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Minimum variance unbiased estimation in the Pareto distribution of first kind under progressive Type II censored data with binomial removals

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Abstract. This paper deals with the problem of uniformly minimum variance unbiased estimation of the parameter of Pareto distribution of first kind based on progressive Type II censored data with binomial removals. We obtain the uniformly minimum variance unbiased estimator (UMVUE) for powers of the shape parameter and its functions. The UMVUE of the variance of these estimators are also given. The UMVUE of (i) p.d.f. (ii) c.d.f. (iii) reliability function (iv) hazard function (v) geometric mean (vi) the logarithm of p^{th} quantile (vii) the r^{th} moment (viii) mean and the variance of Pareto distribution are derived. Further an exact $(1-\alpha)100\%$ confidence interval for the p^{th} quantile is obtained. Finally the UMVUE of lifetime performance index is given.An illustrative numerical example is presented.

1. Introduction

A Type II censored sample is one for which only m smallest observations in a sample of n items are observed. A generalization of Type II censoring is a progressive Type II censoring. Under this scheme, n units of the same kind are placed on test at time zero, and m failures are observed. When the first failure is observed, a number r_1 of surviving units are randomly withdrawn from the test; at the second failure time, r_2 surviving units are selected at random and taken out of the experiment, and so on. At the time of m^{th} failure, the remaining $r_m = n - r_1 - r_2 - ... - r_{m-1} - m$ units are removed. Balakrishnan et al. [6] indicated that such scheme can arise in clinical trials where the drop out of patients may be caused by migration or by lack of interest. In such situations, the progressive censoring scheme with random removals is required. For a detailed discussion of progressive censoring we refer to Balakrishnan and Aggarwala [5]. If $r_1 = r_2 = ... = r_{m-1} = 0$, then, this scheme reduces to the Type II censoring scheme. Also note that if $r_1 = r_2 = ... = r_m = 0$, so that $m = n$, this scheme reduces to the case of no censoring that is the case of a complete sample. In this paper, we use progressive Type II censoring scheme with binomial removals where the number of units removed at each failure time follows a binomial distribution.

To study income distributions Vilferdo Pareto introduced the Pareto distribution in 1897. Since then investigators have been using the Pareto distribution or some related forms of it in industrial, engineering and economic studies. Several such situations have been discussed by Harris [13], Davis and Feldstein [10], Freiling [11], Berger and Mandelbrot [8] and Hassan et al. [14]. For a review of literature on estimating

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parameters of the Pareto distribution one may refer to Amin [1], Arnold and Press [2] Asrabadi [4], Baxter [7], Kern [20], Malik [22], Hosking and Wallis [18], Rytagaard [25] and many others.

Inference for Pareto distribution based on progressive Type II censored data were discussed by many authors. Wu et al. [29] obtained the maximum likelihood estimators of two parameter Pareto distribution under progressive Type II censoring with random removals where the number of units removed at each failure time follows a binomial or uniform distribution. They had also given the expected test time to complete the censoring test. Kus and Kaya [21] obtained the exact confidence region for the parameters of the Pareto distribution under progressive Type II censored data. Parsi et al. [24] introduced a new approach for constructing simultaneous confidence interval of the parameters of the Pareto distribution under progressive Type II censoring.

In the manufacturing or service industry, process capability indices are useful to assess whether the quality of product meets the desired level. The study of assessing the lifetime performance index of the products is the subject of investigation of many authors. Tong et al. [28] constructed a UMVUE of process capability index C_L, based on the complete sample from an exponential distribution , where L is the lower specification limit. Hong et al. [15] constructed a maximum likelihood estimator of C_L and using this estimator they developed the new testing procedure in the condition of known L , with the right type II censored sample for Pareto distribution. Hong et al. [16] constructed a maximum likelihood estimator of C_L and developed $(1-\alpha)100\%$ confidence interval for lifetime performance index in Pareto distribution based on type II right censored samples. Hong et al. [17] constructed a UMVUE of C_L based on right type II censored sample from the Pareto distribution. Using the UMVUE of C_L they developed a novel hypothesis testing procedure in the condition of known L.

