Recurrence relations for marginal and joint moment generating functions of generalized logistic distribution based on lower k record values and its characterization

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Abstract. In this study we give exact expressions and some recurrence relations for marginal and joint moment generating functions of lower k record values from generalized logistic distribution. Further a characterization of this distribution by considering recurrence relations for marginal moment generating functions of the lower k record values is presented.

1. Introduction

The model of record statistics defined by Chandler [8] as a model for successive extremes in a sequence of independent and identically distributed (*iid*) random variables. This model takes a certain dependence structure into consideration. That is, the life-length distribution of the components in the system may change after each failure of the components. For this type of model, we consider the lower record statistics. If various voltages of equipment are considered, only the voltages less than the previous one can be recorded. These recorded voltages are the lower record value sequence.

Record values are found in many situations of daily life as well as in many statistical applications. Often we are interested in observing new records and in recording them: e.g. Olympic records or world records in sports. Motivated by extreme weather conditions, Feller [9] gave some examples of record values with respect to gambling problems. Resnick [20] discussed the asymptotic theory of records. Theory of record values and its distributional properties have been extensively studied in the literature. See Ahsanullah [1], Arnold and Balakrishnan [3], Arnold et al. [4, 5], Nevzorov [15] and Kamps [12] for reviews on various developments in the area of records.

Let $X_1, X_2, \ldots$ be a sequence of *iid* random variables with distribution function (df) $F(x)$ and probability density function (pdf) $f(x)$. Suppose $Y_n = \min\{X_1, X_2, \ldots, X_n\}$ for $n \geq 1$. We say $X_j, j \geq 1$, is lower record value of this sequence, if $Y_j < Y_{j-1}$ for $j > 1$. We suppose that X_1 is a first lower record value. The indices at which the lower record values occur are given by record times $\{L(n), n \geq 1\}$, where

$$L(n) = \min\{j | j > L(n-1), \ X_j < X_{(n-1)}\}, \ n > 1,$$

with $L(1) = 1$. For more details and references, see Ahsanullah [1] and Arnold et al. [5].

For a fixed $k \geq 1$ we define the sequence $\{L_n^{(k)}, n \geq 1\}$ of k lower record times of $\{X_n^{(k)}, n \geq 1\}$ as follows

$$L_1^{(k)} = 1, \quad L_{n+1}^{(k)} = \min\{j > L_n^{(k)}, X_{k:L_k(n)+k-1} > X_{k:j+k-1}\}.$$

For $k = 1$ and $n = 1, 2, \ldots$ we write $L_1^{(1)} = L_n$. Then $\{L_n, n \geq 1\}$ is the sequence of record times of $\{X_n, n \geq 1\}$. The sequence $\{Y_n^{(k)}, n \geq 1\}$, where $Y_n^{(k)} = X_{X_{L(n)}^{(k)}}$ is called the sequence of k lower record values of $\{X_n, n \geq 1\}$. For convenience, we shall also take $Y_0^{(k)} = 0$. Note that $k = 1$ we have $Y_n^{(1)} = X_{L_n}$, $n \geq 1$, which are record value of $\{X_n, n \geq 1\}$. Moreover $Y_1^{(k)} = \min\{X_1, X_2, \ldots, X_k\} = X_{1:k}$.

Let $\{X_n^{(k)}, n \geq 1\}$ be the sequence of k lower record values from (1). Then the *pdf* of $X_{L(n)}^{(k)}$, $n \geq 1$, is given by

$$f_{X_{L(n)}^{(k)}}(x) = \frac{k^n}{(n-1)!}[-ln(F(x))]^{n-1}[F(x)]^{k-1}f(x) \tag{1}$$

and the joint *pdf* of $X_{L(m)}^{(k)}$ and $X_{L(n)}^{(k)}$, $1 \leq m < n, n > 2$, is given by

$$f_{X_{L(m)}^{(k)}, X_{L(n)}^{(k)}}(x, y) = \frac{k^n}{(m-1)!(n-m-1)!}[-ln(F(x))]^{m-1}[-ln(F(y)) + ln(F(x))]^{n-m-1}[F(y)]^{k-1}\frac{f(x)}{F(x)}f(y), \ x < y. \tag{2}$$

Ahsanullah and Raqab [2], Raqab and Ahsanullah [18, 19] have established recurrence relations for moment generating functions of record values from Pareto and Gumbel, power function and extreme value distributions.

