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# A note on entropies of l-max stable, p-max stable, generalized Pareto and generalized log-Pareto distributions

**Sreenivasan Ravi, Ali Saeb**

*Department of Studies in Statistics, University of Mysore, Manasagangotri, Mysore 570006, India*

**Abstract.** Limit laws of partial maxima of independent, identically distributed random variables under linear normalization are called extreme value laws or l-max stable laws and those under power normalization are called p-max stable laws. We derive entropies of these and related laws and also of associated generalized Pareto, generalized log-Pareto and related distributions. Some illustrative graphs are also given.

## 1. Entropies of limit laws of normalized partial maxima

Shannon (1948) defines entropy of an absolutely continuous random variable (rv) $X$ with distribution function (df) $F_X$ and probability density function (pdf) $f_X$ as

$$H(X) = E(-\ln f_X(x)) = -\int_{-\infty}^{\infty} f_X(x)\ln f_X(x)dx, \tag{1}$$

with the convention that the integral is over all real values for which the density is positive. Since then many authors have studied entropy and its properties. A df that maximizes entropy within a class of dfs often turns out to have favourable properties as is known that the normal df has maximum entropy in the class of dfs having a specified variance. Johnson (2006) is a good reference to the application of entropy and information theory to limit theorems, especially the central limit theorem. See also Barron (1986) and Gnedenko and Korolev (1996) for important work on entropy related to limit theorems.

Extreme value laws have found applications in modelling partial maxima of independent, identically distributed (iid) random variables (rvs). Extreme value laws are limit laws of linearly normalized partial maxima of iid rvs. Limit laws of power normalized partial maxima of iid rvs have been called p-max stable laws.

In this note, our main interest is to derive entropies of extreme value laws, p-max stable laws and the related generalized Pareto, generalized log-Pareto distributions and a few of their generalizations. In the next sub-section we state and prove a few preliminary results on entropies which will be used in subsequent sections. Section 1.2 introduces extreme value laws followed by a sub-section on entropies of these laws. Section 1.4 is an introduction to the p-max stable laws followed by a sub-section on their entropies. Section 2 is on entropies of generalized Pareto and generalized log-Pareto laws. After introducing generalized

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Received: 14 May 2012; Revised: 02 June 2012; Accepted: 11 June 2012
*Email address:* ravi@statistics.uni-mysore.ac.in, sreenivasanravi@yahoo.com (Sreenivasan Ravi)

Pareto laws in Section 2.1, we discuss their entropies in Section 2.2. Section 2.3 introduces the generalized log-Pareto laws and Section 2.4 is on the entropies of these laws. An appendix contains some illustrative three dimensional graphs of entropies discussed in this note.

**Notation 1.1.** *Unless otherwise specified, throughout this note, we shall use the following notations. Let $X$ be an absolutely continuous rv with df $F_X$ and pdf $f_X$ and let $a > 0$, $b \in \mathbb{R}$, and $c > 0$ denote scale, location and shape constants, respectively. The df of rv $Y$ is denoted by $F_Y$ and its pdf by $f_Y$. Let $A = \{x : f_X(x) > 0\}$. The Euler's constant with approximate value 0.577 is denoted by $\gamma = -\int_0^\infty (\ln u)e^{-u}du$.*

### 1.1. A few preliminary results on entropy

**Lemma 1.2.** *If $Y = (X-b)/a$, then the entropy of $Y$ is given by $H(Y) = -(\ln a) + H(X)$.*

*Proof.* We have

$$\begin{aligned} F_Y(y) &= P(X \le ay+b) = F_X(ay+b), \quad f_Y(y) = af_X(ay+b), \quad \text{so that from (1),} \\ H(Y) &= -\int_{-\infty}^{\infty} (\ln(af_X(ay+b))\, af_X(ay+b)dy \\ &= -\int_{-\infty}^{\infty} (\ln(af_X(z)))f_X(z)dz \\ &= -(\ln a) + H(X). \end{aligned}$$

□

**Lemma 1.3.** *If $X$ is positive valued and $Y = (X^c - b)/a$, then the entropy of $Y$ is given by*

$$H(Y) = -\left(\ln \frac{a}{c}\right) + (c-1)E_X(\ln X) + H(X).$$

*Proof.* We have

$$\begin{aligned} F_Y(y) &= P\left(X \le (ay+b)^{\frac{1}{c}}\right) = F_X\left((ay+b)^{\frac{1}{c}}\right), \; y > -\frac{b}{a}, \\ f_Y(y) &= \frac{a}{c}(ay+b)^{\frac{1-c}{c}} f_X\left((ay+b)^{\frac{1}{c}}\right), \; y > -\frac{b}{a}, \end{aligned} \tag{2}$$

and as in the proof of the previous lemma, we have

$$\begin{aligned} H(Y) &= -\int_0^\infty \left(\ln\left(\frac{a}{c}z^{1-c}f_X(z)\right)\right) f_X(z)dz, \quad \text{upon simplification,} \\ &= -\left(\ln \frac{a}{c}\right) + (c-1)E_X(\ln X) + H(X). \end{aligned}$$

□

**Lemma 1.4.** *If $X$ is negative valued and $Y = -((-X)^c - b)/a$, then the entropy of $Y$ is given by*

$$H(Y) = -\left(\ln \frac{a}{c}\right) + (c-1)E_X(\ln(-X)) + H(X).$$

*Proof.* We have

$$\begin{aligned} F_Y(y) &= P\left(X \le -(-ay+b)^{\frac{1}{c}}\right) = F_X\left(-(-ay+b)^{\frac{1}{c}}\right), \quad y < b/a, \\ f_Y(y) &= \frac{a}{c}(-ay+b)^{\frac{(1-c)}{c}} f_X\left(-(-ay+b)^{\frac{1}{c}}\right), \; y < b/a, \end{aligned} \tag{3}$$

and as in the proof of Lemma 1.2, we have

$$\begin{aligned} H(Y) &= -\int_{-\infty}^{0}\left(\ln\left(-\frac{a}{c}(-z)^{1-c}f_X(z)\right)\right)f_X(z)dz, \text{ upon simplification,} \\ &= -\left(\ln\frac{a}{c}\right)+(c-1)E_X(\ln(-X))+H(X). \end{aligned}$$

□

**Remark 1.5.** *Note that the entropies in the above lemmata do not depend on the location parameter b.*

*1.2. Extreme value laws*

If $X_1, X_2, \ldots, X_n$, $n \geq 1$, are iid rvs with common df $F$, $M_n = X_1 \vee \ldots \vee X_n$ and

$$\lim_{n\to\infty} P(M_n \leq a_n x + b_n) = \lim_{n\to\infty} F^n(a_n x + b_n) = G(x), \quad x \in C(G), \tag{4}$$

for some norming constants $a_n > 0$, $b_n \in \mathbb{R}$, and some non-degenerate df $G$, where $C(G)$ is the set of all continuity points of $G$, we then say that $F$ belongs to the max domain of attraction of $G$ under linear normalization and denote it by $F \in \mathcal{D}_l(G)$. Using logarithms and the fact that $\lim_{x\to 1}\frac{(-\ln x)}{1-x} = 1$, it is easy to see that (4) is equivalent to

$$\lim_{n\to\infty} n\{1 - F(a_n x + b_n)\} = -\ln G(x), \quad x \in \{y : G(y) > 0\}. \tag{5}$$