In this paper we discuss the problem of UMVUE for shape parameter θ of the Pareto distribution of first kind based on progressive Type II censored data with binomial removals. This paper is organized as follows. In Section 2,the likelihood function is given. In Section 3, the UMVUE of θ and its functions are derived. Also, the UMVUE of the (i) geometric mean (ii) the logarithm of p^{th} quantile (iii) hazard function (iv) p.d.f (v) positive power of reliability function (vi) c.d.f. of Pareto distribution are obtained. In Section 4 the UMVUE of the r^{th} moment and mean of Pareto distribution are obtained. Further an exact $(1-\alpha)100\%$ confidence interval for p^{th} quantile is constructed. Section 5, then presents the UMVUE of lifetime performance index. In Section 6, an illustrative numerical example is given.

2. The model

Let the failure time distribution be Pareto with probability density function

$$f(x) = \begin{cases} \theta\mu^{\theta}\, x^{-\theta-1}, & \theta,\ \mu > 0\ x \geq \mu, \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

where θ is shape parameter and μ is known. The cumulative distribution function is given by

$$F(x) = 1 - \left(\frac{\mu}{x}\right)^{\theta}, x \geq \mu. \tag{2}$$

The survival function is given by

$$S(x) = \left(\frac{\mu}{x}\right)^{\theta}, x \geq \mu. \tag{3}$$

The density given in (1) can be written as

$$f(x) = \frac{a(x)[h(\theta)]^{d(x)}}{[g(\theta)]}, \tag{4}$$

where

$$a(x) = \frac{1}{x}, h(\theta) = \exp(-\theta), d(x) = \log\left(\frac{x}{\mu}\right) \text{ and } g(\theta) = \left(\frac{1}{\theta}\right), \tag{5}$$

such that $a(x) > 0$ and $g(\theta) = \int_{x>\mu} a(x)[h(\theta)]^{d(x)} dx$.

Let (X_1, R_1), (X_2, R_2), $\ldots$, (X_m, R_m), denote a progressively Type II censored sample where $X_i = X_i, m, n$ for $i = 1, 2, ..., m$ and $X_1 < X_2 < ... < X_m$. The conditional likelihood function can be written as, see Cohen[9],

$$L(\theta; \underline{x}/\underline{R} = \underline{r}) = c \prod_{i=1}^{m} f(x_i)[s(x_i)]^{r_i}, \tag{6}$$

where $c = n(n - r_1 - 1)(n - r_1 - r_2 - 2) \cdots (n - r_1 - r_2 - ... - r_{m-1} - m + 1)$ and $0 \le r_i \le (n - m - r_1 - r_2 - ... - r_{i-1})$ for $i = 1, 2, \ldots, m - 1$. Substituting (1) and (3) in (6), we get

$$L(\theta; \underline{x}/\underline{R} = \underline{r}) = c\theta^m \left(\prod_{i=1}^{m} \frac{1}{x_i}\right) \exp\left\{-\theta \sum_{i=1}^{m} (1 + r_i) \log\left(\frac{x_i}{\mu}\right)\right\}. \tag{7}$$

We assume that X_i and R_i are independent for all i. We further suppose that the number of units removed at each failure time follows a binomial distribution with probability p. From Wu et al. [29] the joint probability mass function of $r_1, r_2, \ldots, r_{m-1}$ is given by

$$P(\underline{R} = \underline{r}) = \frac{(n-m)! p^{\sum_{i=1}^{m-1} r_i} (1-p)^{(m-1)(n-m) - \sum_{i=1}^{m-1} (m-i) r_i}}{\left(n - m - \sum_{i=1}^{m-1} r_i\right)! \prod_{i=1}^{m-1} r_i!}. \tag{8}$$

That is

$$P(\underline{R} = \underline{r}) = \frac{(n-m)! \left[\frac{p}{(1-p)^m}\right]^{\sum_{i=1}^{m-1} r_i} (1-p)^{\sum_{i=1}^{m-1} i r_i}}{\prod_{i=1}^{m} r_i! [(1-p)^{-(n-m)}]^{(m-1)}}. \tag{9}$$

