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Limit of the ratio of risks of James-Stein estimators with unknown variance

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Abstract. We study the estimate of the mean θ of a Gaussian random variable $X \sim N_p(\theta, \sigma^2 I_p)$ in IR^p, σ^2 unknown and estimated by the chi-square variable S^2 ($S^2 \sim \sigma^2\chi^2_n$). We particularly study the bounds and limits of the ratios of the risks, of the James-Stein estimator $\delta_{JS}(X)$ and of its positive-part $\delta^+_{JS}(X)$, with that of the maximum likelihood estimator X when $p \to \infty$. If $\lim_{p\to+\infty} \frac{\|\theta\|^2}{p\sigma^2} = c$, we show that the ratios of the risks of the James-Stein estimator $\delta_{J.S}(X)$ and its positive-part $\delta^+_{J.S}(X)$, with that of the maximum likelihood estimator X tend to the same value $\frac{\frac{2}{n+2}+c}{1+c}$ when $p \to \infty$. If n and p tend to infinity we show that the ratios of the risks tend to $\frac{c}{1+c}$ We graphically illustrate the ratios of the risks corresponding to the James-Stein estimators $\delta_{J.S}(X)$ and its positive-part $\delta^+_{j.}(X)$, with that of the maximum likelihood estimator X for diverse values of n and p.

1. Introduction

The estimate of the average θ of a multidimensional Gaussian law $N_p(\theta, \sigma^2 I_p)$ in IR^p, I_p means the matrix unit, has known many developments since the articles of C. Stein [7], [8] and W. James and C. Stein [5].

We cite also others works and generalizations papers [2–4]. In these works one has estimated the average of a Multidimensional Gaussian law $N_p(\theta, \sigma^2 I_p)$ in IR^p by estimators with retrecissor deduced from the empirical average those are proved better in quadratic cost than the empirical average. These studies for the majority were made when σ^2 is known.

More precisely, if X represents an observation or a sample of multidimensional Gaussian law $N_p(\theta, \sigma^2 I_p)$, the aim is to estimate θ by an estimator $\delta(X)$ relatively at the quadratic cost:

$$L(\delta, \theta) = \|\delta(X) - \theta\|^{\theta}_p,$$

where $\|\cdot\|_p$ is the usual norm in IR^p. We associate his function of risk:

$$R(\delta, \theta) = E_\theta(L(\delta, \theta)).$$

W. James and C. Stein [8], have introduced a class of James-Stein estimators improving $\delta(X) = X$, when the dimension of the space of the observations p is ≥ 3, noted

$$\delta_{JS}(X) = \left(1 - \frac{p-2}{\|X\|^2}\right)X.$$

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A. J. Baranchik [1] proposes the positive-part of the James-Stein estimator dominating the James-Stein estimator when $p \geq 3$:

$$\delta_{JS}^{+}(X) = \max\left(0, 1 - \frac{p-2}{\|X\|^2}\right) X.$$

G. Casella and J. T. Hwang [3] studied the case where σ^2 is known and shown that if the limit of the ratio $\frac{\|\theta\|^2}{p}$, when p tends to infinity is a constant $c > 0$, then

$$\lim_{p\to+\infty} \frac{R(\theta, \delta_{J.S}(X))}{R(\theta, X)} = \lim_{p\to+\infty} \frac{R(\theta, \delta_{J.S}^{+}(X))}{R(\theta, X)} = \frac{c}{1+c}, \quad c > 0.$$

Li Sun [6] has considered the following model: $(y_{ij}|\theta_j, \sigma^2) \sim N(\theta_j, \sigma^2)$ $i = 1, \ldots, n$, $j = 1, \ldots, m$ where $E(y_{ij}) = \theta_j$ for the group j and $Var(y_{ij}) = \sigma^2$ is unknown. The James-Stein estimators is written in this case

$$\delta^{JS} = (\delta_1^{JS}, \ldots, \delta_m^{JS})^t \quad \text{avec} \quad \left(1 - \frac{(m-3)S^2}{(N+2)T^2}\right)(\bar{y}_i - \bar{y}) + \bar{y},\ j = 1, \ldots, m,$$

where

$$S^2 = \sum_{i=1}^{n} \sum_{j=1}^{m} (y_{ij} - \bar{y}_j),\ T^2 = n \sum_{j=1}^{m} (\bar{y}_j - \bar{y}),\ \bar{y}_j = \frac{\sum_{i=1}^{n} y_{ij}}{n}, \bar{y} = \frac{\sum_{j=1}^{m} y_j}{m},$$

$N = (n-1)m$, he gives a lower bound for the ratio $\frac{R(\theta,\delta_{J.S}(X))}{R(\theta,X)}$, that we find in (10). We give in (14) an upper bound of the same ratio.

It shows after that if: $q = \frac{lim_{m\to+\infty} \sum_{j=1}^{m} (\theta_j - \bar{\theta})^2}{m}$ exists, then $lim_{m\to+\infty} \frac{R(\theta,\delta^{JS}(X))}{R(\theta,X)} = \lim_{m\to+\infty} \frac{R(\theta,\delta^{JS}(X))}{R(\theta,X)} = \frac{q}{q+\frac{\sigma^2}{n}}$.

In Section 2 we recall a lemma of G. Casella and J. T. Hwang [3] which we generalize if σ^2 is unknown and a technical lemma to calculate a lower bound for the ratio $\frac{R(\theta,\delta^{JS}(X))}{R(\theta,X)}$. We give, indeed, another demonstration of the limit of the ratio $\frac{R(\theta,\delta^{JS}(X))}{R(\theta,X)}$ when $p \to \infty$, because this last enabled us to more easily deduce the limit from the ratio $\frac{R(\theta,\delta^{JS}(X))}{R(\theta,X)}$ when n and p tend simultaneously to infinity.

We generalize the results of G. Casella and J. T. Hwang [3] in Section 3 (case where σ^2 is unknown), by giving a lower bound and an upper bound of ratio $\frac{R(\theta,\delta^{JS}(X))}{R(\theta,X))}$, and its limit when p tends to infinity and n fixes on the one hand, and on the other hand, when n and p tend simultaneously to infinity and this by supposing that $\lim_{p\to+\infty} \frac{\|\theta\|^2}{p\sigma^2} = c > 0$.

