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On discrete distributions with gaps having ALM property

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Abstract. Almost lack of memory property of discrete distributions with gaps in their support is defined and it is observed that a random variable cX has lack of memory property on $\{0, c, 2c, \ldots\}$, $c > 1$ integer *iff* X has lack of memory property on $\{0, 1, 2, \ldots\}$. Also infinite divisibility of discrete distributions having almost lack of property is discussed.

1. Introduction

Geometric distribution plays a vital role both in distribution theory and reliability theory because of its remarkable lack of memory (LM) property, which characterizes the distribution. This property leads to similar properties in reliability context such as constancy of hazard rate, measure of memory being equal to zero and so on. Since the property and its manifestations characterize the distribution, the extensions of the property should lead to a wider class of distributions of which the geometric distribution becomes a special case.

One direction in which this property has been extended is in defining the concept of almost lack of memory (ALM) property by Chukova and Dimitrov [1] and Chukova et al. [2]. Chukova et al. [2] showed that ALM property is equivalent to periodic failure rate of the corresponding distributions and gave a characterization of ALM property in terms of service time properties. Dimitrov et al. [3] showed that non-stationary Poisson process with periodic failure rate is the closest extension of homogeneous Poisson process to model the number of events imbedded in to random environment of periodic failure. They also mention some possible applications in reliability, queues, environmental studies etc.

Just like the role of geometric distribution in discussing LM property here we consider an extension of geometric distribution and discuss its role in ALM property. This extension is called extended geometric distribution on the set of integers $\{0, k, 2k, \ldots\}$, $k \geq 1$, integer. The corresponding random variable (*r.v.*) has probability mass function (*p.m.f.*) defined by

$$P(X = kx) = pq^x, \quad x = 0, 1, 2, \ldots \tag{1}$$

Johnson et al. [6] mentions that geometric distribution may be extended to cover the case of a variable taking values θ_0, $\theta_0 + \delta$, $\theta_0 + 2\delta$, $\ldots$ ($\delta > 0$) with p.m.f.

$$P(X = \theta_0 + \delta j) = pq^j, \; j = 0, 1, 2, \ldots \tag{2}$$

We see that probabilities of extended geometric distribution coincide with that of geometric distribution, but the support of the former is the set of integers $\{0, k, 2k, \ldots\}$ instead of $\{0, 1, 2, \ldots\}$. When $k = 1$, the extended

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geometric distribution reduces to a geometric distribution. For more properties and characterizations of extended geometric distribution see, Sandhya et al. [9].

This paper is organized as follows. In Section 2 we define LM property on $\{0, k, 2k, \ldots\}$ and characterize extended geometric distribution. Next in Section 3 we discuss ALM property for distributions with gaps in their support. Section 4 deals with infinite divisibility of distributions with ALM property, where we show that distributions with ALM property need not be infinitely divisible.

2. Extended geometric distribution and ALM property

The LM property is said to be satisfied by a discrete *r.v.* if for all $x, j \geq 0$, integers

$$P(X \geq x + j/X \geq j) = P(X \geq x). \tag{3}$$

Johnson et al. [6] mentions that the characterization (3) also applies to the *r.v.* X taking values $(\theta_0 + \delta j)$, $j = 0, 1, 2, \ldots$. We know that the above property holds good *iff* X follows a geometric distribution on $\{0, 1, 2, \ldots\}$. We now have

Definition 2.1. *For a r.v. X on* $\{0, k, 2k, \ldots\}$, $P(X \geq (j + x)k/X \geq jk) = P(X \geq xk)$ *for all* $j, x \geq 0$, *integers describes LM property on* $\{0, k, 2k, \ldots\}$.

Consider a geometric distribution on $\{0, k, 2k, \ldots\}$ with $P(X = jk) = pq^j$, $j = 0, 1, 2, \ldots$. For this distribution we have

$$P(X \geq (j + x)k/X \geq jk) = q^x = P(X \geq xk).$$

Now arguing on the lines similar to the proof of Theorem 5 in Rohatgi and Saleh [8, p.189] the converse of the above follows resulting in a geometric distribution on $\{0, k, 2k, \ldots\}$. Hence we have:

Theorem 2.2. *A r.v. X on* $\{0, k, 2k, \ldots\}$ *has LM property iff it is extended geometric.*

It is worth mentioning that for a geometric *r.v.* with *p.m.f.* given by $P(X = x) = pq^{x-1}$, $x = 1, 2, \ldots$, $P(X \geq x + j/X \geq j) = q^x \neq P(X \geq x)$. Hence (3) does not characterize the geometric distribution on $\{1, 2, \ldots\}$.