The unconditional likelihood function is

$$L(\theta, p : \underline{x}, \underline{r}) = L(\theta; \underline{x}/\underline{R} = \underline{r}) P(\underline{R} = \underline{r}). \tag{10}$$

Using (7)and (10) in (**??**), we can write the likelihood function as

$$L(\theta, p : \underline{x}, \underline{r}) = c\theta^m \left(\prod_{i=1}^{m} \frac{1}{x_i}\right) \exp\left\{-\theta \sum_{i=1}^{m} (1 + r_i) \log\left(\frac{x_i}{\mu}\right)\right\} \frac{(n-m)! \left[\frac{p}{(1-p)^m}\right]^{\sum_{i=1}^{m-1} r_i} (1-p)^{\sum_{i=1}^{m-1} i r_i}}{\prod_{i=1}^{m} r_i! [(1-p)^{-(n-m)}]^{(m-1)}}.$$

3. Unbiased estimation

Let

$$Y_i = \log\left(\frac{X_i}{\mu}\right), \ i = 1, 2, \ldots, m. \tag{11}$$

Then Y_i have exponential distribution with mean $1/\theta$. Now consider the following transformation

$$Z_1 = nY_1, \quad Z_i = (n - i + 1 - r_1 - r_2 - \cdots - r_{i-1})(Y_i - Y_{i-1}), \ i = 2, 3, \ldots, m. \tag{12}$$

In order to derive the distribution of Z_i, for $i = 1, 2, \ldots, m$, consider the inverse transformation $Y_1 = \frac{Z_1}{n}$ and $Y_i = \sum\limits_{i=2}^{m} \frac{Z_i}{(n-r_{i-1}-i+1)}$, $i = 2, 3, ..., m$. The variables $Z_1, Z_2, \ldots, Z_m$ defined in (12) are all independent and identically distributed with exponential distribution with mean $1/\theta$, see[27]. The joint density of $Z_1, Z_2, \ldots, Z_m$ is

$$f(\underline{z}, \theta/\underline{R} = \underline{r}) = \theta^m \exp\left\{-\theta \sum_{i=1}^{m} z_i\right\}. \tag{13}$$

It can be seen that

$$\sum_{i=1}^{m} Z_i = \sum_{i=1}^{m} (1 + r_i) Y_i. \tag{14}$$

Using (11) in (14), we have

$$\sum_{i=1}^{m} Z_i = \sum_{i=1}^{m} (1 + r_i) \log\left(\frac{X_i}{\mu}\right). \tag{15}$$

Let

$$T = \sum_{i=1}^{m} Z_i = \sum_{i=1}^{m} (1 + r_i) \log\left(\frac{X_i}{\mu}\right). \tag{16}$$

Since (13) is a member of exponential family of distributions, T is a complete sufficient statistic for θ. The distribution of T is gamma with parameters θ and m, which is again a member of exponential family of distributions. The p.d.f. of T is given by

$$f(t, \theta) = \frac{B(t, m)[h(\theta)]^t}{[g(\theta)]^m}, \tag{17}$$

where $B(t, m) = \frac{t^{m-1}}{\Gamma m}$, $h(\theta) = \exp(-\theta)$, $g(\theta) = 1/\theta$.

Jani and Dave [19] have studied the problem of minimum variance unbiased estimation in a class of exponential family of distributions. They have shown that if $X_1, X_2, \ldots, X_n$ be a random sample from density of the type given in (4) and the p.d.f. of its complete sufficient statistic can be written as the one given in (17) then the UMVUE of $[h(\theta)]^k$ is given by

$$H_{k,n} = \frac{B(t-k, n)}{B(t, n)}, t > k, \tag{18}$$

and the UMVUE of $[g(\theta)]^k$ is

$$G_{k,n} = \frac{B(t, n+k)}{B(t, n)}. \tag{19}$$

Following the results derived in Jani and Dave [19], we get the UMVUE of some important parametric functions as given below.