In Section 4, we give lower and upper bounds of the ratio $\frac{R(\theta,\delta_{J.S}^{+}(X))}{R(\theta,X))}$ and its limit when p tends to infinity and n fixes on the one hand, and on the other hand, when n and p tend simultaneously to infinity, and this in all the cases where σ^2 is unknown.

We take as estimator of σ^2 the statistics S^2 independing of X and of law $\sigma^2\chi_n^2$ in IR^+. In this case the James-Stein estimator and his positive-part are written respectively

$$\delta_{JS}(X) = \left(1 - \frac{(p-2)S^2}{(n+2)\|X\|^2}\right) X \tag{1}$$

$$\delta_{JS}^{+}(X) = \max\left(0, 1 - \frac{(p-2)S^2}{(n+2)\|X\|^2}\right) X. \tag{2}$$

In Section 5, we give a graphic illustration of different ratios of risks and the lower and upper bounds associeted for various values of n and p.

2. Preliminary

Let us recall that the risk of the maximum likelihood estimator X is $p\sigma^2$ the risk of the James Stein estimator (given in (1) of θ is

$$R(\theta,\delta_{J.S}(X)) = \sigma^2\left\{p - \frac{n}{n+2}(p-2)^2 E\left(\frac{1}{p-2+2K}\right)\right\},$$

where $K \sim P\left(\frac{\|\theta\|^2}{2\sigma^2}\right)$ being the law of Poisson of parameter $\frac{\|\theta\|^2}{2\sigma^2}$.

When $X \sim N_p(\theta, I_p)$ G. Casella and J. T. Hwang [3] have given the lemma 1 which expresses the following inequalities:

$$\frac{1}{(p-2+\|\theta\|^2)} \le E\left(\frac{1}{\|X\|^2}\right) \le \frac{p}{(p-2)(p+\|\theta\|^2)}, p \ge 3.$$

In the following lemma, we generalise this result when $X \sim N_p(\theta, \sigma^2 I_p)$ and σ^2 is unknown.

Lemma 2.1. *Let $X \sim N_p(\theta, \sigma^2 I_p)$. If $p \ge 3$ then*

$$\frac{1}{\sigma^2\left(p-2+\frac{\|\theta\|^2}{\sigma^2}\right)} \le E\left(\frac{1}{\|X\|^2}\right) \le \frac{p}{\sigma^2(p-2)\left(p+\frac{\|\theta\|^2}{\sigma^2}\right)}.$$

Proof. We have

$$X \sim N_p(\theta,\sigma^2 I_p) \Rightarrow \frac{X}{\sigma} \sim N_p\left(\frac{\theta}{\sigma}, I_p\right) \Rightarrow \frac{1}{\sigma^2}\|X\|^2 \sim \chi^2_p\left(\frac{\|\theta\|^2}{\sigma^2}\right)$$

$$\begin{aligned} E\left(\frac{1}{\|X\|^2}\right) &= \frac{1}{\sigma^2}E\left(\frac{1}{\frac{\|X\|^2}{\sigma^2}}\right) = \frac{1}{\sigma^2}\sum_{k\ge 0}\int_0^{+\infty}\frac{1}{\omega}\chi^2_{p+2k}(d\omega)\pi\left(\frac{\|\theta\|^2}{2\sigma^2}\right) \\ &= \frac{1}{\sigma^2}E\left(\frac{1}{p-2+2K}\right) \end{aligned} \tag{3}$$

where $K \sim P\left(\frac{\|\theta\|^2}{2\sigma^2}\right)$ and (3) being the definition of the law of χ^2 no centred. According to the inequality of Jensen we have

$$E\left(\frac{1}{\|X\|^2}\right) = \frac{1}{\sigma^2}E\left(\frac{1}{p-2+2K}\right) \ge \frac{1}{\sigma^2(p-2+\frac{\|\theta\|^2}{\sigma^2})}.$$

In the other hand, for any real function h such that $E(h)\chi^2_q((\lambda))\chi^2_q(\lambda))$ exists (see G. Casella [2]), we have

$$E(h(\chi^2_q(\lambda))\chi^2_q(\lambda)) = qE(h(\chi^2_{q+2}(\lambda))) + 2\lambda E(h(\chi^2_{q+4}(\lambda))) \tag{4}$$

for $q = p-2$, $h(\omega) = \frac{1}{\omega}$, $\lambda = \frac{\|\theta\|^2}{2\sigma^2}$ we obtain

$$\int_0^{+\infty}\chi^2_{p-2}(\lambda, d\omega) = (p-2)\int_0^{+\infty}\frac{1}{\omega}\chi^2_p(\lambda, d\omega) + \frac{\|\theta\|^2}{\sigma^2}\int_0^{+\infty}\frac{1}{\omega}\chi^2_{p+2}(\lambda, d\omega).$$

Then

$$1 = (p-2)E\left(\frac{1}{\frac{\|X\|^2}{\sigma^2}}\right) + \frac{\|\theta\|^2}{\sigma^2}E\left(\frac{1}{p+2K}\right) \text{ with } K \sim P\left(\frac{\|\theta\|^2}{2\sigma^2}\right). \tag{5}$$

Thus

$$\sigma^2 E\left(\frac{1}{\|X\|^2}\right) = \frac{1}{p-2}\left[1 - \frac{\|\theta\|^2}{\sigma^2} E\left(\frac{1}{p+2K}\right)\right].$$

Hence

$$E\left(\frac{1}{\|X\|^2}\right) \leq \frac{1}{\sigma^2}\frac{1}{p-2}\left[1 - \frac{\|\theta\|^2}{\sigma^2}\frac{1}{p+\frac{\|\theta\|^2}{\sigma^2}}\right]. \tag{6}$$

Thus

$$E\left(\frac{1}{\|X\|^2}\right) \leq \frac{1}{\sigma^2(p-2)}\left[\frac{p}{p+\frac{\|\theta\|^2}{\sigma^2}}\right].$$

The equality (5) came from $E\left(\chi^2_{p-2}(\lambda)\right) = 1$, with $K \sim P\left(\frac{\|\theta\|^2}{2\sigma^2}\right)$ and the inequality (6) came from Jensen's inequality. Hence the result. □

We recall that if X is of a random variable of multidimensional Gaussian law $X \sim N_p(\theta, \sigma^2 I_p)$ in IR^p, then $U = \|X\|^2 \sim \chi^2_p(\lambda)$ where $\chi^2_p(\lambda)$ designates a noncentral χ^2 distribution with p degrees of freedom and noncentrality parameter $\lambda \left(= \frac{\|\theta\|^2}{\sigma^2}\right)$.