Limit dfs $G$ satisfying (4) are the well known extreme value types of distributions, namely,

$$\begin{aligned} \text{the Fréchet law: } \Phi_\alpha(x) &= \begin{cases} 0, & x < 0, \\ e^{-x^{-\alpha}}, & 0 \leq x; \end{cases} \\ \text{the Weibull law: } \Psi_\alpha(x) &= \begin{cases} e^{-(-x)^{\alpha}}, & x < 0, \\ 1, & 0 \leq x; \end{cases} \\ \text{and the Gumbel law: } \Lambda(x) &= e^{-e^{-x}}, \quad x \in \mathbb{R}; \end{aligned}$$

$\alpha > 0$ being a parameter, with respective pdfs,

$$\begin{aligned} \text{the Fréchet density: } \phi_\alpha(x) &= \begin{cases} 0, & x \leq 0, \\ \alpha x^{-(\alpha+1)}e^{-x^{-\alpha}}, & 0 < x; \end{cases} \\ \text{the Weibull density: } \psi_\alpha(x) &= \begin{cases} \alpha|x|^{\alpha-1}e^{-|x|^{\alpha}}, & x < 0, \\ 0, & 0 \leq x; \end{cases} \\ \text{and the Gumbel density: } \lambda(x) &= e^{-x}e^{-e^{-x}}, \quad x \in \mathbb{R}. \end{aligned}$$

Here two dfs $G$ and $H$ are said to be of the same type if $G(x) = H(Ax + B)$, $x \in \mathbb{R}$, for constants $A > 0$, $B \in \mathbb{R}$. Criteria for $F \in \mathcal{D}_l(G)$ are well known (see, for example, Embrechts et al. (1997); Galambos (1987); Resnick (1987)).

It is well known that if $G$ is an extreme value distribution, then $G$ is max stable in the sense that $G$ satisfies $G^n(a_n x + b_n) = G(x)$, $x \in \mathbb{R}$; for some constants $a_n > 0$, $b_n \in \mathbb{R}$. If rv $X$ has a max stable distribution $G$ then for scale and location constants $a > 0$, $b \in \mathbb{R}$, $Y = \frac{X-b}{a}$ has df $F_Y(y) = P(X \leq ay + b) = G(ay + b)$, $y \in \mathbb{R}$. For constants $c_n > 0$, $d_n \in \mathbb{R}$, we then have

$$\begin{aligned} F_Y^n(c_n y + d_n) &= G^n(a(c_n y + d_n) + b) = G^n(ac_n y + ad_n + b), \\ &= G^n(a_n(ay + b) + b_n) = G(ay + b) = F_Y(y), \ y \in \mathbb{R}, \end{aligned}$$

with $a_n = ac_n > 0$, and $b_n = ad_n + b(1 - a_n)$. Hence the df of $Y$, $F_Y$ is also max stable.

If rv $X$ has the Fréchet law $\Phi_\alpha$, then the rv $Y = X^c$, $c > 0$, has df $\Phi_{\frac{\alpha}{c}}$, since $P(Y \le y) = P(X \le y^{\frac{1}{c}}) = e^{-y^{-\frac{\alpha}{c}}}$, $y > 0$. Similarly, if rv $X$ has the Weibull law $\Psi_\alpha$, then the rv $Y = -|X|^c$ has df $\Psi_{\frac{\alpha}{c}}$, since $P(Y \le y) = P(-|X|^c \le y) = P(X \le -\left|y\right|^{\frac{1}{c}}) = e^{-|y|^{\frac{\alpha}{c}}}$, $y \le 0$.

*1.3. Entropies of extreme value laws*

**Theorem 1.6.** *If rv X has*

*(i) Fréchet law then its entropy is given by* $H(X) = -\ln\alpha + \frac{\alpha+1}{\alpha}\gamma + 1$;

*(ii) Weibull law then its entropy is given by* $H(X) = -\ln\alpha + \frac{\alpha-1}{\alpha}\gamma + 1$;

*(iii) Gumbel law then its entropy is given by* $H(X) = \gamma + 1$.

*Proof.* (i) From (1), we have

$$\begin{aligned} H(X) &= -\int_0^\infty (\ln\phi_\alpha(x))\phi_\alpha(x)dx = -\int_0^\infty \left(\ln\left(\alpha x^{-\alpha-1}e^{-x^{-\alpha}}\right)\right)\alpha x^{-\alpha-1}e^{-x^{-\alpha}}dx, \\ &= -(\ln\alpha) + (\alpha+1)\int_0^\infty (\ln x)\alpha x^{-\alpha-1}e^{-x^{-\alpha}}dx + \int_0^\infty x^{-\alpha}\alpha x^{-\alpha-1}e^{-x^{-\alpha}}dx, \\ &= -(\ln\alpha) - \frac{\alpha+1}{\alpha}\int_0^\infty (\ln u)e^{-u}du + \int_0^\infty ue^{-u}du, \quad \text{upon simplification,} \\ &= -(\ln\alpha) + \frac{\alpha+1}{\alpha}\gamma + 1. \end{aligned}$$

(ii) As in the proof of (i) above, we have

$$\begin{aligned} H(X) &= -\int_{-\infty}^0 \left(\ln\left(\alpha(-x)^{\alpha-1}e^{-(-x)^\alpha}\right)\right)\alpha(-x)^{\alpha-1}e^{-(-x)^\alpha}dx, \\ &= -(\ln\alpha) - \frac{\alpha-1}{\alpha}\int_0^\infty (\ln u)e^{-u}du + \int_0^\infty ue^{-u}du, \quad \text{upon simplification,} \\ &= -(\ln\alpha) + \frac{\alpha-1}{\alpha}\gamma + 1. \end{aligned}$$

(iii) Similar to the proof of (i) above, we have

$$\begin{aligned} H(X) &= -\int_{-\infty}^\infty \left(\ln\left(e^{-x}e^{-e^{-x}}\right)\right)e^{-x}e^{-e^{-x}}dx, \\ &= -\int_0^\infty (\ln u)e^{-u}dx + \int_0^\infty ue^{-u}du, \quad \text{upon simplification,} \\ &= \gamma + 1. \end{aligned}$$

□

**Remark 1.7.** *If* $U \sim \Phi_\alpha$, *then* $V = -\frac{1}{U} \sim \Psi_\alpha$, *and* $W = \alpha\ln U \sim \Lambda$. *Note that* $H(V) = H(U) - \frac{2\gamma}{\alpha}$, *and* $H(W) = H(U) + (\ln\alpha) - \frac{\gamma}{\alpha}$. *Also,* $H(U)$ *and* $H(V)$ *decrease as* $\alpha$ *increases which can also be seen from the graphs given in the Appendix.*

**Theorem 1.8.** *If rv X has*

*(a) Fréchet law, then the entropy of* $Y = \frac{X^c - b}{a}$ *is given by*

$$H(Y) = -\left(\ln\frac{a\alpha}{c}\right) + \frac{c+\alpha}{\alpha}\gamma + 1;$$

(b) *Weibull law, then the entropy of* $Y = -\frac{|X|^c - b}{a}$ *is given by*

$$H(Y) = -\left(\ln \frac{a\alpha}{c}\right) + \frac{\alpha - c}{\alpha}\gamma + 1;$$