(i) Using (18), the UMVUE of $\exp(-\theta k)$ is

$$H_{k,m} = \left[1 - \frac{k}{\sum_{i=1}^{m} (1 + r_i) \log\left(\frac{x_i}{\mu}\right)}\right]^{(m-1)}, \sum_{i=1}^{m} (1 + r_i) \log\left(\frac{x_i}{\mu}\right) > k. \tag{20}$$

Remark 3.1. *Substituting $k = 1$ in (20), we get the UMVUE of* $\exp(-\theta)$ *as*

$$H_{1,m} = \left[1 - \frac{1}{\sum_{i=1}^{m}(1+r_i)\log\left(\frac{x_i}{\mu}\right)}\right]^{(m-1)}, \sum_{i=1}^{m}(1+r_i)\log\left(\frac{x_i}{\mu}\right) > 1$$

Remark 3.2. *Substituting $k = -1$ in (20), we get the UMVUE of* $\exp(\theta)$ *as*

$$H_{-1,m} = \left[1 + \frac{1}{\sum_{i=1}^{m}(1+r_i)\log\left(\frac{x_i}{\mu}\right)}\right]^{(m-1)}, \sum_{i=1}^{m}(1+r_i)\log\left(\frac{x_i}{\mu}\right) > 0.$$

(ii) Using (20), the UMVUE of the variance of $H_{k,m}$ is given by, see [19],

$$\widetilde{\mathrm{Var}}[H_{k,m}] = \left[1 - \frac{k}{\sum\limits_{i=1}^{m}(1+r_i)\log\left(\frac{x_i}{\mu}\right)}\right]^{(2m-2)} - \left[1 - \frac{2k}{\sum\limits_{i=1}^{m}(1+r_i)\log\left(\frac{x_i}{\mu}\right)}\right]^{(m-1)}.$$

(iii) Using (19), the UMVUE of $\left(\frac{1}{\theta}\right)^k$ is given by

$$G_{k,m} = \frac{\Gamma(m)}{\Gamma(m+k)}\left[\sum_{i=1}^{m}(1+r_i)\log\left(\frac{x_i}{\mu}\right)\right]^k. \tag{21}$$

Remark 3.3. *Substituting $k = 1$ in (21), we get the UMVUE of* $1/\theta$ *as*

$$G_{1,m} = \frac{1}{m}\left[\sum_{i=1}^{m}(1+r_i)\log\left(\frac{x_i}{\mu}\right)\right]. \tag{22}$$

Remark 3.4. *Substituting $k = -1$ in (21), we get the UMVUE of θ as*

$$G_{-1,m} = \frac{(m-1)}{\left[\sum\limits_{i=1}^{m}(1+r_i)\log\left(\frac{x_i}{\mu}\right)\right]}. \tag{23}$$

(iv) Using (21), the UMVUE of the variance of $G_{k,m}$ is given by

$$\widetilde{\mathrm{Var}}[G_{k,m}] = \left[\sum_{i=1}^{m}(1+r_i)\log\left(\frac{x_i}{\mu}\right)\right]^{2k}\left[\left(\frac{\Gamma(m)}{\Gamma(m+k)}\right)^2 - \left(\frac{\Gamma(m)}{\Gamma(m+2k)}\right)\right].$$

Remark 3.5. *Using (21), the UMVUE of geometric mean $GM = \mu\exp\left(\frac{1}{\theta}\right)$ is given by*

$$\widetilde{GM} = \mu\sum_{j=0}^{\infty}\frac{\Gamma(m)}{\Gamma(m+j)}\left[\sum_{i=1}^{m}(1+r_i)\log\left(\frac{x_i}{\mu}\right)\right]^j\frac{1}{j!}.$$

Remark 3.6. *Using (23), the UMVUE of logarithm of the p^{th} quantile is given by*

$$\widetilde{\log}\xi_p = \left(\frac{\left[\sum\limits_{i=1}^{m}(1+r_i)\log\left(\frac{x_i}{\mu}\right)\right]\log\left\{(1-p)^{-1}\right\} + m\log(\mu)}{m}\right). \tag{24}$$