Chukova and Dimitrov [1] discuss in detail the concept of ALM property as an extension to LM property. It is shown that *r.v.*s X exist, not exponentially or geometrically distributed such that (3) holds for all $x > 0$ and infinitely many different values of j. Also, Chukova et al. [2] define LM property at a given point $c > 0$.

Definition 2.3. *A non-negative integer-valued r.v. X has the LM property at the point* $c > 0$ *integer iff*

$$P(X \geq x + c/X \geq c) = P(X \geq x), \text{ for all } x \geq 0. \tag{4}$$

Definition 2.4. *A non-negative integer-valued r.v. X has ALM property if there exists a sequence of distinct constants* $\{a_n\}_{n=1}^{\infty}$ *such that* (4) *holds for any integer* $c = a_n$, $n = 1, 2, \ldots$ *and for all* $x \geq 0$.

Also from Marsaglia and Tubilla [7], a *r.v.* possesses ALM property only at points $\{a_n\}_{n=1}^{\infty}$ which form a lattice. That is, in the case of integer-valued *r.v.*s, there exists an integer $c > 0$ such that all a_n satisfying Definition 2.4 are in the form $a_n = k_n c$ where $\{k_n\}$ is a sequence of positive integers. Now we have the following result by Chukova et al. [2].

Lemma 2.5. *If a r.v. X has the LM property at a point* $c > 0$ *integer, then X has the ALM property over the sequence* $\{a_n = nc\}_{n=0}^{\infty}$.

We know that a geometric *r.v.* X has LM property at integers $\{0, 1, 2, \ldots\}$. Then X has ALM property over $\{0, c, 2c, \ldots\}$ by the above lemma. Hence,

$$P(X \geq (x + j)c/X \geq jc) = P(X \geq xc), \text{ for all } x \geq 0 \text{ and } \frac{P(X \geq xc + jc)}{P(X \geq jc)} = q^x = P(X \geq xc). \tag{5}$$

Thus (5) characterizes an extended geometric distribution on $\{0, c, 2c, \ldots\}$ defined by (2). Hence it follows that an extended geometric distribution has LM property on $\{0, c, 2c, \ldots\}$. We record this as:

Theorem 2.6. *A r.v. cX has LM property on* $\{0, c, 2c, \ldots\}$, $c \geq 1$ *integer iff X has LM property on* $\{0, 1, 2, \ldots\}$.

Note. Theorem 2.6 characterizes geometric and extended geometric distributions simultaneously.

Next is a stochastic representation of *r.v.*s having LM property at $c > 0$ as the sum of two independent *r.v.*s given in Chukova et al [2].

Theorem 2.7. *A r.v. X has the LM property at a point* $c > 0$ *integer, (X has the ALM property over the sequence* $\{a_n = nc\}_{n=0}^{\infty}$*) iff it is decomposable in the form* $X = Y_c + cZ$*, where* Y_c *and Z are independent r.v.s,* Y_c *is concentrated on the interval* $[0, c)$ *and Z has a geometric distribution on* $\{0, 1, 2, \ldots\}$ *with parameter* $\alpha = P(X \geq c)$.

In the case of non-negative integer-valued *r.v.*s the above representation holds with Y_c is concentrated on $\{0, \ldots, c-1\}$, $c > 0$ integer. We know that (Sandhya et al. [10]) a direct relationship between geometric and extended geometric distributions is given by: A *r.v.* X follows geometric distribution on $\{0, 1, 2, \ldots\}$ iff cX follows an extended geometric distribution on $\{0, c, 2c, \ldots\}$. Hence the above stochastic representation can be equivalently written as: $X = Y_c + G_c$ where G_c is an extended geometric *r.v.* on $\{0, c, 2c, \ldots\}$. Now it follows that a *r.v.* can have LM at points $\{0, c, 2c, \ldots\}$ and this set of points coincides with the support of the extended geometric distribution. Taking the probability generating function (*p.g.f*) on both sides we get,

$$P(s) = Q_c(s)\frac{1-\alpha}{1-\alpha s^c}.$$

Chukova and Dimitrov [1] mention that the geometric distribution on $\{0, 1, 2, \ldots\}$, belongs to the class of distributions with ALM property with $Q_c(s) = \frac{p}{1-\alpha}\frac{1-\alpha s^c}{1-qs}$ where $\alpha = q^c$. Also, an extended geometric distribution on $\{0, c, 2c, \ldots\}$ belongs to this class with $Q_c(s) = 1$ and $q = \alpha$. They also mention that when $Q_c(s)$ corresponds to the p.g.f of a uniform distribution on $\{0, 1, 2, \ldots, c-1\}$ we get a distribution having ALM property that is neither geometric nor extended geometric. It can be easily seen that when $Q_c(s) = (q+ps)^{c-1}$, (the p.g.f of a binomial distribution with parameters $c-1$ and p) we get a p.g.f $P(s) = (q+ps)^{c-1}\frac{1-\alpha}{1-\alpha s^c}$ having ALM property with conditions on p, α and c.