Recurrence relations for marginal and joint moment generating functions of generalized order statistics from power function distribution are derived by Saran and Singh [21]. Recurrence relations for single and product moments of record values from exponential and generalized extreme value distribution are derived by Balakrishnan and Ahsanullah [6] and Balakrishnan et al. [7]. Pawlas and Szynal [16, 17] and Saran and Singh [22] have established recurrence relations for single and product moments of k record values from Weibull, Gumbel and linear exponential distribution. Kumar [14] have established explicit expression and recurrence relations for single and product moments of k lower record values from exponentiated log-logistic distribution. Kamps [13] investigated the importance of recurrence relations of order statistics in characterization.

In the present study, we established some explicit expressions and recurrence relations for marginal and joint moment generating functions of lower k record values from generalized logistic distribution. A characterization of this distribution has also been obtained on using a recurrence relation for marginal moment generating function.

A random variable X is said to have generalized logistic distribution if its *pdf* is of the form

$$f(x) = \frac{e^{-\frac{x-\mu}{\sigma}}}{\sigma\left(1 + e^{-\frac{x-\mu}{\sigma}}\right)^2}, \ -\infty < x < \infty, \ \mu, \sigma > 0, \tag{3}$$

and the corresponding *df* is

$$F(x) = \frac{1}{\sigma\left(1 + e^{-\frac{x-\mu}{\sigma}}\right)}, \ -\infty < x < \infty, \ \mu, \sigma > 0. \tag{4}$$

Here σ is the shape parameter and μ is the location parameter. Generalized logistic distribution can be used to model data exhibiting a unimodal density function having some skewness present in the data, a feature which is common in practice. For more details on this distribution and its application one may refer to Gupta and Kundu [10].

2. Relations for marginal moment generating function

Note that for generalized logistic distribution defined in (4)

$$F(x) = \sigma\left(1 + e^{\frac{x+\mu}{\sigma}}\right)f(x). \tag{5}$$

The relation in (5) will be used to derive some recurrence relations for the moment generating function of lower k record values from generalized logistic distribution.

Let us denote the marginal moment generating functions of $X_L(n)$ by $M_{X_{L(n)}}(t)$ and its jth derivative by $M_{X_{L(n)}}^{(j)}(t)$.

We shall first establish the explicit expression for marginal moment generating functions of lower k record values $M_{X_{L(n):k}}(t)$. Using (1), we have

$$M_{X_{L(n):k}}(t) = \frac{k^n}{(n-1)!} \int_{-\infty}^{\infty} e^{tx}[F(x)]^{k-1}[-ln(F(x))]^{n-1}f(x)dx \tag{6}$$

on using the transformation $z = F(x)$ in (6), we get

$$M_{X_{L(n):k}}(t) = \frac{e^{-\mu t}k^n}{(n-1)!} \int_0^1 z^{k+\sigma t-1}(1-z)^{-\sigma t}[-lnz]^{n-1}dz.$$

On using the Maclaurine series expansion $(1-z)^{-t} = \sum_{p=0}^{\infty} \frac{(t)_p z^p}{p!}$, where

$$(t)_p = \begin{cases} 1, & p = 0, \\ t(t+1)\dots(t+p-1), & p = 1, 2, \dots \end{cases}$$

and integrating the resulting expression, we obtain

$$M_{X_{L(n):k}}(t) = e^{-\mu t}k^n \sum_{p=0}^{\infty} \frac{(\sigma t)_p}{p!(k+p+\sigma t)^n}, \quad t \neq 0. \tag{7}$$

Remark 2.1. *Setting $k = 1$ in (7), we deduce the explicit expression for marginal moment generating functions of lower record values from the generalized logistic distribution.*

Recurrence relations for marginal moment generating functions of lower k record values from df (1) can be derived in the following theorem.

Theorem 2.2. *For positive integers $k \geq 1$ and $n \geq 1$ and $r = 1, 2, \dots,$*

$$\left(1 + \frac{\sigma t}{k}\right)M_{X_{L(n):k}}^{(j)}(t) = M_{X_{L(n-1):k}}^{(j)}(t) - \frac{j\sigma}{k}M_{X_{L(n):k}}^{(j-1)}(t) - \frac{\sigma e^{\mu/\sigma}}{k}\left\{tM_{X_{L(n):k}}^{(j)}(t+1/\sigma) + jM_{X_{L(n):k}}^{(j-1)}(t+1/\sigma)\right\}. \tag{8}$$

Proof. From (1), we have

$$M_{X_{L(n):k}}(t) = \frac{k^n}{(n-1)!} \int_{-\infty}^{\infty} e^{tx}[F(x)]^{k-1}[-ln(F(x))]^{n-1}f(x)dx. \tag{9}$$