Lemma 2.2. *Let f be a real function defined on IR and X a random variable of multidimensional Gaussian law $X \sim N_p(\theta, \sigma^2 I_p)$ in IR^p. If for $p \geq 1$, $E[(f(U)\chi^2_p(\lambda)]$ exists, then:*
a) If f nonincreasing we have

$$E[(f(U)\chi^2_{p+2}(\lambda)] \leq E[(f(U)\chi^2_p(\lambda)]. \tag{7}$$

b) If f nondecreasing we have:

$$E[(f(U)\chi^2_{p+2}(\lambda)] \geq E[(f(U)\chi^2_p(\lambda)]. \tag{8}$$

Proof. a) We have

$$E[(f(U)\chi^2_{p+2}(\lambda)] - E[(f(U)\chi^2_p(\lambda)] = E\left[f(U)\left(\frac{u}{(p+2K)} - 1\right)\chi^2_p(\lambda)\right] E[f(U)\chi^2_p(\lambda)] E\left[\left(\frac{u}{(p+2K)} - 1\right)\chi^2_p(\lambda)\right]$$

because the covariance of two functions one increasing and the other decreasing is negative or null, with $K \sim P\left(\frac{\|\theta\|^2}{2\sigma^2}\right)$. However

$$E\left[\left(\frac{u}{(p+2K)} - 1\right)\chi^2_p(\lambda)\right] = 0,$$

then

$$E[(f(U)\chi^2_{p+2}(\lambda)] \leq E[(f(U)\chi^2_p(\lambda)].$$

Hence the result a), (in the same manner we get b). Thus the result. □

3. Bounds and limit of the ratio of the risks of James-Stein estimator to the maximum likelihood estimator

Theorem 3.1. *If* $\lim_{p\to+\infty} \frac{\|\theta\|^2}{p\sigma^2} = c > 0$, *then*

$$\lim_{p\to+\infty} \frac{R(\theta,\delta_{J.S}(X))}{R(\theta,X))} = \frac{\frac{2}{n+2}+c}{1+c}.$$

Proof. We have

$$R(\theta,\delta_{J.S}(X)) = \sigma^2\left\{p - \frac{n}{n+2}(p-2)^2 E\left(\frac{1}{p-2+2K}\right)\right\} \tag{9}$$

where $K \sim P\left(\frac{\|\theta\|^2}{2\sigma^2}\right)$. Then

$$\begin{aligned}
\frac{R(\theta,\delta_{J.S}(X))}{R(\theta,X))} &= 1 - \frac{n}{n+2}\frac{(p-2)^2}{p}E\left(\frac{1}{p-2+2K}\right)\\
&= 1 - \frac{n}{n+2}\frac{\sigma^2(p-2)^2}{p}E\left(\frac{1}{\|X\|^2}\right)\\
&\le 1 - \frac{n}{n+2}\frac{(p-2)^2}{p}\frac{1}{\left(p-2+\frac{\|\theta\|^2}{\sigma^2}\right)} &(10)\\
&\le 1 - \frac{n}{n+2}\frac{(p-2)^2}{p^2}\frac{1}{\frac{p-2}{p}+\frac{\|\theta\|^2}{p\sigma^2}}. &(11)
\end{aligned}$$

The inequality (10) is obtained from lemma 2.1. Hence

$$\begin{aligned}
\lim_{p\to+\infty} \frac{R(\theta,\delta_{J.S}(X))}{R(\theta,X))} &\le \lim_{p\to+\infty}\left\{1 - \frac{n}{n+2}\frac{(p-2)^2}{p^2}\frac{1}{\frac{p-2}{p}+\frac{\|\theta\|^2}{p\sigma^2}}\right\}\\
&\le 1 - \frac{n}{(n+2)}\frac{1}{(1+c)}\\
&\le \frac{(n+2)(1+c)-n}{(n+2)(1+c)} &(12)\\
&\le \frac{\frac{2}{n+2}+c}{1+c}. &(13)
\end{aligned}$$

Where $\lim_{p\to+\infty} \frac{\|\theta\|^2}{p\sigma^2} = c$. In the other hand

$$\begin{aligned}
\frac{R(\theta,\delta_{J.S}(X))}{R(\theta,X))} &= 1 - \frac{n}{n+2}\frac{(p-2)^2}{p}E\left(\frac{1}{p-2+2K}\right)\\
&= 1 - \frac{n}{n+2}\frac{(p-2)^2}{p}\sigma^2 E\left(\frac{1}{\|X\|^2}\right).
\end{aligned}$$

According to lemma 2.1 we have

$$\begin{aligned}
\frac{R(\theta,\delta_{J.S}(X))}{R(\theta,X))} &\ge 1 - \frac{n}{n+2}\frac{(p-2)^2}{(p-2)p}\frac{p}{\left(p+\frac{\|X\|^2}{\sigma^2}\right)}\\
&\ge 1 - \frac{n}{n+2}\frac{(p-2)^2}{p}\frac{1}{\left(1+\frac{\|X\|^2}{p\sigma^2}\right)}. &(14)
\end{aligned}$$

Thus we obtain lower and upper bounds of the ratio $\frac{R(\theta,\delta_{J.S}(X))}{R(\theta,X))}$ deduced from (10) and (14)

$$1-\frac{n}{n+2}\frac{(p-2)}{p}\frac{1}{\left(1+\frac{\|\theta\|^2}{p\sigma^2}\right)}\le\frac{R(\theta,\delta_{J.S}(X))}{R(\theta,X))}$$
$$\le 1-\frac{n}{n+2}\frac{(p-2)^2}{p}\frac{1}{\left(p-2+\frac{\|\theta\|^2}{\sigma^2}\right)}. \tag{15}$$