3. ALM property for distributions with gaps in their support

Distributions with gaps in their support have a key role in random summation schemes in connection with their stability and infinite divisibility see Satheesh [11] and Satheesh et al. [12]. It may be noted that from Lemma 2.5 and Theorem 2.7 it also follows that a *r.v.* X having ALM property has the stochastic representation,

$$X = Y_c + cZ. \tag{6}$$

Further, to discuss the ALM property of distributions with gaps we cannot start with the stochastic representation (6) since here the support of the distribution on the LHS cannot have gaps. This also justifies the requirement for the following definition of ALM property of distributions with gaps in their support.

Definition 3.1. *A r.v. X on* $\{0, c, 2c, \ldots\}$, $c \geq 1$ *integer, is said to have LM property at a given integer* $l = jc$*, if* $P\{X \geq cx + l / X \geq l\} = P\{X \geq cx\}$ *for all* $x \geq 0$ *integer.*

Definition 3.2. *A r.v. X on* $\{0, c, 2c, \ldots\}$, $c \geq 1$ *integer, is said to have ALM property over the sequence* $\{nl\}_{n=0}^{\infty}$ *if it has LM property at l.*

Hence it follows that if X has LM property at $l = jc$, then it has ALM property over $\{njc\}_{n=0}^{\infty}$.

Theorem 3.3. *Let* $W = cX$, $c \geq 1$ *integer, be a r.v. on* $\{0, c, 2c, \ldots\}$*. Then W has LM property at* $l = jc$ *iff X has LM property at j.*

Proof. Suppose that a *r.v.* X has LM property at a point $j > 0$. Then

$$P(X \geq x + j/X \geq j) = P(X \geq x),\ \forall x \geq 0.$$

That is,

$$P(cX \geq c(x + j)/cX \geq cj) = P(cX \geq cx),\ \forall x \geq 0.$$

That is,

$$P(W \geq cx + l)/W \geq l) = P(W \geq cx),\ \forall x \geq 0.$$

Thus W has LM property at l by Definition 3.1. The converse is straight forward. □

Corollary 3.4. *If a r.v.* $W = cX$, $c \geq 1$ *integer with support on* $\{0, c, 2c, \ldots\}$ *has ALM property over the sequence over* $\{nl\}_{n=0}^{\infty}$ *or* $\{njc\}_{n=0}^{\infty}$ *iff* X *has ALM property on* $\{nj\}_{n=0}^{\infty}$.

Thus it follows that the ALM property of cX at the point $l = jc$ is equivalent to the ALM property of X at the point $\frac{l}{c} = j$. Hence in order to discuss the ALM property of $W = cX$, we need discuss the ALM property of X. However, if W has support on $\{0, c, 2c, \ldots\}$, then it has the following stochastic representation, $cX = cY_c + c^2Z$ or $W = cY_c + G_{c^2}$, where G_{c^2} has an extended geometric distribution on $\{0, c^2, 2c^2, \ldots\}$. Taking the *p.g.f.*s on both sides stochastic representation we get, $P(s^c) = Q(s^c)\frac{1-\alpha}{1-\alpha s^{c^2}}$. Here $Q(s^c)$ corresponds to a finite discrete distribution on $\{0, c^2, 2c^2, \ldots, c^2 - c\}$. Thus to discuss the ALM property of an extended geometric distribution on $\{0, c, 2c, \ldots\}$ it is enough to discuss the ALM property of geometric distribution on $\{0, 1, 2, \ldots\}$. But we have already seen that the above geometric distribution has the ALM property with

$$Q_c(s) = \frac{p}{1-\alpha}\frac{1-\alpha s^c}{1-qs}$$

where $\alpha = q^c$. Also, the p.g.f.

$$P(s^c) = (q + ps^c)^{c^2-c}\frac{1-\alpha}{1-\alpha s^{c^2}}.$$

can be shown to have ALM property using the same argument.