Remark 3.7. *Using (24), the UMVUE of hazard function can be obtained as*

$$\widetilde{h}(x) = \frac{(m-1)}{\left[\sum_{i=1}^{m}(1+r_i)\log\left(\frac{x_i}{\mu}\right)\right]}\log\left(\frac{x}{\mu}\right).$$

Remark 3.8. *In case of Type II censored sample that is when $r_i = 0, i = 1, 2, \ldots, m-1$ and $r_m = n-m$, the UMVUE of θ can be obtained from (24) as*

$$\widetilde{\theta_m} = \frac{(m-1)}{\left[\sum_{i=1}^{m}\log\left(\frac{x_i}{\mu}\right) + (n-m)\log\left(\frac{x_m}{\mu}\right)\right]}.$$

In case of complete sample that is when $r_i = 0, i = 1, 2, \ldots, m$ and $m = n$, the UMVUE of θ reduces to

$$\widetilde{\theta_n} = \frac{(n-1)}{\left[\sum_{i=1}^{n}\log\left(\frac{x_i}{\mu}\right)\right]}. \tag{25}$$

The result given in (25) coincides with Asrabadi [4] who obtained it by using usual techniques.

(v) The UMVUE of density $f(x)$ given in (1), for fixed x is given by

$$\phi_{x,m} = \frac{1}{x}\left(\frac{m-1}{\sum_{i=1}^{m}(1+r_i)\log\left(\frac{x_i}{\mu}\right)}\right)\left(1 - \frac{\log\left(\frac{x}{\mu}\right)}{\sum_{i=1}^{m}(1+r_i)\log\left(\frac{x_i}{\mu}\right)}\right)^{(m-2)}, \ \log\left(\frac{x}{\mu}\right) < \sum_{i=1}^{m}(1+r_i)\log\left(\frac{x_i}{\mu}\right), m > 2. \tag{26}$$

Remark 3.9. *For complete sample case that is when $r_1 = r_2 = \cdots = r_m = 0$ and $m = n$, the result (26) reduces to*

$$\phi_{x,n} = (n-1)\frac{\left[\sum_{i=1}^{m}\log(x_i) - \log(x) - (n-1)\log\mu\right]}{x\left[\sum_{i=1}^{m}\log(x_i) - n\log\mu\right]^{n-1}}. \tag{27}$$

The result given in (27) coincides with Asrabadi [4].

(vi) The UMVUE of variance of $\phi_{x,m}$ for $m > 2$ is given by

$$\widetilde{\mathrm{Var}}[\phi_{x,m}] = \begin{cases} \left[\frac{m-1}{tx}\right]^2\left[1 - \frac{\log(\frac{x}{\mu})}{t}\right]^{(2m-4)} - \left[\frac{m-1}{tx}\right]\left[1 - \frac{\log(\frac{x}{\mu})}{t}\right]^{(m-2)}\left[\frac{1}{x}\frac{m-2}{t-\log(\frac{x}{\mu})}\right]\left[1 - \frac{\log(\frac{x}{\mu})}{t-\log(\frac{x}{\mu})}\right]^{(m-3)} & , \ t > 2\log(\frac{x}{\mu}) \\ \left[\frac{1}{x}\frac{(m-1)}{t}\right]^2\left[1 - \frac{\log(\frac{x}{\mu})}{t}\right]^{(2m-4)} & , \ \log(\frac{x}{\mu}) < t \leq 2\log(\frac{x}{\mu}) \\ 0 & , \ \text{otherwise}, \end{cases}$$

where $t = \sum_{i=1}^{m}(1+r_i)\log(\frac{x}{\mu})$.