Passaging to the limit when $p\to+\infty$ we obtain

$$\lim_{p\to+\infty}\frac{R(\theta,\delta_{J.S}(X))}{R(\theta,X))}\ge\lim_{p\to+\infty}\left\{1-\frac{n}{n+2}\frac{(p-2)}{p}\frac{1}{\left(1+\frac{\|\theta\|^2}{p\sigma^2}\right)}\right\}$$
$$\ge 1-\frac{n}{(n+2)}\frac{1}{(1+c)}$$
$$\ge\frac{\frac{2}{n+2}+c}{1+c} \tag{16}$$

where $\lim\limits_{p\to+\infty}\frac{\|\theta\|^2}{p\sigma^2}=c$. Combining (13) and (16) we obtain

$$\lim_{p\to+\infty}\frac{R(\theta,\delta_{J.S}(X))}{R(\theta,X))}=\frac{\frac{2}{n+2}+c}{1+c}.$$

Hence the result. □

Corollary 3.2. *If* $\lim\limits_{p\to+\infty}\frac{\|\theta\|^2}{p\sigma^2}=c>0$, *we have*

$$\lim_{n,p\to+\infty}\frac{R(\theta,\delta_{J.S}(X))}{R(\theta,X))}=\frac{c}{1+c}.$$

Proof. It is obtained immediately from (11) and (14). □

4. Bounds and limit of the ratio of the risks of the positive-part James-Stein estimator to the maximum likelihood estimator

The results of the positive -part James-Stein estimator $\delta^+_{J.S}(X)$ are similar to the results obtained on the James-Stein estimator $\delta_{J.S}(X)$.
Indeed, we denote $\alpha=\frac{p-2}{n+2}$, and recall that:

$$\delta^+_{J.S}(X)=\left(1-\alpha\frac{S^2}{\|X\|^2}\right)^+X=\phi^+_{JS}(X)X\left(1-\alpha\frac{S^2}{\|X\|^2}\right)XI_{\left(\alpha\frac{S^2}{\|X\|^2}\le 1\right)}$$
$$\delta^-_{J.S}(X)=\left(1-\alpha\frac{S^2}{\|X\|^2}\right)^-X=\phi^-_{JS}(X)X\left(\alpha\frac{S^2}{\|X\|^2}-1\right)XI_{\left(\alpha\frac{S^2}{\|X\|^2}\ge 1\right)}$$

$I_{\left(\alpha\frac{S^2}{\|X\|^2}\ge 1\right)}$ showing the indicating function of the set $\left(\alpha\frac{S^2}{\|X\|^2}\ge 1\right)$.

We will denote for the needs for demonstration and each time that it will be necessary, for the sequel, by: $Y=\frac{X}{\sigma},\beta=\frac{\theta}{\sigma},T^2=\frac{S^2}{\sigma^2}$ and $U=\|Y\|^2$.

Then we have $Y\sim N(\beta,I_p)$, $T^2\sim\chi^2_n$ and $U\sim\chi^2_p(\lambda)$ where χ^2_n designates the law of the chi square centered with n degrees of freedom and $\chi^2_p(\lambda)$ designates a noncentral χ^2 distribution with p degrees of freedom and noncentrality parameter $\lambda\left(=\frac{\|\theta\|^2}{\sigma^2}\right)$.

The following lemma is a recall of the expression of the risk of $\delta^+_{J.S}(X)$.

Lemma 4.1. *We have that*

$$R(\theta,\delta^{+}_{J.S}(X)) = R(\theta,\delta_{J.S}(X)) + E\left\{\left[\|X\|^2 + 2(p-2)\sigma^2\alpha\frac{S^2}{\|X\|^2} - \alpha^2\frac{(S^2)^2}{\|X\|^2} - 2p\sigma^2\right] I_{\alpha\frac{S^2}{\|X\|^2}\geq 1}\right\}. \tag{17}$$

Proof. We have

$$\begin{aligned}
R(\theta,\delta_{J.S}(X)) &= E\left(\|\delta_{J.S}(X)-\theta\|^2\right)\\
&= E\left(\|\phi^{+}_{JS}(X)X-\theta-\phi^{-}_{JS}(X)X\|^2\right)\\
&= E\left(\|\phi^{+}_{JS}(X)X-\theta\|^2\right) + E\left(\left[\phi^{-}_{JS}(X)\right]^2\|X\|^2\right) - 2E\left\{\langle\phi^{+}_{JS}(X)X-\theta,\phi^{-}_{JS}(X)X\rangle\right\}\\
&= R(\theta,\delta^{+}_{J.S}(X)) + E\left(\left[\phi^{-}_{JS}(X)\right]^2\|X\|^2\right) - 2E\left\{\langle-\theta,\phi^{-}_{JS}(X)X\rangle\right\}\\
&= R(\theta,\delta^{+}_{J.S}(X)) + E\left(\left[\phi^{-}_{JS}(X)\right]^2\|X\|^2\right) - 2E\left\{\langle X-\theta,\phi^{-}_{JS}(X)X\rangle\right\} + 2E\left\{\langle X,\delta^{-}_{J.S}(X)X\rangle\right\}.
\end{aligned}$$

Then

$$\begin{aligned}
R(\theta,\delta_{J.S}(X)) = R(\theta,\delta^{+}_{J.S}(X)) &+ E\left\{\left(\left(\alpha\frac{S^2}{\|X\|^2}-1\right)I_{\alpha\frac{S^2}{\|X\|^2}\geq 1}\right)^2\|X\|^2\right\}\\
&- 2E\left\{{}^t(X-\theta)X\left(\left(\alpha\frac{S^2}{\|X\|^2}-1\right)I_{\alpha\frac{S^2}{\|X\|^2}\geq 1}\right)\right\} + 2E\left\{\left(\left(\alpha\frac{S^2}{\|X\|^2}-1\right)I_{\alpha\frac{S^2}{\|X\|^2}\geq 1}\right)\|X\|^2\right\}.
\end{aligned}$$