Integrating by parts taking $[F(x)]^{k-1}f(x)$ as the part to be integrated and the rest of the integrand for differentiation, we get

$$M_{X_{L(n):k}}(t) = M_{X_{L(n-1):k}}(t) - \frac{tk^n}{k(n-1)!} \int_{-\infty}^{\infty} e^{tx}[F(x)]^k[-ln(F(x))]^{n-1}dx$$

the constant of integration vanishes since the integral considered in (9) is a definite integral. On using (5), we obtain

$$
\begin{aligned}
M_{X_{L(n):k}}(t) &= M_{X_{L(n-1):k}}(t) - \frac{\sigma t k^n}{k(n-1)!}\Big\{ \int_{-\infty}^{\infty} e^{tx}[F(x)]^{k-1}[-ln(F(x))]^{n-1} f(x)dx \\
&\quad + e^{\mu/\sigma} \int_{-\infty}^{\infty} e^{(t+1/\sigma)x}[F(x)]^{k-1}[-ln(F(x))]^{n-1} f(x)dx\Big\} M_{X_{L(n):k}}(t) \\
&= M_{X_{L(n-1):k}}(t) - \frac{\sigma t}{k} M_{X_{L(n):k}}(t) - \frac{\sigma t e^{\mu/\sigma}}{k} M_{X_{L(n):k}}(t + 1/\sigma).
\end{aligned}
\tag{10}
$$

Differentiating both the sides of (10) j times with respect to t, we get

$$
M_{X_{L(n):k}}^{(j)}(t) = M_{X_{L(n-1):k}}^{(j)}(t) - \frac{\sigma t}{k} M_{X_{L(n):k}}^{(j)}(t) - \frac{j\sigma}{k} M_{X_{L(n):k}}^{(j-1)}(t) - \frac{t\sigma e^{\mu/\sigma}}{k} M_{X_{L(n):k}}^{(j)}(t + 1/\sigma) - \frac{j\sigma e^{\mu/\sigma}}{k} M_{X_{L(n):k}}^{(j-1)}(t + 1/\sigma).
$$

The recurrence relation in equation (8) is derived simply by rewriting the above equation.

By differentiating both sides of equation (8) with respect to t and then setting $t = 0$, we obtain the recurrence relations for single moment of lower k record values from generalized logistic distribution in the form

$$
E\big(X_{L(n):k}^j\big) = E\big(X_{L(n-1):k}^j\big) - \frac{j\sigma}{k}\Big\{ E\big(X_{L(n):k}^{j-1}\big) + e^{\mu/\sigma} E\big(\phi(X_{L(n):k})\big)\Big\},
\tag{11}
$$

where $\phi(x) = x^{j-1} e^{x/\sigma}$. $\quad\square$

Remark 2.3. *Setting $k = 1$ in (9), we deduce the recurrence relation for marginal moment generating functions of lower record values from the generalized logistic distribution.*

3. Relations for joint moment generating function

On using (2), the explicit expression for the joint moment generating of lower k record values $M_{X_{L(m,n):k}}(t_1, t_2)$ can be obtained

$$
M_{X_{L(m,n):k}}(t_1, t_2) = \frac{k^n}{(m-1)!(n-m-1)!} \int_{-\infty}^{\infty} e^{t_1 x}[-ln(F(x))]^{m-1} \frac{f(x)}{F(x)} I(x)dx
\tag{12}
$$

where

$$
I(x) = \int_{-\infty}^{x} e^{t_2 y}[-ln(F(y)) + ln(F(x))]^{n-m-1}[F(x)]^{k-1} f(y)dy.
\tag{13}
$$

By setting $w = ln(F(x)) - ln(F(y))$ in (13), we obtain

$$
I(x) = e^{-\mu t_2} \sum_{p=0}^{\infty} \frac{(\sigma t_2)_p \, [F(y)]^{k+p+\sigma t_2} \Gamma(n-m)}{p! [k + p + \sigma t_2]^{n-m}}.
$$

On substituting the above expression of $I(x)$ in (12) and simplifying the resulting equation, we obtain

$$
M_{X_{L(m,n):k}}(t_1, t_2) = k^n e^{-\mu(t_1+t_2)} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(\sigma t_2)_p}{p! [k + p + \sigma t_2]^{n-m}} \cdot \frac{(\sigma t_1)_q}{q! [k + p + q + \sigma(t_1 + t_2)]^m}.
\tag{14}
$$

Remark 3.1. *Setting $k = 1$ in (14), we deduce the explicit expression for joint moment generating function of lower record value for generalized logistic distribution.*

Making use of (1), we can drive recurrence relations for joint moment generating function of lower k record values from (4).