(c) *Gumbel law, then the entropy of* $Y = \frac{X-b}{a}$ *is given by*

$$H(Y) = -(\ln a) + \gamma + 1.$$

*Proof.* Using Lemma 1.3 and (i) of Theorem 1.6, we have

$$\begin{aligned} H(Y) &= -\left(\ln \frac{a}{c}\right) + (c-1)E_X(\ln X) + H(X), \\ &= -\left(\ln \frac{a\alpha}{c}\right) + \frac{c-1}{\alpha}\gamma + \frac{\alpha+1}{\alpha}\gamma + 1, \\ &= -\left(\ln \frac{a\alpha}{c}\right) + \frac{c+\alpha}{\alpha}\gamma + 1, \end{aligned}$$

since $E_X(\ln X) = \int_0^\infty \alpha x^{-\alpha-1} e^{-x^{-\alpha}} (\ln x) dx = -\frac{1}{\alpha}\int_0^\infty (\ln u) e^{-u} du = \dfrac{\gamma}{\alpha}$, proving (a).
To prove (b), we use Lemma 1.4 and (ii) of Theorem 1.6, to get

$$\begin{aligned} H(Y) &= -\left(\ln \frac{a}{c}\right) + (c-1)E_X(\ln(-X)) + H(X), \\ &= -\left(\ln \frac{a\alpha}{c}\right) + \frac{\alpha-c}{\alpha}\gamma + 1, \end{aligned}$$

since $E_X(\ln(-X)) = \int_{-\infty}^0 \alpha(-x)^{\alpha-1} e^{-(-x)^\alpha} (\ln(-x)) dx = -\frac{\gamma}{\alpha}$.
Similarly, to prove (c), we use Lemma 1.2 and (iii) of Theorem 1.6, to get

$$H(Y) = -(\ln a) + H(X) = -(\ln a) + 1 + \gamma.$$

□

### 1.4. *The p-max stable laws*

If the normalization in (4) is of the form $\left(\delta_n \mid x \mid^{\beta_n} \text{sign}(x)\right)$ with $\text{sign}(x) = -1,\ 0$ or $1$ according as $x < 0,\ = 0$ or $> 0$ instead of $(a_n x + b_n)$, for some norming constants $\delta_n > 0$, $\beta_n > 0$, then

$$\lim_{n\to\infty} P(M_n \le \delta_n |x|^{\beta_n} \text{sign}(x)) = \lim_{n\to\infty} F^n(\delta_n |x|^{\beta_n} \text{sign}(x)) = \mathcal{K}(x), \quad x \in C(\mathcal{K}), \tag{6}$$

for some non-degenerate df $\mathcal{K}$ and we say that $F$ belongs to the p-max domain of attraction of $\mathcal{K}$ and denote it by $F \in \mathcal{D}_p(\mathcal{K})$. Similar to (5), note that (6) is equivalent to

$$\lim_{n\to\infty} n\{1 - F(\delta_n |x|^{\beta_n} \text{sign}(x))\} = -\ln \mathcal{K}(x),\ x \in \{y : \mathcal{K}(y) > 0\}. \tag{7}$$

The limit laws $\mathcal{K}$ satisfying (6) are the p-types of the six p-max stable laws, namely,

$$\begin{aligned} \text{the log-Fréchet law: } \mathcal{K}_{1,\alpha}(x) &= \begin{cases} 0, & x < 1, \\ e^{-(\ln x)^{-\alpha}}, & 1 \le x; \end{cases} \\ \text{the log-Weibull law: } \mathcal{K}_{2,\alpha}(x) &= \begin{cases} 0, & x < 0, \\ e^{-(-\ln x)^{\alpha}}, & 0 \le x < 1, \\ 1, & 1 \le x; \end{cases} \\ \text{the standard Fréchet law: } \mathcal{K}_3(x) &= \Phi_1(x), x \in \mathbb{R}; \end{aligned}$$

$$
\begin{aligned}
\text{the negative log Fréchet law: } \mathcal{K}_{4,\alpha}(x) &= \begin{cases} 0, & x < -1, \\ e^{-(-\ln(-x))^{-\alpha}}, & -1 \le x < 0, \\ 1, & 0 \le x; \end{cases} \\
\text{the negative log-Weibull law: } \mathcal{K}_{5,\alpha}(x) &= \begin{cases} e^{-(\ln(-x))^{\alpha}} & x < -1, \\ 1, & -1 \le x; \end{cases} \\
\text{the standard Weibull law: } \mathcal{K}_{6}(x) &= \Psi_1(x), x \in \mathbb{R};
\end{aligned}
$$

$\alpha > 0$ being a parameter, with pdfs,

$$
\begin{aligned}
\text{the log-Fréchet density: } \kappa_{1,\alpha}(x) &= \begin{cases} \alpha x^{-1}(\ln x)^{-\alpha-1}e^{-(\ln x)^{-\alpha}} & x \ge 1, \\ 0, & x < 1; \end{cases} \\
\text{the log-Weibull density: } \kappa_{2,\alpha}(x) &= \begin{cases} \alpha x^{-1}(-\ln x)^{\alpha-1}e^{-(-\ln x)^{\alpha}} & 0 \le x < 1, \\ 0, & x < 0, 1 \le x; \end{cases} \\
\text{the standard Fréchet density: } \kappa_{3}(x) &= \phi_1(x), x \in \mathbb{R}; \\
\text{the negative log Fréchet density: } \kappa_{4,\alpha}(x) &= \begin{cases} \alpha(-x)^{-1}(-\ln(-x))^{-\alpha-1}e^{-(-\ln(-x))^{-\alpha}}, & -1 \le x < 0, \\ 0, & x < -1, 0 \le x; \end{cases} \\
\text{the negative log-Weibull density: } \kappa_{5,\alpha}(x) &= \begin{cases} \alpha(-x)^{-1}(\ln(-x))^{\alpha-1}e^{-(\ln(-x))^{\alpha}}, & x < -1, \\ 0, & -1 \le x; \end{cases} \\
\text{the standard Weibull density: } \kappa_{6}(x) &= \psi_1(x), x \in \mathbb{R}.
\end{aligned}
$$

Here two dfs $\mathcal{K}$ and Ł are said to be of the same $p$-type if $\mathcal{K}(x) = Ł(A|x|^B \operatorname{sign}(x))$, $x \in \mathbb{R}$, for constants $A > 0$, $B > 0$.

Pancheva (1984) studied limit laws of partial maxima of iid rvs under non-linear normalization and in particular, power normalization. For criteria for a df to belong to a p-max domain, we refer to Mohan and Ravi (1993), wherein it was also shown that if a df $F \in \mathcal{D}_l(G)$ for some max stable law $G$ then there always exists a p-max stable law $\mathcal{K}$ such that $F \in \mathcal{D}_p(\mathcal{K})$ and that the converse need not hold always. This shows that p-max stable laws attract more dfs to their max domains than the l-max stable laws. See also Falk et al. (2004) for criteria for dfs to belong to p-max domain of p-max stable laws.

It is well known that if $\mathcal{K}$ is a p-max stable law, then $\mathcal{K}$ satisfies $\mathcal{K}^n(\delta_n |x|^{\beta_n} \operatorname{sign}(x)) = \mathcal{K}(x)$, $x \in \mathbb{R}$; for some constants $\delta_n > 0$, $\beta_n > 0$. If rv $X$ has a p-max stable distribution $\mathcal{K}$ then for scale and shape constants $a > 0$ and $c > 0$, the rv $Y = \frac{|X|^c}{a}\operatorname{sign}(X)$, has df $F_Y(y) = P(X \le a\left|y\right|^{\frac{1}{c}} \operatorname{sign}(y)) = \mathcal{K}(a\left|y\right|^{\frac{1}{c}} \operatorname{sign}(y))$, $y \in \mathbb{R}$. For constants $\delta_n^* > 0$, $\beta_n^* > 0$, we then have

$$
\begin{aligned}
F_Y^n(\delta_n^* \left|y\right|^{\beta_n^*} \operatorname{sign}(y)) &= \mathcal{K}^n\left(a(\delta_n^*)^{\frac{1}{c}} \left|y\right|^{\frac{\beta_n^*}{c}} \operatorname{sign}(y)\right), \qquad (8) \\
&= \mathcal{K}^n\left(\delta_n\left(a\left|y\right|^{\frac{1}{c}}\right)^{\beta_n} \operatorname{sign}(y)\right), \\
&= \mathcal{K}(a\left|y\right|^{\frac{1}{c}} \operatorname{sign}(y)) = F_Y(y),\ y \in \mathbb{R},
\end{aligned}
$$

with $\delta_n = a^{1-\beta_n}(\delta_n^*)^{\frac{1}{c}}$, and $\beta_n = \beta_n^*$. Hence the df of $Y$, $F_Y$ is also p-max stable.

