ProbStat Forum, Volume 05, July 2012, Pages 80–85
ISSN 0974-3235

ProbStat Forum is an e-journal.
For details please visit www.probstat.org.in

Conditional expectation of function of dual generalized order statistics An alternative approach

M.J.S. Khan[a], M.I. Khan[b]

[a]*Department of Applied Mathematics and Statistics, Baba Saheb Bheem Rao Ambedkar University, Lucknow–226 025, India*
[b]*Department of Statistics and Operations Research, Aligarh Muslim University, Aligarh–202 002, India*

Abstract. A general form of continuous distribution has been characterized through the conditional expectation of function of dual generalized order statistics and lower record values using Meijer's G-function.

1. Introduction

Burkschat *et al.* (2003) introduced the concept of dual generalized order statistics ($dgos$) as: Let X_1, X_2, ..., X_n is a sequence of independent and identically distributed random variables with distribution function (df) $F(x)$ and with probability density function pdf $f(x)$. Further, let $n \in N$, $n \geq 2$, $k > 0$, $\tilde{m} = (m_1, m_2, \dots, m_{n-1}) \in \Re^{n-1}$, $M_r = \sum\limits_{j=r}^{n-1} m_j$, such that $\gamma_r = k + n - r + M_r > 0$, $\forall\ r \in \{1, 2, \dots, n-1\}$. Then $X'(r, n, \tilde{m}, k)$, $r = 1, 2, \dots, n$ are called $dgos$ if their joint pdf is given by

$$k\left(\prod_{j=1}^{n-1} \gamma_j\right)\left(\prod_{i=1}^{n-1} [F(x_i)]^{m_i} f(x_i)\right)[F(x_n)]^{k-1} f(x_n) \tag{1}$$

for $F^{-1}(1) > x_1 \geq x_2 \geq \ldots \geq x_n > F^{-1}(0)$.

Conditional expectation of dual generalized order statistics are extensively used in characterizing the probability distributions. Various approaches are available in the literature. For detailed survey and discussion of characterization results one may refer to Ahasanullah (2004), Mbah and Ahsanullah (2007), Khan *et al.* (2009), Khan *et al.* (2010a, b) and Faizan and Khan (2011). In this paper we have characterized the distribution through conditional expectation of $dgos$ conditioned on non-adjacent $dgos$ using Meijer's G-function.

The pdf of $X'(r, n, m, k)$ with respect to a measure P_F is given by

$$f_r(x) = c_{r-1} G_r(F(x)|\gamma_1, \dots, \gamma_r) I_{(\alpha,\beta)}(x). \tag{2}$$

Here I_A denotes the indicator function and $G_r(x) = G_{r,r}^{r,0}(x|\gamma_1, \dots, \gamma_r) = G_{r,r}^{r,0}\left(x\middle|\begin{matrix}\gamma_1, \dots, \gamma_r \\ \gamma_1 - 1, \dots, \gamma_r - 1\end{matrix}\right)$ is the parti-

2010 *Mathematics Subject Classification*. Primary: 62E15; Secondary: 62E10, 62G30
Keywords. Characterization, Meijer's G-function, dual generalized order statistics, lower records
Received: 03 December 2011; Accepted: 12 June 2012
Email addresses: jahangirskhan@gmail.com (M.J.S. Khan), izhar.stats@gmail.com (Corresponding author) (M.I. Khan)

cular Meijer's G-function defined by

$$G_{r,r}^{r,0}\left(s\Big|_{\gamma_1-1,\ldots,\gamma_r-1}^{\gamma_1,\ldots,\gamma_r}\right)=\frac{1}{2\pi i}\int_L\frac{s^z}{\prod_{j=1}^r(\gamma_j-1-z)}dz, \tag{3}$$

where L is an appropriate chosen contour of integration. See Mathai (1993, Chapter 3) for the definition of Meijer's G-function and its numerous properties and applications.