(vii) Considering x as fixed, the UMVUE of positive integer power, $R^h(x)$ of reliability function $R(X) = P(X > x)$, $x \geq 0$ is obtained as follows. Since $R^h(x) = [h(\theta)]^{h\log\left(\frac{x}{\mu}\right)}$, where $h(\theta)$ is given in (5) and using (20) with $k = h\log\left(\frac{x}{\mu}\right)$, the UMVUE $\tilde{R^h}(x)$ of $R^h(x)$ is given by

$$\widetilde{R}^h(x) = \left[1 - \frac{h\log\left(\frac{x}{\mu}\right)}{\sum_{i=1}^{m}(1+r_i)\log\left(\frac{x_i}{\mu}\right)}\right]^{(m-1)}, \ h\log\left(\frac{x}{\mu}\right) < \sum_{i=1}^{m}(1+r_i)\log\left(\frac{x_i}{\mu}\right). \tag{28}$$

Substituting $h = 1$ in (28), one gets the UMVUE of reliability for fixed x .

Remark 3.10. *Using (28), the UMVUE of the variance of $\widetilde{R}(x)$ is given by*

$$\widetilde{Var}[\widetilde{R}(x)] = \begin{cases} \left[1 - \frac{\log\left(\frac{x}{\mu}\right)}{\sum_{i=1}^{m}(1+r_i)\log\left(\frac{x_i}{\mu}\right)}\right]^{(2m-2)} - \left[1 - \frac{2\log\left(\frac{x}{\mu}\right)}{\sum_{i=1}^{m}(1+r_i)\log\left(\frac{x_i}{\mu}\right)}\right]^{(m-1)} & , 0 < x < \mu e^{\frac{t}{2}}, \\ \left[1 - \frac{\log\left(\frac{x}{\mu}\right)}{\sum_{i=1}^{m}(1+r_i)\log\left(\frac{x_i}{\mu}\right)}\right]^{(2m-2)} & , \mu e^{\frac{t}{2}} < x < \mu e^{t}, \\ 0 & , \text{otherwise.} \end{cases}$$

(viii) The UMVUE of cumulative distribution function given in (2) is

$$\widetilde{F}(x) = \begin{cases} 0, & , x < \mu, \\ 1 - \left(1 - \frac{\log\left(\frac{x}{\mu}\right)}{\sum_{i=1}^{m}(1+r_i)\log\left(\frac{x_i}{\mu}\right)}\right)^{(m-1)} & , \log\left(\frac{x}{\mu}\right) < \sum_{i=1}^{m}(1+r_i)\log\left(\frac{x_i}{\mu}\right), \\ 1 & , \text{otherwise.} \end{cases}$$

Shanubhogue and Jain [26] have studied the problem of minimum variance unbiased estimation in exponential distribution under progressive Type II censored data with binomial removals. They have given the UMVUE for parameter p and various functions of p. Since the joint density $P(\underline{R} = \underline{r})$ given in (10) is independent of θ one gets the same estimators of p and its various functions as given in Shanubhogue and Jain [26].

4. UMVU estimator of the r^{th} moment and mean.

Theorem 4.1. *Under progressive Type II censored data,the r^{th} moment of $\phi_{x,m}$ is the UMVUE estimator of the r^{th} moment of the Pareto distribution.*

Proof. We have

$$\begin{aligned} \widetilde{E}(X^r) &= \int_{\mu}^{\mu\exp(t)} x^r \phi_{x,m} dx \\ &= \left(\frac{m-1}{t}\right)\int_{\mu}^{\mu\exp(t)} x^{r-1}\left(1 - \frac{\log\left(\frac{x}{\mu}\right)}{t}\right)^{(m-2)} dx \end{aligned} \tag{29}$$

where t is given in (**??**). Further simplification of (29) results to

$$\widetilde{E}(X^r) = \mu^r(m-1)\exp[rt]\left(\int_0^1 u^{m-2}\exp[-rtu]du\right). \tag{30}$$