*1.5. Entropies of p-max stable laws*

The following theorem gives the entropies of the p-max stable laws.

**Theorem 1.9.** *If rv X has*

*(i) the log-Fréchet law, then its entropy is given by*

$$
H(X) = -(\ln\alpha) + \frac{\alpha+1}{\alpha}\gamma + \Gamma\left(1 - \frac{1}{\alpha}\right) + 1, \quad \alpha > 1;
$$

*(ii) the log-Weibull law, then its entropy is given by*

$$H(X) = -(\ln\alpha) + \frac{\alpha-1}{\alpha}\gamma - \Gamma\left(1+\frac{1}{\alpha}\right) + 1;$$

*(iii) the negative log-Fréchet law, then its entropy is given by*

$$H(X) = -(\ln\alpha) + \frac{\alpha+1}{\alpha}\gamma - \Gamma\left(1-\frac{1}{\alpha}\right) + 1, \quad \alpha > 1;$$

*(iv) the negative log-Weibull law, then its entropy is given by*

$$H(X) = -(\ln\alpha) + \frac{\alpha-1}{\alpha}\gamma + \Gamma\left(1+\frac{1}{\alpha}\right) + 1.$$

**Remark 1.10.** *Note that in the above and the next theorem, for $\alpha \leq 1$, the entropies of the log-Fréchet and the negative log-Fréchet laws do not exist.*

*Proof.* (i) From (1), we have

$$\begin{aligned} H(X) &= -\int_1^\infty (\ln\kappa_{1,\alpha}(x))\kappa_{1,\alpha}(x)dx, \\ &= -\int_1^\infty \ln\left(\alpha x^{-1}(\ln x)^{-\alpha-1}e^{-(\ln x)^{-\alpha}}\right)\left(\alpha x^{-1}(\ln x)^{-\alpha-1}e^{-(\ln x)^{-\alpha}}\right)dx, \\ &= -\int_0^\infty \ln\left(\alpha e^{-u}u^{-\alpha-1}e^{-u^{-\alpha}}\right)\left(\alpha u^{-\alpha-1}e^{-u^{-\alpha}}\right)du, \quad \text{upon simplification}, \\ &= -(\ln\alpha) + \frac{\alpha+1}{\alpha}\gamma + 1 + \Gamma\left(1-\frac{1}{\alpha}\right), \quad \alpha > 1. \end{aligned}$$

(ii) As in the proof of (i), we get

$$\begin{aligned} H(X) &= -\int_{-\infty}^0 \ln\left(\alpha e^{-u}(-u)^{\alpha-1}e^{-(-u)^{\alpha}}\right)\left(\alpha(-u)^{\alpha-1}e^{-(-u)^{\alpha}}\right)du, \quad \text{upon simplification}, \\ &= -(\ln\alpha) + \frac{\alpha-1}{\alpha}\gamma + 1 - \Gamma\left(1+\frac{1}{\alpha}\right). \end{aligned}$$

(iii) Similar to the proof of (i), we observe that

$$\begin{aligned} H(X) &= -\int_0^\infty \ln\left(\alpha e^{u}u^{-\alpha-1}e^{-u^{-\alpha}}\right)\left(\alpha u^{-\alpha-1}e^{-u^{-\alpha}}\right)du, \quad \text{upon simplification}, \\ &= -(\ln\alpha) + \frac{\alpha+1}{\alpha}\gamma + 1 - \Gamma\left(1-\frac{1}{\alpha}\right), \quad \alpha > 1. \end{aligned}$$

(iv) Here we get

$$\begin{aligned} H(X) &= -\int_{-\infty}^0 \ln\left(\alpha e^{u}(-u)^{\alpha-1}e^{-(-u)^{\alpha}}\right)\left(\alpha(-u)^{\alpha-1}e^{-(-u)^{\alpha}}\right)du, \quad \text{upon simplification}, \\ &= -(\ln\alpha) + \frac{\alpha-1}{\alpha}\gamma + 1 + \Gamma\left(1+\frac{1}{\alpha}\right). \end{aligned}$$

□

**Theorem 1.11.** *If rv X has*

(a) *the log-Fréchet distribution, then the entropy of* $Y = \frac{X^c - b}{a}$ *is given by*

$$H(Y) = -\left(\ln \frac{a\alpha}{c}\right) + c\Gamma\left(1 - \frac{1}{\alpha}\right) + 1 + \frac{\alpha+1}{\alpha}\gamma, \quad \alpha > 1;$$

(b) *the log-Weibull distribution, then the entropy of* $Y = \frac{X^c - b}{a}$ *is given by*

$$H(Y) = -\left(\ln \frac{a\alpha}{c}\right) - c\Gamma\left(1 + \frac{1}{\alpha}\right) + 1 + \frac{\alpha-1}{\alpha}\gamma;$$

(c) *the standard Fréchet distribution, then the entropy of* $Y = \frac{X^c - b}{a}$ *is given by*

$$H(Y) = -\left(\ln \frac{a}{c}\right) + (c+1)\gamma + 1.$$

(d) *the negative log-Fréchet distribution, then the entropy of* $Y = -\frac{(-X)^c - b}{a}$ *is given by*

$$H(Y) = -\left(\ln \frac{a\alpha}{c}\right) - c\Gamma\left(1 - \frac{1}{\alpha}\right) + 1 + \frac{\alpha+1}{\alpha}\gamma, \quad \alpha > 1;$$

(e) *the negative log-Weibull distribution, then the entropy of* $Y = -\frac{(-X)^c - b}{a}$ *is given by*

$$H(Y) = -\left(\ln \frac{a\alpha}{c}\right) + c\Gamma\left(1 + \frac{1}{\alpha}\right) + 1 + \frac{\alpha-1}{\alpha}\gamma;$$

(f) *the standard Weibull distribution, then the entropy of* $Y = -\frac{(-X)^c - b}{a}$ *is given by*

$$H(Y) = -\left(\ln \frac{a}{c}\right) + (1-c)\gamma + 1.$$

*Proof.* Using Lemma 1.3 and (i) of Theorem 1.9, we have

$$\begin{aligned} H(Y) &= -\left(\ln \frac{a}{c}\right) + (c-1)E_X(\ln X) + H(X), \\ &= -\left(\ln \frac{a\alpha}{c}\right) + c\Gamma\left(1 - \frac{1}{\alpha}\right) + 1 + \frac{\alpha+1}{\alpha}\gamma, \end{aligned}$$

since $E_X(\ln X) = \int_1^\infty \alpha x^{-1}(\ln x)^{-\alpha-1}e^{-(\ln x)^{-\alpha}}dx = \Gamma\left(1 - \frac{1}{\alpha}\right), \quad \alpha > 1$, proving (a).
To prove (b), we use Lemma 1.3 and (ii) of Theorem 1.9, to get

$$\begin{aligned} H(Y) &= -\left(\ln \frac{a}{c}\right) + (c-1)E_X(\ln X) + H(X), \\ &= -\left(\ln \frac{a\alpha}{c}\right) - c\Gamma\left(1 + \frac{1}{\alpha}\right) + 1 + \frac{\alpha-1}{\alpha}\gamma, \end{aligned}$$