According to the lemma of P. Shao and W. E. Strawderman ([7, Lemma 2.1]) we have,

$$\begin{aligned}
&E\left\{{}^t(X-\theta)X\left(\left(\alpha\frac{S^2}{\|X\|^2}-1\right)I_{\alpha\frac{S^2}{\|X\|^2}\geq 1}\right)\right\}\\
&= \sigma^2 E\left\{\left(-2\alpha\frac{S^2}{\sigma^2\|Y\|^2} + p\left(\alpha\frac{S^2}{\sigma^2\|Y\|^2}-1\right)\right)I_{\alpha\frac{S^2}{\sigma^2\|Y\|^2}\geq 1}\right\}\\
&= \sigma^2 E\left\{\left((p-2)\alpha\frac{S^2}{\sigma^2\|Y\|^2}-p\right)I_{\alpha\frac{S^2}{\sigma^2\|Y\|^2}\geq 1}\right\}\\
&= \sigma^2 E\left\{\left((p-2)\alpha\frac{S^2}{\sigma^2\|Y\|^2}-p\right)I_{\alpha\frac{S^2}{\sigma^2\|Y\|^2}\geq 1}\right\}\\
&= \sigma^2 E\left\{\left((p-2)\alpha\frac{S^2}{\|X\|^2}-p\right)I_{\alpha\frac{S^2}{\|X\|^2}\geq 1}\right\}
\end{aligned}$$

thus

$$\begin{aligned}
R(\theta,\delta_{J.S}(X)) = R(\theta,\delta^{+}_{J.S}(X)) &+ E\left\{\left[\alpha^2\frac{(S^2)^2}{\|X\|^2} - 2\alpha S^2 + \|X\|^2\right]I_{\alpha\frac{S^2}{\|X\|^2}\geq 1}\right\}\\
&- 2\sigma^2 E\left\{\left((p-2)\alpha\frac{S^2}{\|X\|^2}-p\right)I_{\alpha\frac{S^2}{\|X\|^2}\geq 1}\right\} + 2E\left\{\left(\left(\alpha\frac{S^2}{\|X\|^2}-1\right)I_{\alpha\frac{S^2}{\|X\|^2}\geq 1}\right)\|X\|^2\right\}.
\end{aligned}$$

Therefore

$$R(\theta,\delta^{+}_{J.S}(X)) = R(\theta,\delta_{J.S}(X)) + E\left\{\left[\|X\|^2 + 2\sigma^2(p-2)\alpha\frac{S^2}{\|X\|^2} - \alpha^2\frac{(S^2)^2}{\|X\|^2} - 2p\sigma^2\right]I_{\alpha\frac{S^2}{\|X\|^2}\geq 1}\right\}.$$

Hence the result. □

The fact that $R(\theta,\delta^+_{J.S}(X)) \le R(\theta,\delta_{J.S}(X))$ (Baranchick [1]), an upper bound of the ratio $\frac{R(\theta,\delta^+_{J.S}(X))}{R(\theta,X))}$ would be for example the upper bound of the ratio $\frac{R(\theta,\delta_{J.S}(X))}{R(\theta,X))}$ that we know the asymptotic character. Thus we will interesting ourselves in what follows on a lower bound of the ratio $\frac{R(\theta,\delta^+_{J.S}(X))}{R(\theta,X))}$.

We have the following result.

Theorem 4.2. *For all $p \ge 3$, we have the following minoration of the ratio of the risks $\frac{R(\theta,\delta^+_{J.S}(X))}{R(\theta,X))}$:*

$$\begin{aligned}\frac{R(\theta,\delta^+_{J.S}(X))}{p\sigma^2} &\ge \frac{R(\theta,\delta_{J.S}(X))}{p\sigma^2} + \frac{(p+\lambda)}{p}\int_0^{+\infty} IP\left(\chi^2_n \ge \frac{u}{\alpha}\right)\chi^2_{p+4}(\lambda,du)\frac{(\alpha n+4)}{p}\int_0^{+\infty} IP\left(\chi^2_{n+4} \ge \frac{u}{\alpha}\right)\chi^2_{p-2}(\lambda,du)\\ &= \frac{R(\theta,\delta^+_{J.S}(X))}{p\sigma^2} + \frac{(p+\lambda)}{p}F_{(p+4,n,\lambda)}(\alpha) - \frac{(\alpha n+4)}{p}F_{(p-2,n+4,\lambda)}(\alpha) \qquad (18)\end{aligned}$$

where $F_{(p,n,\lambda)}(\alpha)$ is a function to distribution of Fisher with p and n degrees of freedom and parameter of noncentrality $\lambda\left(=\frac{\|\theta\|^2}{\sigma^2}\right)$.

Proof. Taking again the equality (17) we have

$$R(\theta,\delta^+_{J.S}(X)) = R(\theta,\delta_{J.S}(X)) + E\left\{\left[\|X\|^2 + 2(p-2)\sigma^2\alpha\frac{S^2}{\|X\|^2} - \alpha^2\frac{(S^2)^2}{\|X\|^2} - 2p\sigma^2\right]I_{\alpha\frac{S^2}{\|X\|^2}\ge 1}\right\}.$$

Then

$$\begin{aligned}I_1 &= E\left(\|X\|^2 I_{\alpha\frac{S^2}{\|X\|^2}\ge 1}\right)\\ &= \sigma^2 E\left(\frac{\|X\|^2}{\sigma^2} I_{\alpha\frac{S^2}{\|X\|^2}\ge 1}\right)\\ &= \sigma^2\int_0^{+\infty}\left(\int_{\frac{u}{\alpha}}^{+\infty}\chi^2_n(0,ds)\right)u\chi^2_p(\lambda,du). \qquad (19)\end{aligned}$$

Taking $h(u) = \int_{\frac{u}{\alpha}}^{+\infty}\chi^2_n(0,ds)$ and $q = p$, and applying (4) we obtain

$$\begin{aligned}I_1 &= \sigma^2 p\int_0^{+\infty} IP\left(\chi^2_n \ge \frac{u}{\alpha}\right)\chi^2_{p+2}(\lambda,du) + \sigma^2\lambda\int_0^{+\infty} IP\left(\chi^2_n \ge \frac{u}{\alpha}\right)\chi^2_{p+4}(\lambda,du)\\ &\ge \sigma^2(p+\lambda)\int_0^{+\infty} IP\left(\chi^2_n \ge \frac{u}{\alpha}\right)\chi^2_{p+4}(\lambda,du). \qquad (20)\end{aligned}$$