Theorem 3.2. *For $1 \leq m \leq n - 2$ and $r, s = 1, 2, \ldots,$*

$$\left(1 + \frac{\sigma t_2}{k}\right) M^{(i,j)}_{X_{L(m,n):k}}(t_1, t_2) = M^{(i,j)}_{X_{L(m,n-1):k}}(t_1, t_2) - \frac{j\sigma}{k} M^{(i,j-1)}_{X_{L(m,n):k}}(t_1, t_2)$$
$$- \frac{\sigma e^{\mu/\sigma}}{k}\left\{t_2 M^{(i,j)}_{X_{L(m,n):k}}(t_1, t_2 + 1/\sigma) + j M^{(i,j-1)}_{X_{L(m,n):k}}(t_1, t_2 + 1/\sigma)\right\}. \tag{15}$$

Proof. From (2) for $1 \leq m \leq n - 1$ and $r, s = 1, 2, \ldots,$

$$M_{X_{L(m,n):k}}(t_1, t_2) = \frac{k^n}{(m-1)!(n-m-1)!} \int_{-\infty}^{\infty} [-ln(F(x))]^{m-1} \frac{f(x)}{F(x)} G(x)dx, \tag{16}$$

where

$$G(x) = \int_{-\infty}^{x} e^{t_1 x + t_2 y} [-ln(F(y)) + ln(F(x))]^{n-m-1} [F(y)]^{k-1} f(y)dy.$$

Integrating $G(x)$ by parts treating $[F(y)]^{k-1}f(y)$ for integration and the rest of the integrand for differentiation, and substituting the resulting expression in (16), we get

$$M_{X_{L(m,n):k}}(t_1, t_2) = M_{X_{L(m,n-1):k}}(t_1, t_2) - \frac{t_2 k^n}{k(m-1)!(n-m-1)!} \int_{-\infty}^{\infty} \int_{-\infty}^{x} e^{t_1 x + t_2 y}$$
$$\times [-ln(F(x))]^{m-1} [-ln(F(y)) + ln(F(x))]^{n-m-1} [F(y)]^{k} \frac{f(x)}{F(x)} dydx$$

the constant of integration vanishes since the integral in $G(x)$ is a definite integral. On using the relation (5), we obtain

$$M_{X_{L(m,n):k}}(t_1, t_2) = M_{X_{L(m,n-1):k}}(t_1, t_2) - \frac{\sigma t_2 k^n}{k(m-1)!(n-m-1)!}\left\{\int_{-\infty}^{\infty} \int_{-\infty}^{x} e^{t_1 x + t_2 y} [-ln(F(x))]^{m-1}\right.$$
$$\times [-ln(F(y)) + ln(F(x))]^{n-m-1} [F(y)]^{k-1} \frac{f(x)}{F(x)} dydx$$
$$\left. + e^{\mu/\sigma} \int_{-\infty}^{\infty} \int_{-\infty}^{x} e^{t_1 x + (t_2 + 1/\sigma)y} [-ln(F(x))]^{m-1} [-ln(F(y)) + ln(F(x))]^{n-m-1} [F(y)]^{k-1} \frac{f(x)}{F(x)} dydx\right\},$$

$$M_{X_{L(m,n):k}}(t_1, t_2) = M_{X_{L(m,n-1):k}}(t_1, t_2) - \frac{\sigma t_2}{k} M_{X_{L(m,n):k}}(t_1, t_2) - \frac{\sigma t_2 e^{\mu/\sigma}}{k} M_{X_{L(m,n):k}}(t_1, t_2 + 1/\sigma). \tag{17}$$

Differentiating both the sides of (17) i times with respect to t_1 and then j times with respect to t_2, we get

$$M^{(i,j)}_{X_{L(m,n):k}}(t_1, t_2) = M^{(i,j)}_{X_{L(m,n-1):k}}(t_1, t_2) - \frac{\sigma t_2}{k} M^{(i,j)}_{X_{L(m,n):k}}(t_1, t_2) - \frac{j\sigma}{k} M^{(i,j-1)}_{X_{L(m,n):k}}(t_1, t_2)$$
$$- \frac{\sigma t_2 e^{\mu/\sigma}}{k} M^{(i,j)}_{X_{L(m,n):k}}(t_1, t_2 + 1/\sigma) - \frac{j\sigma e^{\mu/\sigma}}{k} M^{(i,j-1)}_{X_{L(m,n):k}}(t_1, t_2 + 1/\sigma),$$

which, when rewritten gives the recurrence relation in (15).