since $E_X(\ln X) = \int_0^1 \alpha(\ln x)x^{-1}(-\ln x)^{\alpha-1}e^{-(-\ln x)^\alpha}dx = -\Gamma\left(1 + \frac{1}{\alpha}\right)$.
We prove (c) by using Lemma 1.3 and (i) of Theorem 1.6, to get

$$\begin{aligned} H(Y) &= -\left(\ln \frac{a}{c}\right) + (c-1)E_X(\ln X) + H(X), \\ &= -\left(\ln \frac{a}{c}\right) + (c+1)\gamma + 1, \end{aligned}$$

since $E_X(\ln X) = \int_0^\infty x^{-2}e^{-x^{-1}} \ln x dx = \gamma$.
(d) is proved by using Lemma 1.4 and (iii) of Theorem 1.9, to get

$$\begin{aligned} H(Y) &= -\left(\ln \frac{a}{c}\right) + (c-1)E_X(\ln(-X)) + H(X), \\ &= -\left(\ln \frac{a\alpha}{c}\right) - c\Gamma\left(1 - \frac{1}{\alpha}\right) + 1 + \frac{\alpha+1}{\alpha}\gamma, \end{aligned}$$

since $E_X(\ln(-X)) = \int_{-1}^0 \alpha \ln(-x)(-x)^{-1}(-\ln(-x))^{-\alpha-1}e^{-(-\ln(-x))^{-\alpha}}dx = -\Gamma\left(1-\frac{1}{\alpha}\right), \quad \alpha > 1$.
To prove (e), we use Lemma 1.4 and (iv) Theorem 1.9, and get

$$\begin{aligned} H(Y) &= -\left(\ln \frac{a}{c}\right) + (c-1)E_X(\ln(-X)) + H(X), \\ &= -\left(\ln \frac{a\alpha}{c}\right) + c\Gamma\left(1 + \frac{1}{\alpha}\right) + 1 + \frac{\alpha-1}{\alpha}\gamma, \end{aligned}$$

since $E_X(\ln(-X)) = \int_{-\infty}^{-1} \alpha \ln(-x)(-x)^{-1}(\ln(-x))^{\alpha-1}e^{-(\ln(-x))^{\alpha}}dx = \Gamma\left(1+\frac{1}{\alpha}\right)$.
Finally, (f) is proved by using Lemma 1.4 and (ii) of Theorem 1.6, to get

$$\begin{aligned} H(Y) &= -\left(\ln \frac{a}{c}\right) + (c-1)E_X(\ln(-X)) + H(\psi_1), \\ &= -\left(\ln \frac{a}{c}\right) - (c-1)\gamma + 1, \end{aligned}$$

since $E_X(\ln(-X)) = \int_{-\infty}^0 e^x \ln(-x)dx = -\gamma$. $\square$

## 2. Entropies of generalized Pareto laws

### 2.1. *The generalized Pareto laws*

The generalized Pareto distributions (gPDs) are limit laws of linearly normalized conditional excesses over a high threshold as the threshold tends to infinity. These have been used to model exceedances over high thresholds and were first studied by Balkema and de Haan (1974). It is well known that the extreme value laws or l-max stable laws and the gPDs are related by the relation $W(x) = 1 + \ln G(x), x \in \mathbb{R}$, where $G$ is an extreme value law and $W$ is the corresponding gPD. Because of this relationship, it is easy to see that location and scale versions of gPDs are gPDs and introduction of a shape parameter in a gPD will again lead to a gPD as is true for extreme value laws as seen earlier in Section 1.2. The gPDs also satisfy stability relations similar to the extreme value laws and we omit these details as these can be derived from the corresponding stability relations for the l-max stable laws given in Section 1.2.

We give below the gPDs and their entropies are given in the next section. Corresponding to the three types of extreme value laws, there are three possible types of gPDs, namely,

$$\begin{aligned} \text{the Pareto law: } W_{1,\alpha}(x) &= \begin{cases} 0, & x < 1, \\ 1 - x^{-\alpha}, & 1 \le x; \end{cases} \\ \text{the negative Beta (1,1) law: } W_{2,\alpha}(x) &= \begin{cases} 0, & x < -1, \\ 1-(-x)^{\alpha}, & -1 \le x < 0, \\ 1, & 0 \le x; \end{cases} \\ \text{and the standard exponential law: } W_3(x) &= \begin{cases} 0, & x < 0, \\ 1 - e^{-x}, & 0 \le x; \end{cases} \end{aligned}$$

$\alpha > 0$ being a parameter, with respective pdfs

$$\begin{aligned}
\text{the Pareto density: } w_{1,\alpha}(x) &= \begin{cases} 0, & x < 1, \\ \alpha x^{-\alpha-1}, & 1 \le x; \end{cases} \\
\text{the negative Beta (1,1) density: } w_{2,\alpha}(x) &= \begin{cases} 0, & x < -1 \ \& \ 0 \le x, \\ \alpha(-x)^{\alpha-1}, & -1 \le x < 0, \end{cases} \\
\text{and the standard exponential density: } w_3(x) &= \begin{cases} 0, & x < 0, \\ e^{-x}, & 0 \le x. \end{cases}
\end{aligned}$$

### 2.2. *Entropies of the gPDs*

**Theorem 2.1.** *If rv X has*

*(i) the Pareto law then its entropy is given by* $H(X) = -(\ln \alpha) + \frac{\alpha+1}{\alpha}$;

*(ii) the negative Beta(1,1) law then its entropy is given by* $H(X) = -(\ln \alpha) + \frac{\alpha-1}{\alpha}$;

*(iii) the standard exponential law then its entropy is given by* $H(X) = 1$.

*Proof.* For proving (i), we have, from (1),

$$\begin{aligned}
H(X) &= -\int_1^\infty \ln w_{1,\alpha}(x) w_{1,\alpha}(x)dx = -\int_1^\infty \left(\ln\left(\alpha x^{-\alpha-1}\right)\right)\alpha x^{-\alpha-1}dx, \\
&= -(\ln \alpha) + \frac{\alpha+1}{\alpha}\int_0^\infty ue^{-u}du, \quad \text{upon simplification,} \\
&= -(\ln \alpha) + \frac{\alpha+1}{\alpha}.
\end{aligned}$$

(ii) follows since

$$\begin{aligned}
H(X) &= -(\ln \alpha) - (\alpha - 1)\int_{-1}^0 (\ln(-x))\alpha(-x)^{\alpha-1}dx, \\
&= -(\ln \alpha) + \frac{\alpha-1}{\alpha}, \quad \text{upon simplification.}
\end{aligned}$$