The inequality (20) cames from inequality (7) of lemma 2.2. On the other hand

$$\begin{aligned}I_2 &= \sigma^2 E\left(\left[2(p-2)\alpha\frac{S^2}{\|X\|^2} - 2p\right]I_{\left(\alpha\frac{S^2}{\|X\|^2}\ge 1\right)}\right)\\ &\ge -4\sigma^2 E\left(I_{\left(\alpha\frac{S^2}{\|X\|^2}\ge 1\right)}\right)\\ &\ge -4\sigma^2\int_0^{+\infty}\left(\int_0^{+\infty}\chi^2_n(0,ds)\right)\chi^2_p(\lambda,du)\\ &\ge -4\sigma^2\int_0^{+\infty} IP\left(\chi^2_n \ge \frac{u}{\alpha}\right)\chi^2_p(\lambda,du)\\ &\ge -4\sigma^2\int_0^{+\infty} IP\left(\chi^2_{n+4} \ge \frac{u}{\alpha}\right)\chi^2_{p-2}(\lambda,du). \qquad (21)\end{aligned}$$

The inequality (21) cames from inequality (7) of lemma 2.2 and by observing that $IP\left(\chi^2_{n+4} \geq \frac{u}{\alpha}\right) \geq IP\left(\chi^2_{n} \geq \frac{u}{\alpha}\right)$ for all $n \geq 1$. On the other hand

$$\begin{aligned}
I_3 &= -E\left(\alpha^2 \frac{(S^2)^2}{\|X\|^2} I_{\left(\alpha\frac{S^2}{\|X\|^2}\geq 1\right)}\right)\\
I_3 &= -\alpha^2\sigma^2 E\left(\frac{(T^2)^2}{U} I_{\left(T^2\geq \frac{U}{\alpha}\right)}\right)\\
&= -\alpha^2\sigma^2 \int_0^{+\infty}\left(\int_{\frac{u}{\alpha}}^{+\infty}(t^2)^2\chi^2_n(0,dt^2)\right)\frac{1}{u}\chi^2_p(\lambda,du)\\
&\geq -\frac{\sigma^2\alpha}{(n+2)}\int_0^{+\infty}\left(\int_{\frac{u}{\alpha}}^{+\infty}(t^2)^2\chi^2_n(0,dt^2)\right)\chi^2_{p-2}(\lambda,du).
\end{aligned}$$

The last inequality cames from (4) while taking $h(u) = \frac{1}{u}\int_{\frac{u}{\alpha}}^{+\infty}(t^2)^2\chi^2_n(0,ds)$, $q = p-2$ and the fact that $\alpha = \frac{p-2}{n+2}$.

However

$$\begin{aligned}
\int_{\frac{u}{\alpha}}^{+\infty}(t^2)^2\chi^2_n(0,dt^2) &= n\int_{\frac{u}{\alpha}}^{+\infty} t^2\chi^2_{n+2}(0,dt^2)\\
&= n(n+2)\int_{\frac{u}{\alpha}}^{+\infty}\chi^2_{n+4}(0,dt^2)\\
&= n(n+2)IP\left(\chi^2_{n+4}\geq\frac{u}{\alpha}\right). \qquad (22)
\end{aligned}$$

Then we have

$$I_3 \geq -\sigma^2\alpha n\int_0^{+\infty} IP\left(\chi^2_{n+4}\geq\frac{u}{\alpha}\right)\chi^2_{p-2}(\lambda,du). \qquad (23)$$

According to (14), (20), (21) and (23) we have

$$\begin{aligned}
\frac{R(\theta,\delta^+_{J.S}(X))}{p\sigma^2} &= \frac{R(\theta,\delta_{J.S}(X))}{p\sigma^2} + \frac{I_1}{p\sigma^2} + \frac{I_2}{p\sigma^2} + \frac{I_3}{p\sigma^2}\\
&\geq \frac{R(\theta,\delta_{J.S}(X))}{p\sigma^2} + \frac{\sigma^2(p+\lambda)}{p\sigma^2}\int_0^{+\infty}\left(IP\left(\chi^2_n\geq\frac{u}{\alpha}\right)\right)\chi^2_{p+4}(\lambda,du)\\
&\quad -\frac{4\sigma^2}{p\sigma^2}\int_0^{+\infty}\left(IP\left(\chi^2_{n+4}\geq\frac{u}{\alpha}\right)\right)\chi^2_{p-2}(\lambda,du) - \frac{\sigma^2\alpha n}{p\sigma^2}\int_0^{+\infty} IP\left(\chi^2_{n+4}\geq\frac{u}{\alpha}\right)\chi^2_{p-2}(\lambda,du)\\
&\geq \frac{R(\theta,\delta_{J.S}(X))}{p\sigma^2} + \frac{(p+\lambda)}{p}\int_0^{+\infty} IP\left(\chi^2_n\geq\frac{u}{\alpha}\right)\chi^2_{p+4}(\lambda,du) - \frac{(\alpha n+4)}{p}\int_0^{+\infty} IP\left(\chi^2_{n+4}\geq\frac{u}{\alpha}\right)\chi^2_{p-2}(\lambda,du). \qquad (24)
\end{aligned}$$

Thus the result. □

Remark 4.3. *The lower bound given by the inequality* (18) *is a bound enough "fine" and close to the ratio* $\frac{R(\theta,\delta^+_{J.S}(X))}{p\sigma^2}$ *as soon as* λ *moves away from zero. The simulations in Section 5 illustrates it rather well. On the other hand for the passage to limit of the ratio* $\frac{R(\theta,\delta^+_{J.S}(X))}{p\sigma^2}$ *when* p *tends to infinity or when* p *and* n *tend simultaneously to infinity, a less fine bound min would be enough. Then we have the next result on the limit of the ratio of the risks* $\frac{R(\theta,\delta^+_{J.S}(X))}{R(\theta,X))}$.