Using the following result, given in Gradshteyn and Ryzhik [12, page 340],

$$\int_0^1 x^n \exp[-\mu x]dx = \frac{n!}{\mu^{n+1}} - \exp[-u\mu]\sum_{k=0}^{n}\frac{n!}{k!}\frac{u^k}{\mu^{n-k+1}}, u > 0, \mu > 0, n = 0, 1, ...$$

equation (30) further simplifies to

$$\widetilde{E}(X^r) = \frac{\mu^r(m-1)!}{[rt]^{(m-1)}}\left(\exp[rt] - 1 - \sum_{k=1}^{m-2}\frac{[rt]^k}{k!}\right), 0 < r < \theta. \tag{31}$$

We can show that $\widetilde{E}(X^r)$ is an unbiased estimate of $E(X^r) = \frac{\mu^r\theta}{(\theta-r)}$, $0 < r < \theta$, the r^{th} moment of Pareto distribution. The proof is completed by the fact that $\widetilde{E}(X^r)$ is a function of the complete sufficient statistic T. For the complete sample case the result (31) reduces to the one obtained by Asrabadi [4]. □

Corollary 4.2. *Substituting $r = 1$ in (31), we get UMVUE of mean life as*

$$\widetilde{\mathrm{E}}(X) = \frac{\mu(m-1)!}{[t]^{(m-1)}}\left(\exp[t] - 1 - \sum_{k=1}^{m-2}\frac{[t]^k}{k!}\right), 1 < \theta.$$

Theorem 4.3. *The $(1-\alpha)100\%$ confidence interval for p^{th} quantile is given by*

$$\left[\mu\exp\left(\frac{2m\tilde{\log}\left(\frac{\xi_p}{\mu}\right)}{\chi^2_{2m,1-\alpha/2}}\right), \mu\exp\left(\frac{2m\tilde{\log}\left(\frac{\xi_p}{\mu}\right)}{\chi^2_{2m,\alpha/2}}\right)\right]. \tag{32}$$

Proof. Using equation (16) and (21) we have, $t = \dfrac{m\tilde{\log}\left(\frac{\xi_p}{\mu}\right)}{\theta\log\left[(1-p)^{-1}\right]}$ and t has gamma distribution with parameters θ and m. We make the transformation $Q = 2T\theta$. Hence $Q = \dfrac{2m\tilde{\log}\left(\frac{\xi_p}{\mu}\right)}{\log\left(\frac{\xi_p}{\mu}\right)}$ has a chi-square distribution with $2m$ degrees of freedom. Thus

$$P\left[\chi^2_{2m,\frac{\alpha}{2}} \le \frac{2m\tilde{\log}\left(\frac{\xi_p}{\mu}\right)}{\log\left(\frac{\xi_p}{\mu}\right)} \le \chi^2_{2m,1-\frac{\alpha}{2}}\right] = 1-\alpha. \tag{33}$$

Rearranging (33) we get(32). □

5. UMVU of lifetime performance index.

Suppose that the lifetime of units may be given by a Pareto distribution with probability density function given in (1). Using the transformation given in (11), the distribution of Y is exponential with p.d.f.

$$f(y) = \begin{cases}\theta\exp[-\theta y], & y > 0, \theta > 0\\ 0, & \text{otherwise.}\end{cases}$$

Montgomery [23] has given a capability index C_L given by,

$$C_L = \frac{\mu - L}{\sigma}, \tag{34}$$

where μ is process mean, σ is process standard deviation and L is the lower specification limit. Using(11) and since the distribution of Y is exponential distribution, the equation (34) can be written as, see [16], $C_L = 1 - \theta L$, $C_L < 1$. Using (24)the UMVUE of C_L is given by

$$\widetilde{C_L} = 1 - \frac{(m-1)L}{\sum_{i=1}^{m}(1+r_i)\log\left(\frac{x_i}{\mu}\right)}. \tag{35}$$

Remark 5.1. *In case of Type II censored sample that is substituting $r_i = 0, i = 1, 2, \ldots, m-1$ and $r_m = n - m$ in (35), one gets the result given in Hong et al. [17].*

Table 1: Observed times of failure and the dropouts.