Now consider another discrete distribution with gaps in its support and with *p.g.f.* $\frac{1}{(m-(m-1)s^c)^{\frac{1}{c}}}$, $c \geq 1$ integer and $m > 1$, which is called Harris distribution and is denoted by $H_0(m, c, \frac{1}{c})$. For more on this see, Sandhya et al. [10] Now it can be seen that if $Y \sim H_0(m, c, \frac{1}{c})$, then $X = \frac{Y}{c}$ follows a negative binomial distribution on $\{0, 1, 2, \ldots\}$ with p.g.f $\frac{1}{(m-(m-1)s)^{\frac{1}{c}}}$. Thus by Theorem 3.3 it is enough to discuss the ALM property of the above negative binomial distribution. That is, we have to check whether it admits the following representation

$$\frac{1}{(m-(m-1)s)^{\frac{1}{c}}} = Q_c(s)\frac{1}{m-(m-1)s^c},$$

where $Q_c(s)$ corresponds to the p.g.f. of a *r.v.* on $\{0, 1, \ldots, c-1\}$. Thus we have to show that

$$Q_c(s) = \frac{m-(m-1)s^c}{(m-(m-1)s)^{\frac{1}{c}}}$$

is a p.g.f. for the ALM property to be satisfied. But, if $Q_c(s)$ is a *p.g.f.* $Q_c(1) = 1$ and it should be absolutely monotone, see Feller [4, p. 223]. Clearly, $Q_c(1) = 1$ only if $c = 1$, that is, the negative binomial distribution reduces to the geometric distribution. Hence there is no ALM property for the above negative binomial distribution. It may also be noted that for absolute monotonicity $Q_c^{(n)}(s) \geq 0$, for all $n \geq 1$ integer. Now,

$$Q_c^{(1)}(s) = \frac{(m-1)(m-(m-1)s)^{\frac{1}{c}-1}}{(m-(m-1)s)^{\frac{2}{c}}} \left\{ \frac{m}{c} + (m-1)s^c \left(\frac{c^2-1}{c} \right) - cms^{c-1} \right\}.$$

But $Q_c^{(1)}(s) \geq 0$ only if $\frac{m}{c} + (m-1)s^c \left(\frac{c^2-1}{c} \right) - cms^{c-1} \geq 0$; that is when

$$\frac{m-1}{m} \geq \frac{c^2 s^{c-1} - 1}{(c^2-1)s^c}.$$

This is possible only if

$$1 \geq \frac{1}{m} + \frac{1}{s} \left(\frac{c^2}{c^2-1} \right) - \frac{1}{(c^2-1)s^c}.$$

But letting $c = 2$ and $s = 0.5$, we get

$$\frac{1}{s} \left(\frac{c^2}{c^2-1} \right) - \frac{1}{(c^2-1)s^c} = \frac{4}{0.5(4-1)} - \frac{1}{(0.5)^2(4-1)} = \frac{4}{3} > 1.$$

Thus $Q_c^{(1)}(s)$ is negative for these values of c and s and consequently $Q_c(s)$ cannot be a *p.g.f.* Hence negative binomial distribution does not have ALM property. Consequently, Harris distribution also does not have ALM property by Theorem 3.3.

4. Infinite divisibility of discrete distributions with ALM property

We have seen that the *r.v.* X with p.g.f $P(s)$ is said to have ALM property if $P(s) = Q_c(s)(\frac{1-\alpha}{1-\alpha s^c})$. Since $Q_c(s)$ corresponds to the p.g.f of a *r.v.* on $\{0, 1, \ldots, c-1\}$ we can write $Q_c(s) = 1 + sp_1 + s^2 p_2 + \cdots + s^{c-1} p_{c-1}$, and we have

$$P(s) = \sum_{i=0}^{c-1} s^i \frac{1-\alpha}{1-\alpha s^c} p_i. \tag{7}$$

This is a finite mixture of geometric distributions on $\{0, c, 2c, \ldots\}$, $\{1, 1+c, 1+2c, \ldots\}$, $\{2, 2+c, 2+2c, \ldots\}$ respectively. Here the question is: is this mixture infinite divisible (i.d)? The question arises because of the already known result in Feller [5, p. 464] that every mixture of geometric distributions is infinitely divisible, where the mixture is of the type

$$G(s) = \sum_{i=0}^{c-1} \frac{1-\alpha_i}{1-\alpha_i s} p_i. \tag{8}$$

. For $P(s)$ to be i.d. it is necessary that $p_0 > 0$. Suppose that $c = 2$. Then we have

$$G(s) = (1-\alpha) \left[\sum_{i=0}^{1} s^i p_i \right] \left\{ \sum_{j=0}^{\infty} (\alpha s^2)^j \right\} = \sum_{n=0}^{\infty} q_n s^n, \tag{9}$$

where $q_0 = (1-\alpha)p_0$, $q_1 = (1-\alpha)p_1$ and $q_2 = (1-\alpha)\alpha^2 p_0$. For $P(s)$ to be i.d. it is necessary that $2q_2 q_0 \geq q_1^2$. That is, $2\alpha^2 p_0^2 \geq p_1^2$. This obviously does not hold always. Take for example, $\alpha = \frac{1}{2} = p_0$.

It is worth mentioning here that a geometric distribution on {0,1,2, ... } has ALM property and is i.d. The discussion above leads to a natural question (open) whether this characterizes geometric distribution.

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