Similarly, (iii) follows since $H(X) = \int_0^\infty xe^{-x}dx = 1$. □

**Theorem 2.2.** *If rv X has*

*(a) the Pareto law, then the entropy of* $Y = \frac{X^c - b}{a}$ *is given by*

$$H(Y) = -\left(\ln \frac{a\alpha}{c}\right) + \frac{c+\alpha}{\alpha};$$

*(b) the negative Beta(1,1) law, then the entropy of* $Y = -\frac{|X|^c - b}{a}$ *is given by*

$$H(Y) = -\left(\ln \frac{a\alpha}{c}\right) + \frac{\alpha - c}{\alpha};$$

*(c) the standard exponential law, then the entropy of* $Y = \frac{X-b}{a}$ *is given by*

$$H(Y) = -(\ln a) + 1.$$

*Proof.* From Lemma 1.3 and (i) of Theorem 2.1,

$$\begin{aligned} H(Y) &= -\left(\ln\frac{a}{c}\right) + (c-1)E_X(\ln X) + H(X), \\ &= -\left(\ln\frac{a\alpha}{c}\right) + \frac{c-1}{\alpha} + \frac{\alpha+1}{\alpha}, \\ &= -\left(\ln\frac{a\alpha}{c}\right) + \frac{\alpha+c}{\alpha}, \end{aligned}$$

since $E_X(\ln X) = \int_1^\infty \alpha x^{-\alpha-1}(\ln x)dx = \frac{1}{\alpha}$, proving (a).
To prove (b), we use Lemma 1.4 and (ii) of Theorem 2.1, to get

$$\begin{aligned} H(Y) &= -\left(\ln\frac{a}{c}\right) + (c-1)E_X(\ln(-X)) + H(X), \\ &= -\left(\ln\frac{a\alpha}{c}\right) + \frac{\alpha-c}{\alpha}, \end{aligned}$$

since $E_X(\ln(-X)) = \int_{-1}^0 \alpha(-x)^{\alpha-1}(\ln(-x))dx = -\frac{1}{\alpha}$.
For proving (c), from Lemma 1.2 and (iii) of Theorem 2.1, we get

$$H(Y) = -(\ln a) + H(X) = -(\ln a) + 1.$$

□

*2.3. The generalized log-Pareto laws*

Similar to the gPDs, one can introduce the generalized log-Pareto distributions or the glPDs using the *p*-max stable laws in place of the extreme value laws. Some properties of the glPDs were studied in Cormann and Reiss(2009). Properties satisfied by the scale and shape versions of the p-max stable laws similar to those discussed in Section 1.4 are also satisfied by the glPDs and we omit the details as these are straight forward. We list out the glPDs below and give their entropies in the next section. The glPDs are p-types of the following laws:

$$\text{the log-Pareto law: } Q_{1,\alpha}(x) = \begin{cases} 0, & x < e, \\ 1-(\ln x)^{-\alpha}, & x \geq e; \end{cases}$$

$$\text{the log-negative Beta(1,1) law: } Q_{2,\alpha}(x) = \begin{cases} 0, & x < e^{-1}, \\ 1-(-\ln x)^{\alpha}, & e^{-1} \leq x < 1, \\ 1, & 1 \leq x; \end{cases}$$

$$\text{the standard Pareto law: } Q_3(x) = \begin{cases} 0, & x < 1; \\ 1-x^{-1}, & x \geq 1, \end{cases}$$

$$\text{the negative log-Pareto law: } Q_{4,\alpha}(x) = \begin{cases} 0, & x < -e^{-1}, \\ 1-(-\ln(-x))^{-\alpha}, & -e^{-1} \leq x < 0, \\ 1, & 0 \leq x; \end{cases}$$

$$\text{the negative log-negative Beta(1,1) law: } Q_{5,\alpha}(x) = \begin{cases} 0, & x < -e, \\ 1-(\ln(-x))^{\alpha}, & -e \leq x < -1, \\ 1, & -1 \leq x; \end{cases}$$

$$\text{the standard negative Beta(1,1) law: } Q_6(x) = \begin{cases} 0, & x < -1, \\ 1+x, & -1 \leq x < 0, \\ 1, & 0 \leq x; \end{cases}$$

$\alpha > 0$ being a parameter, with respective pdfs,

$$q_{1,\alpha}(x) = \begin{cases} \frac{\alpha}{x}(\ln x)^{-\alpha-1}, & x \geq e, \\ 0, & x < e; \end{cases}$$

$$q_{2,\alpha}(x) = \begin{cases} \frac{\alpha}{x}(-\ln x)^{\alpha-1}, & e^{-1} \leq x < 1, \\ 0, & x < e^{-1} \;\&\; 1 \leq x; \end{cases}$$

$$q_3(x) = \begin{cases} 0, & x < 1; \\ x^{-2}, & x \geq 1, \end{cases}$$

$$q_{4,\alpha}(x) = \begin{cases} -\frac{\alpha}{x}(-\ln(-x))^{-\alpha-1}, & -e^{-1} \leq x < 0, \\ 0, & x < -e^{-1} \;\&\; 0 \leq 1; \end{cases}$$

$$q_{5,\alpha}(x) = \begin{cases} -\frac{\alpha}{x}(\ln(-x))^{\alpha-1}, & -e \leq x < -1, \\ 0, & x < -e \;\&\; -1 \leq x; \end{cases}$$

$$q_6(x) = \begin{cases} 1, & -1 \leq x < 0, \\ 0, & x < -1 \;\&\; 0 \leq x. \end{cases}$$

### *2.4. Entropies of the generalized log-Pareto laws*

**Theorem 2.3.** *If rv X has*

*(i) the log-Pareto law then its entropy is given by*

$$H(X) = -(\ln\alpha) + \frac{2\alpha^2 - 1}{\alpha(\alpha-1)}, \; \alpha > 1;$$

*(ii) the log-negative Beta (1,1) law then its entropy is given by*

$$H(X) = -(\ln\alpha) - \frac{1}{\alpha(\alpha+1)};$$

*(iii) the standard Pareto law then its entropy is given by* $H(X) = 2$;

*(iv) the negative log-Pareto law then its entropy is given by*

$$H(X) = -(\ln\alpha) - \frac{1}{\alpha(\alpha-1)}, \; \alpha > 1;$$

*(v) the negative log-negative Beta (1,1) law then its entropy is given by*

$$H(X) = -(\ln\alpha) + \frac{2\alpha^2 - 1}{\alpha(\alpha+1)};$$

*(vi) the standard negative Beta (1,1) law then its entropy is given by* $H(X) = 0$.

**Remark 2.4.** *Note that in the above and the next theorem, for $\alpha \leq 1$, the entropies of the log-Pareto and the negative log-Pareto laws do not exist.*

*Proof.* From (1), for $\alpha > 1$, we have

$$\begin{aligned}
H(X) &= -\int_0^1 \left(\ln\left(\alpha e^{-u^{-\frac{1}{\alpha}}} u^{\frac{\alpha+1}{\alpha}}\right)\right) du, \quad \text{upon simplification,} \\
&= -(\ln\alpha) + \int_0^1 u^{-\frac{1}{\alpha}} du - \frac{\alpha+1}{\alpha}\int_0^1 (\ln u) du, \\
&= -(\ln\alpha) + \frac{\alpha}{\alpha-1} + \frac{\alpha+1}{\alpha}, \\
&= -(\ln\alpha) + \frac{2\alpha^2-1}{\alpha(\alpha-1)}, \quad \text{proving (i).}
\end{aligned}$$

For proving (ii), from (1), we have

$$\begin{aligned}
H(X) &= -\int_0^1 \left(\ln\left(\alpha e^{u^{\frac{1}{\alpha}}} u^{\frac{\alpha-1}{\alpha}}\right)\right) du, \quad \text{upon simplification,} \\
&= -(\ln\alpha) - \frac{\alpha}{\alpha+1} + \frac{\alpha-1}{\alpha} \; = \; -(\ln\alpha) - \frac{1}{\alpha(\alpha+1)}.
\end{aligned}$$

Similarly, from (1), we have

$$H(X) = 2\int_1^\infty (\ln x) x^{-2} dx = -2\int_0^1 (\ln u) du = 2, \text{proving (iii).}$$

Again from (1), for $\alpha > 1$, we have

$$\begin{aligned}
H(X) &= -\int_0^1 \left(\ln\left(\alpha e^{u^{-\frac{1}{\alpha}}} u^{\frac{\alpha+1}{\alpha}}\right)\right) du, \quad \text{upon simplification,} \\
&= -(\ln\alpha) - \frac{\alpha}{\alpha-1} + \frac{\alpha+1}{\alpha}, \\
&= -(\ln\alpha) - \frac{1}{\alpha(\alpha-1)}, \quad \text{proving (iv).}
\end{aligned}$$

