n+4}^2 \geq \frac{u}{\alpha}\right) \chi_{p-2}^2(\lambda, du). \tag{23}$$

According to (14), (20), (21) and (23) we have

$$\begin{aligned} \frac{R(\theta, \delta_{J.S}^+(X))}{p\sigma^2} &= \frac{R(\theta, \delta_{J.S}(X))}{p\sigma^2} + \frac{I_1}{p\sigma^2} + \frac{I_2}{p\sigma^2} + \frac{I_3}{p\sigma^2} \\ &\geq \frac{R(\theta, \delta_{J.S}(X))}{p\sigma^2} + \frac{\sigma^2(p+\lambda)}{p\sigma^2} \int_0^{+\infty} \left(IP\left(\chi_n^2 \geq \frac{u}{\alpha}\right)\right) \chi_{p+4}^2(\lambda, du) \\ &\quad - \frac{4\sigma^2}{p\sigma^2} \int_0^{+\infty} \left(IP\left(\chi_{n+4}^2 \geq \frac{u}{\alpha}\right)\right) \chi_{p-2}^2(\lambda, du) - \frac{\sigma^2 \alpha n}{p\sigma^2} \int_0^{+\infty} IP\left(\chi_{n+4}^2 \geq \frac{u}{\alpha}\right) \chi_{p-2}^2(\lambda, du) \\ &\geq \frac{R(\theta, \delta_{J.S}(X))}{p\sigma^2} + \frac{(p+\lambda)}{p} \int_0^{+\infty} IP\left(\chi_n^2 \geq \frac{u}{\alpha}\right) \chi_{p+4}^2(\lambda, du) - \frac{(\alpha n + 4)}{p} \int_0^{+\infty} IP\left(\chi_{n+4}^2 \geq \frac{u}{\alpha}\right) \chi_{p-2}^2(\lambda, du). \end{aligned} \tag{24}$$

Thus the result. \square

Remark 4.3. The lower bound given by the inequality (18) is a bound enough “fine” and close to the ratio $\frac{R(\theta, \delta_{J.S}^+(X))}{p\sigma^2}$ as soon as λ moves away from zero. The simulations in Section 5 illustrates it rather well. On the other hand for the passage to limit of the ratio $\frac{R(\theta, \delta_{J.S}^+(X))}{p\sigma^2}$ when p tends to infinity or when p and n tend simultaneously to infinity, a less fine bound min would be enough. Then we have the next result on the limit of the ratio of the risks $\frac{R(\theta, \delta_{J.S}^+(X))}{R(\theta, X)}$.

Proposition 4.4. If $\lim_{p \rightarrow +\infty} \frac{\|0\|^2}{p\sigma^2} = c > 0$, we have:

$$\lim_{p \rightarrow +\infty} \frac{R(\theta, \delta_{J.S}^+(X))}{R(\theta, X)} = \frac{\frac{2}{n+2} + c}{1 + c}.$$

Proof. Baranchick [1] showed that $R(\theta, \delta_{JS}^+(X)) \leq R(\theta, \delta_{JS}(X))$ for $p \geq 3$ and all $\theta, \sigma \in IR^p IR^+$. Thus the upper bound in (11) plays the role of the upper bound of $\frac{R(\theta, \delta_{JS}^+(X))}{R(\theta, X)}$. It is enough to show that the limit of the lower bound is higher or equal to $\frac{\frac{2}{n+2}+c}{1+c}$. Indeed according to (24) we have

$$\begin{aligned} \frac{R(\theta, \delta_{JS}^+(X))}{R(\theta, X)} &\geq \frac{R(\theta, \delta_{JS}(X))}{p\sigma^2} + \frac{(p + \lambda)}{p} \int_0^{+\infty} IP\left(\chi_n^2 \geq \frac{u}{\alpha}\right) \chi_{p+4}^2(\lambda, du) \\ &\quad - \frac{(\alpha n + 4)}{p} \int_0^{+\infty} IP\left(\chi_{n+4}^2 \geq \frac{u}{\alpha}\right) \chi_{p-2}^2(\lambda, du) \\ \frac{R(\theta, \delta_{JS}^+(X))}{R(\theta, X)} &\geq \frac{R(\theta, \delta_{JS}(X))}{p\sigma^2} + \int_0^{+\infty} IP\left(\chi_n^2 \geq \frac{u}{\alpha}\right) \chi_{p+4}^2(\lambda, du) - \frac{4}{p} - 1 \end{aligned} \tag{25}$$

because

$$\begin{aligned} \frac{(p + \lambda)}{p} \int_0^{+\infty} IP\left(\chi_n^2 \geq \frac{u}{\alpha}\right) \chi_{p+4}^2(\lambda, du) &\geq \int_0^{+\infty} IP\left(\chi_n^2 \geq \frac{u}{\alpha}\right) \chi_{p+4}^2(\lambda, du) \\ &\quad - \int_0^{+\infty} IP\left(\chi_n^2 \geq \frac{u}{\alpha}\right) \geq -1 \quad \text{and} \quad -\frac{n(p-2)}{p(n+2)} \geq -1. \end{aligned}$$