i	x_i	r_i
1	308.79	1
2	314.81	1
3	325.83	2
4	369.09	0
5	373.12	0
6	375.88	0
7	376.26	0
8	390.55	1
9	428.54	0
10	465.77	0
11	516.88	1
12	274.34	7

6. Illustrative example

In this section we illustrate the use of the estimation methods given in this article using the data given in Amin [1]. He has generated a progressive Type II censored data with binomial removals for the Pareto distribution with $\mu = 300$ and $\theta = 1.5$ for $n = 25$ units and $m = 12$. The data for the removals r_i were generated from the binomial distribution as follows: r_1 from *B(13, 0.05)* and $r_i / r_1, r_2, \ldots, r_{i-1}$ from $B(13 - \sum_{j=1}^{i-1} r_j, 0.05)$ for $i = 2, 3, \ldots, 12$. The data given in Table 1 is taken from Amin [1].

Using the results given in Sections 3 and 4, the UMVUE estimates of different parametric functions of θ based on data given in Table 1 are given below.

Sr.No.	*Parametric function*	*UMVUE estimate*
1	$\exp[-\theta]$	$H_{1,12} = 0.26898$
2	*Variance of* $H_{1,12}$	$\widetilde{\mathrm{Var}}[H_{1,12}] = 0.01181$
3	$\exp[\theta]$	$H_{-1,12} = 3.23153$
4	*Variance of* $H_{-1,12}$	$\widetilde{\mathrm{Var}}[H_{-1,12}] = 1.11683$
5	$\frac{1}{\theta}$	$G_{1,12} = 0.740574$
6	*Variance of* $G_{1,12}$	$\widetilde{\mathrm{Var}}[G_{1,12}] = 0.04218$
7	θ	$G_{-1,12} = 1.237780$
8	*Variance of* $G_{-1,12}$	$\widetilde{\mathrm{Var}}[G_{-1,12}] = 0.13928$
9	*Geometric mean*	$\widetilde{GM} = 316.8492$
10	*Logarithm of third quartile*	$\widetilde{\log\xi_{0.75}} = 3.17176$
11	*Hazard function at* $x = 310$	$\widetilde{h}(x) = 0.04059$
12	*Density at* $x = 310$	$\phi_{310,12} = 0.00385$
13	*Variance of* $\phi_{310,12}$	$\widetilde{\mathrm{Var}}[\phi_{310,12}] = 0.003695$
14	*Relibility at* $x = 310$	$\widetilde{R}(x) = 0.96015$
15	*Variance of* $\widetilde{R}(x)$	$\widetilde{\mathrm{Var}}[\widetilde{R}(x)] = 0.000139$
16	*c.d.f. at* $x = 310$	$\widetilde{F}(x) = 0.03985$
17	*Mean of Pareto Distribution*	$\widetilde{E}(X) = 890.7971$

Using (32), the 95% confidence interval for third quartile is $(561.00398, 2187.89775)$.

7. Conclusion

We thank one of the referees for bringing to our notice the paper, Asgharzadeh and Valiollahi [3]. We would like to add the following in this regard. The authors have obtained the UMVUE of θ, hazard function and reliability, for proportional hazards family under progressive Type II censored sample. They have also given $100(1-\alpha)\%$ confidence interval for θ. Our approach in getting these estimators is quite different and easy from the approach given in [3]. We have assumed that removals have binomial distribution. We have obtained the UMVUE of (i) $\left(\frac{1}{\theta}\right)^k$, (ii) $\exp[-\theta k]$, (iii) reliability function (iv) density function, where one can choose the appropriate values of k. We have given an elegant expressions for the UMVUE of the variance of UMVUE of these estimators. We have also obtained the UMVUE of geometric mean, r^{th} moment, mean and c.d.f. of Pareto distribution. We have also obtained the UMVUE of lifetime performance index. Using the UMVUE of the logarithm of p^{th} quantile, we have obtained the $(1-\alpha)100\%$confidence interval for p^{th} quantile. Many results given in Asarabadi [4], can be obtained as a particular case of our result. Inference when μ is unknown, as pointed out by the referee, will be investigated separately.

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