By differentiating both sides of equation (17) with respect to t_1, t_2 and then setting $t_1 = t_2 = 0$, we obtain the recurrence relations for product moments of lower k record values from generalized logistic distribution in the form

$$E\left(X^{(i,j)}_{L(m,n):k}\right) = E\left(X^{(i,j)}_{L(m,n-1):k}\right) - \frac{j\sigma}{k}\left\{E\left(X^{(i,j-1)}_{L(m,n):k}\right) + e^{\mu/\sigma} E\left(\phi(X_{L(m,n):k})\right)\right\}, \tag{18}$$

where $\phi(x, y) = x^i y^{j-1} e^{y/\sigma}$. $\quad\square$

Remark 3.3. *Setting $k = 1$ in (15), we deduce the recurrence relation for joint moment generating function of lower record value for generalized logistic distribution.*

4. Characterization

Theorem 4.1. *Let X be a non-negative random variable having an absolutely continuous distribution function $F(x)$ with $F(0) = 0$ and $0 < F(x) < 1$ for all $x > 0$, then*

$$M_{X_{L(n):k}}(t) = M_{X_{L(n-1):k}}(t) - \frac{\sigma t}{k} M_{X_{L(n):k}}(t) - \frac{\sigma t e^{\mu/\sigma}}{k} M_{X_{L(n):k}}(t + 1/\sigma) \tag{19}$$

if and only if

$$F(x) = \frac{1}{\sigma\left(1 + e^{-\frac{x-\mu}{\sigma}}\right)}, \quad -\infty < x < \infty, \ \mu, \sigma > 0.$$

Proof. The necessary part follows immediately from equation (10). On the other hand if the recurrence relation in equation (19) is satisfied, then on using equation (1), we have

$$\frac{k^n}{(n-1)!} \int_{-\infty}^{\infty} e^{tx} [F(x)]^{k-1} [-ln(F(x))]^{n-1} f(x) dx = \frac{(n-1)k^n}{k(n-1)!} \int_{-\infty}^{\infty} e^{tx} [F(x)]^{k-1} [-ln(F(x))]^{n-2} f(x) dx$$

$$- \frac{\sigma t k^n}{k(n-1)!} \int_{-\infty}^{\infty} e^{tx} [F(x)]^{k-1} [-ln(F(x))]^{n-1} f(x) dx$$

$$- \frac{\sigma t\, e^{\mu/\sigma}\, k^n}{k(n-1)!} \int_{-\infty}^{\infty} e^{(t+1/\sigma)x} [F(x)]^{k-1} [-ln(F(x))]^{n-1} f(x) dx. \tag{20}$$

Integrating the first integral on the right hand side of equation (20), by parts, we get

$$\frac{k^n}{(n-1)!} \int_{-\infty}^{\infty} e^{tx} [F(x)]^{k-1} [-ln(F(x))]^{n-1} f(x) dx = \frac{t k^n}{k(n-1)!} \int_{-\infty}^{\infty} e^{tx} [F(x)]^{k} [-ln(F(x))]^{n-1} dx$$

$$+ \frac{k^n}{(n-1)!} \int_{-\infty}^{\infty} e^{tx} [F(x)]^{k-1} [-ln(F(x))]^{n-1} f(x) dx - \frac{\sigma t k^n}{k(n-1)!} \int_{-\infty}^{\infty} e^{tx} [F(x)]^{k-1} [-ln(F(x))]^{n-1} f(x) dx$$

$$- \frac{\sigma t\, e^{\mu/\sigma}\, k^n}{k(n-1)!} \int_{-\infty}^{\infty} e^{(t+1/\sigma)x} [F(x)]^{k-1} [-ln(F(x))]^{n-1} f(x) dx$$

which reduces to

$$\frac{t k^n}{k(n-1)!} \int_{-\infty}^{\infty} e^{tx} [F(x)]^{k-1} [-ln(F(x))]^{n-1} \left\{ F(x) - \sigma f(x) - \sigma e^{(x+\mu)/\sigma} f(x) \right\} dx = 0. \tag{21}$$

Now applying a generalization of the Müntz-Szász Theorem (Hwang and Lin [11]) to equation (21), we get

$$\frac{f(x)}{F(x)} = \frac{1}{\sigma\left(1 + e^{\frac{x+\mu}{\sigma}}\right)}$$

which proves that

$$F(x) = \frac{1}{\sigma\left(1 + e^{-\frac{x-\mu}{\sigma}}\right)}, \quad -\infty < x < \infty, \ \mu, \sigma > 0.$$

□

5. Conclusion

In this study we give exact expressions and some recurrence relations for marginal and joint moment generating functions of lower k record values from generalized logistic distribution. Further, characterization of this distribution has also been obtained on using a recurrence relation for marginal moment generating functions.

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