Proposition 4.4. *If* $\lim_{p\to+\infty}\frac{\|\theta\|^2}{p\sigma^2} = c > 0$*, we have:*

$$\lim_{p\to+\infty}\frac{R(\theta,\delta^+_{J.S}(X))}{R(\theta,X))} = \frac{\frac{2}{n+2}+c}{1+c}.$$

Proof. Baranchick [1] showed that $R(\theta, \delta^{+}_{J.S}(X)) \leq R(\theta, \delta_{J.S}(X))$ for $p \geq 3$ and all $\theta, \sigma \in IR^p IR^+$. Thus the upper bound in (11) plays the role of the upper bound of $\frac{R(\theta,\delta^{+}_{J.S}(X))}{R(\theta,X))}$. It is enough to show that the limit of the lower bound is higher or equal to $\frac{\frac{2}{n+2}+c}{1+c}$. Indeed according to (24) we have

$$\frac{R(\theta, \delta^{+}_{J.S}(X))}{R(\theta, X))} \geq \frac{R(\theta, \delta_{J.S}(X))}{p\sigma^2} + \frac{(p+\lambda)}{p}\int_0^{+\infty} IP\left(\chi^2_n \geq \frac{u}{\alpha}\right)\chi^2_{p+4}(\lambda, du)$$
$$- \frac{(\alpha n + 4)}{p}\int_0^{+\infty} IP\left(\chi^2_{n+4} \geq \frac{u}{\alpha}\right)\chi^2_{p-2}(\lambda, du)$$
$$\frac{R(\theta, \delta^{+}_{J.S}(X))}{R(\theta, X))} \geq \frac{R(\theta, \delta_{J.S}(X))}{p\sigma^2} + \int_0^{+\infty} IP\left(\chi^2_n \geq \frac{u}{\alpha}\right)\chi^2_{p+4}(\lambda, du) - \frac{4}{p} - 1 \tag{25}$$

because

$$\frac{(p+\lambda)}{p}\int_0^{+\infty} IP\left(\chi^2_n \geq \frac{u}{\alpha}\right)\chi^2_{p+4}(\lambda, du) \geq \int_0^{+\infty} IP\left(\chi^2_n \geq \frac{u}{\alpha}\right)\chi^2_{p+4}(\lambda, du)$$
$$-\int_0^{+\infty} IP\left(\chi^2_n \geq \frac{u}{\alpha}\right) \geq -1 \quad \text{and} \quad -\frac{n(p-2)}{p(n+2)} \geq -1.$$

Let us denote that like $\alpha = \frac{p-2}{n+2}$ and thus tends to $+\infty$ when $p \longrightarrow +\infty$, we have according to the theorem of Lebesgue by taking for example, the increasing sequel with $p\left(f_p(u) = \int_{\frac{u}{\alpha}}^{+\infty} \chi^2_n(0, ds) = IP\left(\chi^2_n \geq \frac{u}{\alpha}\right)\right)$

$$\lim_{p\to+\infty} IP\left(\chi^2_n \geq \frac{u}{\alpha}\right) = IP\left(\chi^2_n \geq 0\right) = 1, \quad \forall\ n \geq 1 \tag{26}$$

thus

$$\lim_{p\to+\infty} \int_0^{+\infty} IP\left(\chi^2_n \geq \frac{u}{\alpha}\right)\chi^2_{p+4}(\lambda, du) = 1$$

where $\lambda = \frac{\|\theta\|^2}{\sigma^2}$. Finally we obtains

$$\lim_{p\to+\infty} \frac{R(\theta, \delta^{+}_{J.S}(X))}{R(\theta, X))} \geq \frac{\frac{2}{n+2} + c}{1+c}.$$

Thus the result. □

The case where n and p tend simultaneously to $+\infty$ is given in the following theorem.

Theorem 4.5. *If* $\lim_{p\to+\infty} \frac{\|\theta\|^2}{p\sigma^2} = c > 0$, *we have*

$$\lim_{n,p\to+\infty} \frac{R(\theta, \delta^{+}_{J.S}(X))}{R(\theta, X))} = \frac{c}{1+c}.$$

Proof. On the one hand, we have

$$\lim_{n,p\to+\infty} \frac{R(\theta, \delta^{+}_{J.S}(X))}{R(\theta, X))} \leq \lim_{n,p\to+\infty} \frac{R(\theta, \delta_{J.S}(X))}{R(\theta, X))} = \frac{c}{1+c} \tag{27}$$

because $R(\theta,\delta^+_{J.S}(X)) \leq R(\theta,\delta_{J.S}(X))$ for $p \geq 3$ and all $\theta,\sigma \in IR^p IR^+$. In the other hand, by beginning again (25) we have

$$\lim_{n,p\to+\infty} \frac{R(\theta,\delta^+_{J.S}(X))}{R(\theta,X))} \geq \frac{c}{1+c} + \lim_{n,p\to+\infty} \int_0^{+\infty} \left(IP\left(\chi^2_n \geq \frac{u}{\alpha}\right)\right) \chi^2_{p+4}(\lambda,du) - 1. \tag{28}$$

However

$$\begin{aligned} IP\left(\chi^2_n \geq \frac{u}{\alpha}\right) &= IP\left(\sum_{i=1}^n y_i^2 \geq \frac{u(n+2)}{(p-2)}\right) \\ &= IP\left(\frac{\sum_{i=1}^n y_i^2}{n} \geq \frac{u}{(p-2)} + \frac{2u}{n(p-2)}\right) \end{aligned}$$

where $y_1, y_2, \ldots, y_n$ are independent, Gaussian random variables centered reduced. Thus by the strong law of the great numbers we have

$$\begin{aligned} \lim_{n,p\to+\infty} IP\left(\chi^2_n \geq \frac{u}{\alpha}\right) &= \lim_{n,p\to+\infty} IP\left(\frac{\sum_{i=1}^n y_i^2}{n} \geq \frac{u}{(p-2)} + \frac{2u}{n(p-2)}\right) \\ &= \lim_{n,p\to+\infty} IP\left(\frac{\sum_{i=1}^n y_i^2}{n} \geq 0\right) \\ &= IP(1 \geq 0) = 1 \end{aligned}$$

ainsi

$$\lim_{n,p\to+\infty} \int_0^{+\infty} \left(IP\left(\chi^2_n \geq \frac{u}{\alpha}\right)\right) \chi^2_{p+4}(\lambda,du) = \int_0^{+\infty} \chi^2_{p+4}(\lambda,du) = 1. \tag{29}$$

Combining (27), (28) and (29), there is finally the result. □

5. Simulations

We illustrate graphically in what follows the ratios of the risks $\frac{R(\theta,\delta_{J.S}(X))}{R(\theta,X))}$, $\frac{R(\theta,\delta^+_{J.S}(X))}{R(\theta,X))}$ as well as the evolution of the bound min and max associated, given respectively by the expressions (15) and (18), for $\sigma^2 = 1$ and different values of n and p.