Let us denote that like $\alpha = \frac{p-2}{n+2}$ and thus tends to $+\infty$ when $p \rightarrow +\infty$, we have according to the theorem of Lebesgue by taking for example, the increasing sequel with $p \left(f_p(u) = \int_{\frac{u}{\alpha}}^{+\infty} \chi_n^2(0, ds) = IP\left(\chi_n^2 \geq \frac{u}{\alpha}\right) \right)$

$$\lim_{p \rightarrow +\infty} IP\left(\chi_n^2 \geq \frac{u}{\alpha}\right) = IP(\chi_n^2 \geq 0) = 1, \quad \forall n \geq 1 \tag{26}$$

thus

$$\lim_{p \rightarrow +\infty} \int_0^{+\infty} IP\left(\chi_n^2 \geq \frac{u}{\alpha}\right) \chi_{p+4}^2(\lambda, du) = 1$$

where $\lambda = \frac{\|\theta\|^2}{\sigma^2}$. Finally we obtains

$$\lim_{p \rightarrow +\infty} \frac{R(\theta, \delta_{JS}^+(X))}{R(\theta, X)} \geq \frac{\frac{2}{n+2} + c}{1 + c}.$$

Thus the result. \square

The case where n and p tend simultaneously to $+\infty$ is given in the following theorem.

Theorem 4.5. *If $\lim_{p \rightarrow +\infty} \frac{\|\theta\|^2}{p\sigma^2} = c > 0$, we have*

$$\lim_{n, p \rightarrow +\infty} \frac{R(\theta, \delta_{JS}^+(X))}{R(\theta, X)} = \frac{c}{1 + c}.$$

Proof. On the one hand, we have

$$\lim_{n, p \rightarrow +\infty} \frac{R(\theta, \delta_{JS}^+(X))}{R(\theta, X)} \leq \lim_{n, p \rightarrow +\infty} \frac{R(\theta, \delta_{JS}(X))}{R(\theta, X)} = \frac{c}{1 + c} \tag{27}$$

because $R(\theta, \delta_{JS}^+(X)) \leq R(\theta, \delta_{JS}(X))$ for $p \geq 3$ and all $\theta, \sigma \in IR^p IR^+$. In the other hand, by beginning again (25) we have

$$\lim_{n,p \rightarrow +\infty} \frac{R(\theta, \delta_{JS}^+(X))}{R(\theta, X)} \geq \frac{c}{1+c} + \lim_{n,p \rightarrow +\infty} \int_0^{+\infty} \left(IP \left(\chi_n^2 \geq \frac{u}{\alpha} \right) \right) \chi_{p+4}^2(\lambda, du) - 1. \tag{28}$$

However

$$\begin{aligned} IP \left(\chi_n^2 \geq \frac{u}{\alpha} \right) &= IP \left(\sum_{i=1}^n y_i^2 \geq \frac{u(n+2)}{(p-2)} \right) \\ &= IP \left(\frac{\sum_{i=1}^n y_i^2}{n} \geq \frac{u}{(p-2)} + \frac{2u}{n(p-2)} \right) \end{aligned}$$

where y_1, y_2, \dots, y_n are independent, Gaussian random variables centered reduced. Thus by the strong law of the great numbers we have

$$\begin{aligned} \lim_{n,p \rightarrow +\infty} IP \left(\chi_n^2 \geq \frac{u}{\alpha} \right) &= \lim_{n,p \rightarrow +\infty} IP \left(\frac{\sum_{i=1}^n y_i^2}{n} \geq \frac{u}{(p-2)} + \frac{2u}{n(p-2)} \right) \\ &= \lim_{n,p \rightarrow +\infty} IP \left(\frac{\sum_{i=1}^n y_i^2}{n} \geq 0 \right) \\ &= IP(1 \geq 0) = 1 \end{aligned}$$

ainsi

$$\lim_{n,p \rightarrow +\infty} \int_0^{+\infty} \left(IP \left(\chi_n^2 \geq \frac{u}{\alpha} \right) \right) \chi_{p+4}^2(\lambda, du) = \int_0^{+\infty} \chi_{p+4}^2(\lambda, du) = 1. \tag{29}$$

Combining (27), (28) and (29), there is finally the result. \square

5. Simulations

We illustrate graphically in what follows the ratios of the risks $\frac{R(\theta, \delta_{JS}(X))}{R(\theta, X)}$, $\frac{R(\theta, \delta_{JS}^+(X))}{R(\theta, X)}$ as well as the evolution of the bound min and max associated, given respectively by the expressions (15) and (18), for $\sigma^2 = 1$ and different values of n and p .