For proving (v), we have, from (1),

$$\begin{aligned}
H(X) &= -\int_0^1 \left(\ln\left(\alpha e^{-u^{\frac{1}{\alpha}}} u^{\frac{\alpha-1}{\alpha}}\right)\right) du, \quad \text{upon simplification,} \\
&= -(\ln\alpha) + \frac{\alpha}{\alpha+1} + \frac{\alpha-1}{\alpha} = -(\ln\alpha) + \frac{2\alpha^2-1}{\alpha(\alpha+1)}.
\end{aligned}$$

Similarly, for proving (vi), we have, from (1), $H(X) = 0$. □

**Theorem 2.5.** *If rv X has*

*(a) the log-Pareto distribution, then the entropy of* $Y = \frac{X^c - b}{a}$ *is given by*

$$H(Y) = -\left(\ln\frac{a\alpha}{c}\right) + \frac{\alpha(c-1)}{\alpha-1} + \frac{2\alpha^2-1}{\alpha(\alpha-1)}, \; \alpha > 1;$$

*(b) the log-negative Beta(1,1) distribution, then the entropy of* $Y = \frac{X^c - b}{a}$ *is given by*

$$H(Y) = -\left(\ln\frac{a\alpha}{c}\right) - \frac{\alpha(c-1)}{\alpha+1} - \frac{1}{\alpha(\alpha+1)};$$

*(c) the standard Pareto distribution, then the entropy of* $Y = \frac{X^c - b}{a}$ *is given by*

$$H(Y) = -\left(\ln \frac{a}{c}\right) + c + 1;$$

*(d) the negative log-Pareto distribution, then the entropy of* $Y = -\frac{(-X)^c - b}{a}$ *is given by*

$$H(Y) = -\left(\ln \frac{a\alpha}{c}\right) - \frac{\alpha(c-1)}{\alpha - 1} - \frac{1}{\alpha(\alpha-1)},\ \alpha > 1;$$

*(e) the negative log-negative Beta(1,1) distribution, then the entropy of* $Y = -\frac{(-X)^c - b}{a}$ *is given by*

$$H(Y) = -\left(\ln \frac{a\alpha}{c}\right) + \frac{\alpha(c-1)}{\alpha + 1} + \frac{2\alpha^2 - 1}{\alpha(\alpha+1)};$$

*(f) the standard negative Beta(1,1) distribution, then the entropy of* $Y = -\frac{(-X)^c - b}{a}$ *is given by*

$$H(Y) = -\left(\ln \frac{a}{c}\right) - c + 1.$$

*Proof.* Using Lemma 1.3 and (i) of Theorem 2.3, for $\alpha > 1$, we have

$$\begin{aligned} H(Y) &= -\left(\ln \frac{a}{c}\right) + (c-1)E_X(\ln X) + H(X), \\ &= -\left(\ln \frac{a\alpha}{c}\right) + \frac{\alpha(c-1)}{\alpha-1} + \frac{2\alpha^2 - 1}{\alpha(\alpha-1)}, \end{aligned}$$

since $E_X(\ln X) = \int_e^\infty \alpha x^{-1}(\ln x)^{-\alpha-1}dx = \int_0^1 u^{-\frac{1}{\alpha}} = \frac{\alpha}{\alpha-1}$, for proving (a).
For proving (b), we use Lemma 1.3 and (ii) of Theorem 2.3, to get

$$\begin{aligned} H(Y) &= -\left(\ln \frac{a}{c}\right) + (c-1)E_X(\ln X) + H(X), \\ &= -\left(\ln \frac{a\alpha}{c}\right) - (c-1)\frac{\alpha}{\alpha+1} - \frac{1}{\alpha(\alpha+1)}, \end{aligned}$$

since $E_X(\ln X) = \int_{e^{-1}}^1 \alpha(\ln x)x^{-1}(-\ln x)^{\alpha-1}dx = -\int_0^1 u^{\frac{1}{\alpha}} = -\frac{\alpha}{\alpha+1}$.
To prove (c), we use Lemma 1.3 and (iii) of Theorem 2.3, to get

$$\begin{aligned} H(Y) &= -\left(\ln \frac{a}{c}\right) + (c-1)E_X(\ln X) + H(X), \\ &= -\left(\ln \frac{a}{c}\right) + (c-1) + 2 \ = \ -\left(\ln \frac{a}{c}\right) + c + 1, \end{aligned}$$

since $E_X(\ln X) = \int_1^\infty x^{-2}(\ln x)dx = -\int_0^1 (\ln u)du = 1$.
For proving (d), from Lemma 1.4 and (iv) of Theorem 2.3, for $\alpha > 1$, we get

$$\begin{aligned} H(Y) &= -\left(\ln \frac{a}{c}\right) + (c-1)E_X(\ln(-X)) + H(X), \\ &= -\left(\ln \frac{a\alpha}{c}\right) - \frac{\alpha(c-1)}{\alpha-1} - \frac{1}{\alpha(\alpha-1)}, \end{aligned}$$

since $E_X(\ln(-X)) = \int_{-e^{-1}}^0 \alpha(\ln(-x))(-x)^{-1}(-\ln(-x))^{-\alpha-1}dx = -\int_0^1 u^{-\frac{1}{\alpha}} = -\frac{\alpha}{\alpha-1}$.

To prove (e), using Lemma 1.4 and (v) Theorem 2.3, we have

$$\begin{aligned} H(Y) &= -\left(\ln \frac{a}{c}\right) + (c-1)E_X(\ln(-X)) + H(X), \\ &= -\left(\ln \frac{a\alpha}{c}\right) + \frac{\alpha(c-1)}{\alpha+1} + \frac{2\alpha^2-1}{\alpha(\alpha+1)}, \end{aligned}$$

since $E_X(\ln(-X)) = \int_{-e}^{-1} \alpha(\ln(-x))(-x)^{-1}(\ln(-x))^{\alpha-1}dx = \int_0^1 u^{\frac{1}{\alpha}}du = \frac{\alpha}{\alpha+1}$.
Finally, to prove (f), using Lemma 1.4 and (vi) of Theorem 2.3, we have

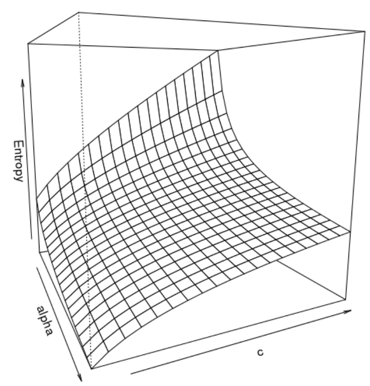
$$H(Y) = -\left(\ln \frac{a}{c}\right) + (c-1)E_X(\ln(-X)) + H(X) = -\left(\ln \frac{a}{c}\right) - c + 1,$$

since $E_X(\ln(-X)) = \int_{-1}^{0}(\ln(-x))dx = -1$. □

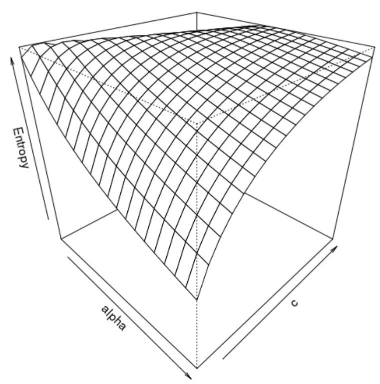
We next give an appendix giving several graphs for chosen values of the $\alpha$ and the shape parameters. Some remarks about the behaviour of entropies discussed in the text are also made .