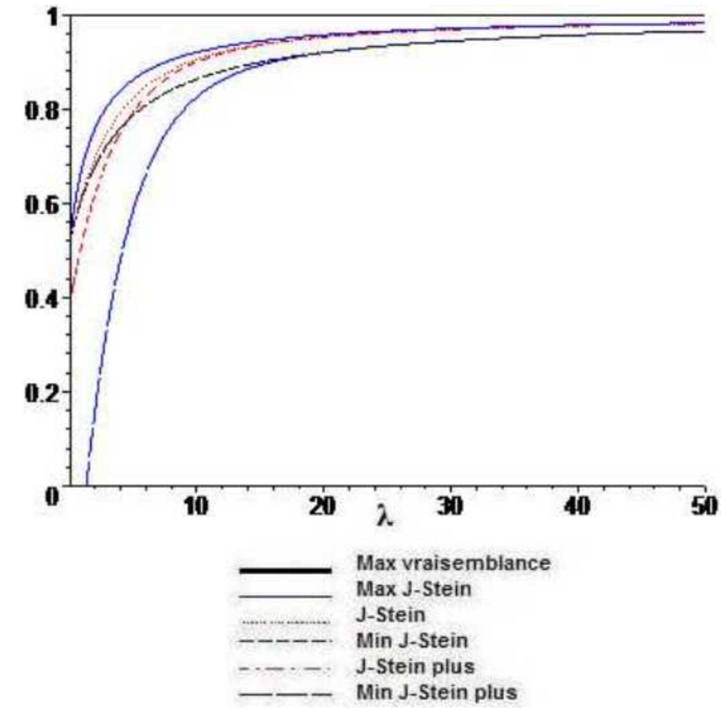

Figure 1: Graph of the relative risks and their minima and maxima for $n = 50$ and $p = 4$, according to $\lambda = \frac{\|\theta\|^2}{\sigma^2}$ and $\sigma^2 = 1$.

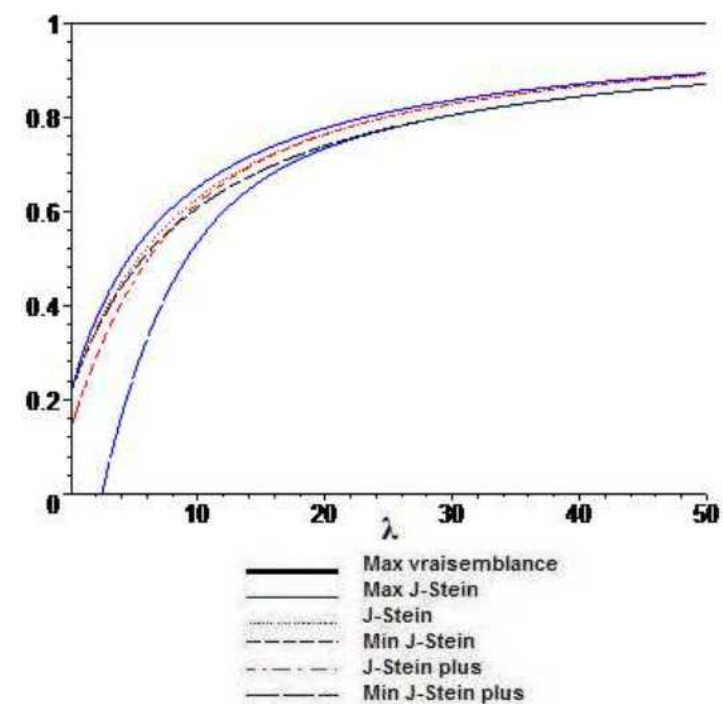

Figure 2: Graph of the relative risks and their minima and maxima for $n = 100$ and $p = 10$, according to $\lambda = \frac{\|\theta\|^2}{\sigma^2}$ and $\sigma^2 = 1$.

Conclusion

In the case of the estimate of the average θ of a multidimensional Gaussian law $N_p(\theta, I_p)$ in IR^p, Casella, G and J, T, Tzon Hwang [3] showed that if $\lim\limits_{p\to+\infty} \frac{\|\theta\|^2}{p} = c_1 > 0$ then the ratio $\frac{R(\theta,\delta^+_{JS}(X))}{R(\theta,X))}$ tends to $\frac{c_1}{1+c_1}$. By taking the same model, namely $X \sim N_p(\theta, \sigma^2 I_p)$ with this time σ^2 unknown, and estimated by the statistic S^2 independent of X and of law $\sigma^2\chi^2_n$ in IR^+. We have for our part, showed that we obtain a similar ratio depend on the size n of the sample, with that found by the latter, as soon as $\lim\limits_{p\to+\infty} \frac{\|\theta\|^2}{p\sigma^2} = c > 0$. Moreover we obtain a ratio constant and independent of n, when n and p tend simultaneously to $+\infty$ and this without taking account of an unspecified relation of order or functional calculus between n and p. Li Sun. [6] is him also interested if σ^2 is unknown, but studied the behavior of the ratio $\frac{R(\theta,\delta^+_{JS}(X))}{R(\theta,X))}$ and $\frac{R(\theta,\delta_{JS}(X))}{R(\theta,X))}$, when only p tends to infinity. An idea would be to see whether we obtain similar ratios in the general case of the symmetrical spherical models.

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