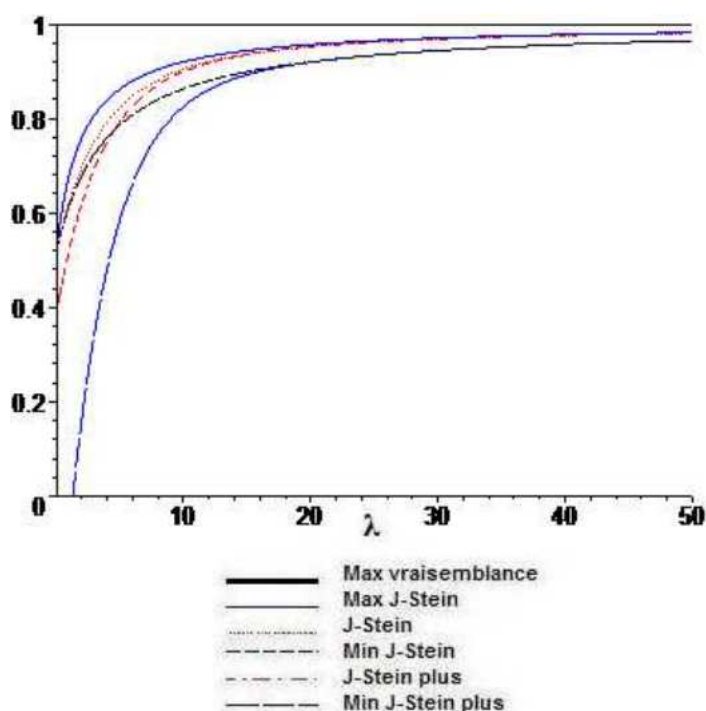


Figure 1: Graph of the relative risks and their minima and maxima for $n = 50$ and $p = 4$, according to $\lambda = \frac{\|\theta\|^2}{\sigma^2}$ and $\sigma^2 = 1$.

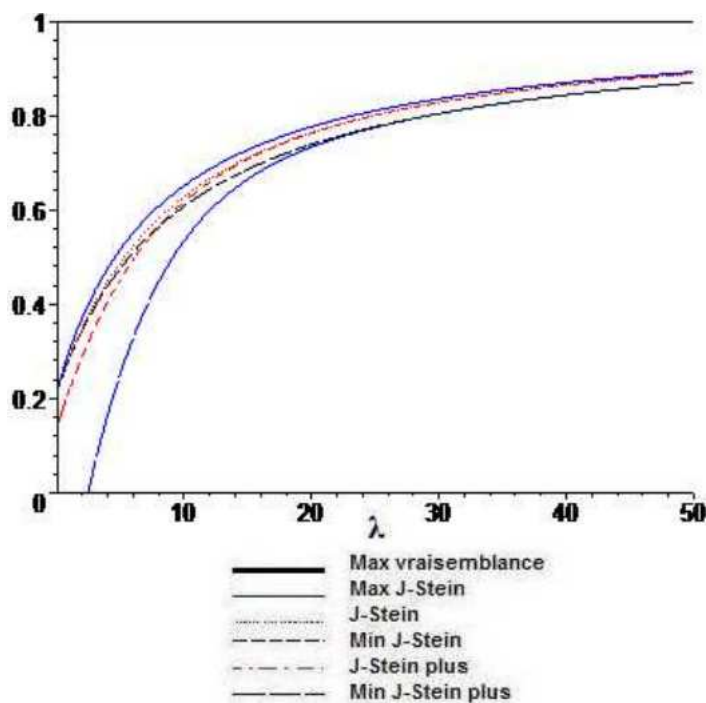


Figure 2: Graph of the relative risks and their minima and maxima for $n = 100$ and $p = 10$, according to $\lambda = \frac{\|\theta\|^2}{\sigma^2}$ and $\sigma^2 = 1$.

Conclusion

In the case of the estimate of the average θ of a multidimensional Gaussian law $N_p(\theta, I_p)$ in IR^p , Casella, G and J, T, Tzon Hwang [3] showed that if $\lim_{p \rightarrow +\infty} \frac{\|\theta\|^2}{p} = c_1 > 0$ then the ratio $\frac{R(\theta, \delta_{JS}^+(X))}{R(\theta, X)}$ tends to $\frac{c_1}{1+c_1}$. By taking the same model, namely $X \sim N_p(\theta, \sigma^2 I_p)$ with this time σ^2 unknown, and estimated by the statistic S^2 independent of X and of law $\sigma^2 \chi_n^2$ in IR^+ . We have for our part, showed that we obtain a similar ratio depend on the size n of the sample, with that found by the latter, as soon as $\lim_{p \rightarrow +\infty} \frac{\|\theta\|^2}{p\sigma^2} = c > 0$. Moreover we obtain a ratio constant and independent of n , when n and p tend simultaneously to $+\infty$ and this without taking account of an unspecified relation of order or functional calculus between n and p . Li Sun. [6] is him also interested if σ^2 is unknown, but studied the behavior of the ratio $\frac{R(\theta, \delta_{JS}^+(X))}{R(\theta, X)}$ and $\frac{R(\theta, \delta_{JS}(X))}{R(\theta, X)}$, when only p tends to infinity. An idea would be to see whether we obtain similar ratios in the general case of the symmetrical spherical models.

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