## 3. Appendix

*3.1. Graphs of entropies of l-max stable laws*



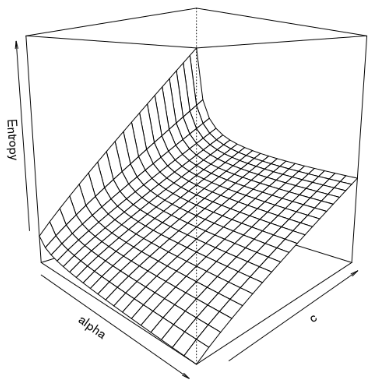

(a) Entropy of Fréchet law with $1 \leq c \leq 10$, $1 \leq \alpha \leq 10$



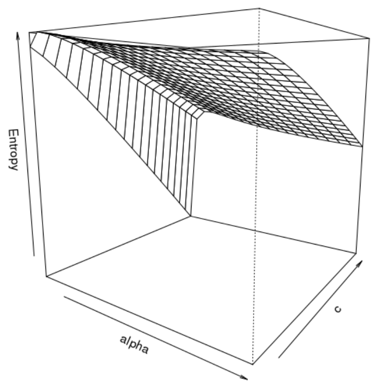

(b) Entropy of Weibull law with $1 \leq c \leq 10$, $1 \leq \alpha \leq 10$

**Remark 3.1.** *Observe that the entropy of the Fréchet law decreases as the parameter $\alpha$ increases for fixed c and the entropy of the Weibull law is decreasing as the parameter $\alpha$ increases for fixed c.*

*3.2. Graphs of entropies of p-max stable laws*

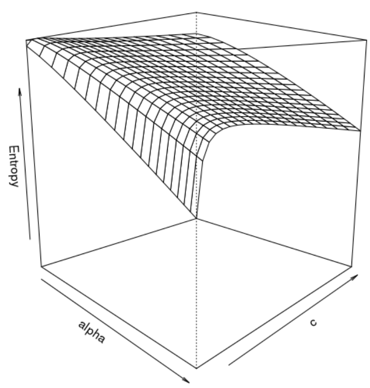



(a) Entropy of log-Fréchet law $1 \le c \le 10, 1 \le \alpha \le 10$

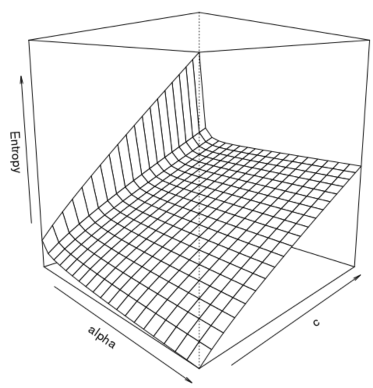



(b) Entropy of Log-Weibull law with $1 \le c \le 10$, $0.5 \le \alpha \le 10$

**Remark 3.2.** *Observe that the entropy of the Log-Fréchet law is increasing as c increases for fixed $\alpha$ and the entropy of the log-Weibull law is decreasing as c increases for fixed $\alpha$.*



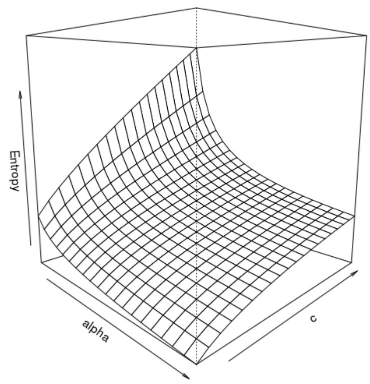

(a) Negative log-Fréchet law with $1 \le c \le 10, 1 \le \alpha \le 10$.

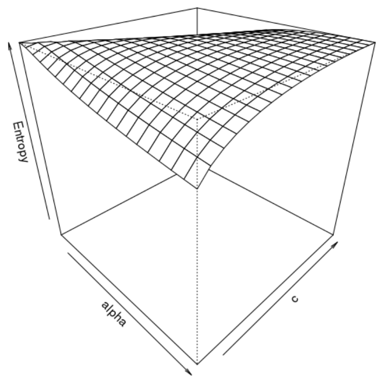



(b) Negative log-Weibull law with $1 \le c \le 10$, $0.5 \le \alpha \le 10$

**Remark 3.3.** *Note that the entropy of the negative log-Fréchet law decreases as c increases for fixed $\alpha$ and the entropy of the negative log-Weibull law is increasing as c increases for fixed $\alpha$.*

## 3.3. *Graphs of entropies of generalized Pareto laws*

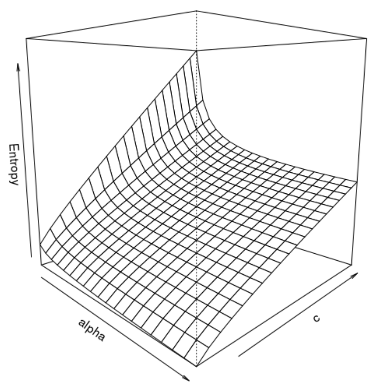


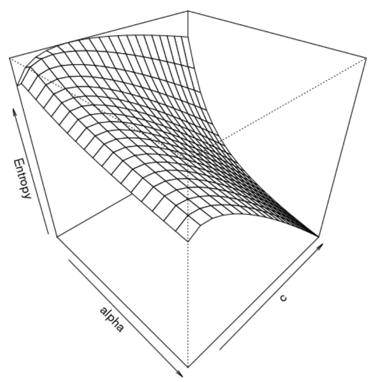




(a) Entropy of Pareto law with $1 \leq c \leq 10$, $1 \leq \alpha \leq 10$. (b) Entropy of negative Beta with $1 \leq c \leq 10$, $1 \leq \alpha \leq 10$.

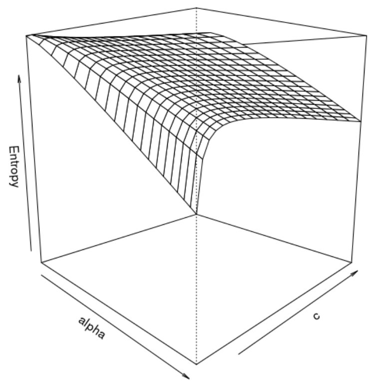
**Remark 3.4.** *Observe that the entropy of the Pareto law is increasing as $c$ increases for $\alpha$ fixed and the entropy of the negative Beta law is decreasing as $\alpha$ increases for fixed $c$.*

## 3.4. *Graphs of entropies of generalized log-Pareto laws*

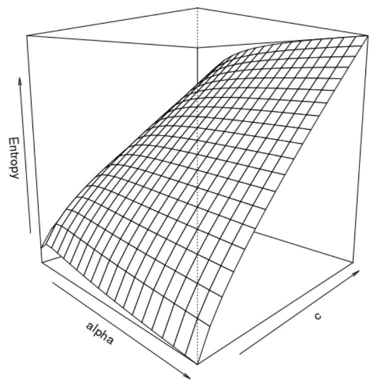



(a) Entropy of Log-Pareto with $1 \leq c \leq 10$, $1 \leq \alpha \leq 10$. (b) Entropy of Log-negative-Beta with $1 \leq c \leq 10$, $0.5 \leq \alpha \leq 10$




(a) Negative log-Pareto with $1 \leq c \leq 10$ and $1 \leq \alpha \leq 10$




(b) Negative log-negative-Beta with $1 \leq c \leq 10$ and $0.5 \leq \alpha \leq 10$.

**Remark 3.5.** *Observe that the entropy of the log-Pareto law is increasing as c increases for $\alpha$ fixed and the entropy of the negative log-Pareto law decreases as c increases for fixed $\alpha$.*

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