The joint $P_F\otimes P_F$ density of $X'(r,n,\tilde{m},k)$ and $X'(s,n,\tilde{m},k)$, $1\le r<s\le n$, is given by

$$f_{r,s}(x,y)=c_{s-1}\frac{1}{F(x)}G_{s-r}\left(\frac{F(y)}{F(x)}\Big|\gamma_{r+1},\ldots,\gamma_s\right)G_r(F(x)|\gamma_1,\ldots,\gamma_r), \qquad \alpha\le y<x\le\beta, \tag{4}$$

where

$$c_{r-1}=\prod_{i=1}^r\gamma_i.$$

Hence the conditional P_F density function of $X'(s,n,\tilde{m},k)$ given $X'(r,n,\tilde{m},k)=x, 1\le r<s\le n$, is

$$f_{s|r}(y|x)=\frac{c_{s-1}}{c_{r-1}}G_{s-r}\left(\frac{F(y)}{F(x)}\Big|\gamma_{r+1},\ldots,\gamma_s\right)\frac{1}{F(x)}I_{(\beta,x)}(y), \qquad \alpha\le y<x\le\beta, \tag{5}$$

and the conditional P_F density function of $X'(r,n,\tilde{m},k)$ given $X'(s,n,\tilde{m},k)=y$, $1\le r<s\le n$, is

$$f_{r|s}(x|y)=\frac{G_{s-r}\left(\frac{F(y)}{F(x)}\Big|\gamma_{r+1},\ldots,\gamma_s\right)G_r(F(x)|\gamma_1,\ldots,\gamma_r)}{F(x)G_s(F(y)|\gamma_1,\ldots,\gamma_s)}I_{(y,\alpha)}(x). \tag{6}$$

Some auxiliary results

Here some results are given that are used in subsequent sections:

i) $G_1(x|\gamma_1)=x^{\gamma_1-1}$

ii) $(\gamma_r-\gamma_1)G_r(x|\gamma_1,\ldots,\gamma_r)=G_{r-1}(x|\gamma_1,\ldots,\gamma_{r-1})-G_{r-1}(x|\gamma_2,\ldots,\gamma_r)$

iii) $x^aG_r(x|\gamma_1,\ldots,\gamma_r)=G_r(x|\gamma_1+a,\ldots,\gamma_r+a)$, $a\in\Re$

iv) $\lim\limits_{x\to1-}G_r(x|\gamma_1,\ldots,\gamma_r)=\begin{cases}1, & r=1\\ 0, & r\ge2\end{cases}$

v) $\lim\limits_{x\to0+}G_r(x|\gamma_1,\ldots,\gamma_r)=\begin{cases}0, & \text{if } \gamma_{1:r}>1\\ \prod_{\substack{j=1\\ j\ne1}}^r\frac{1}{\gamma_j-\gamma_l}, & \text{if } \gamma_{1:r}=1<\gamma_{2:r}\\ \infty, & \text{if } \gamma_{1:r}=\gamma_{2:r}=1 \text{ or } \gamma_{1:r}<1\end{cases}$

vi) $\frac{d}{dx}G_r(x|\gamma_1,\ldots,\gamma_r)=\frac{1}{x}\left[(\gamma_r-1)G_r(x|\gamma_1,\ldots,\gamma_r)-G_{r-1}(x|\gamma_1,\ldots,\gamma_{r-1})\right]$

vii) $\frac{d}{dx}G_r(x|\gamma_1,\ldots,\gamma_r)=\frac{1}{x}\left[(\gamma_1-1)G_r(x|\gamma_1,\ldots,\gamma_r)-G_{r-1}(x|\gamma_2,\ldots,\gamma_{r-1})\right]$ where $\gamma_{1:r}=\min(\gamma_1,\ldots,\gamma_r)$ and $l=\max\{1\le j\le r:\gamma_j=\gamma_{1:r}\}$.

For property (*i*), see Mathai (1993, p. 130), for property (*ii*), see Cramer and Kamps (2003), and property (*iii*), see Mathai (1993, p. 69). Property (*iv*) can easily be deduced from Lemma 2.2 of Cramer *et al.* (2004 b). Whereas (*vi*) and (*vii*) can be established from (3).

2. Characterization of distribution

Theorem 2.1. *Let $X'(i,n,\tilde{m},k)$, $i=1,\ldots,n$ be the dgos from a continuous population with the df $F(x)$ and the pdf $f(x)$ over the support (α,β) and $\xi(x)$ be a monotonic and differentiable function of x. If for two consecutive values r*

and $r+1$, $1 \le r < s-1 < n$

$$g_{s|l}(x) = E\left[\xi\{X'(s,n,\tilde{m},k)\}|X'(l,n,\tilde{m},k) = x\right], \qquad l = r, r+1, \tag{7}$$

exists, then

$$F(x) = e^{-\int_x^\beta A(t)dt}$$
$$A(t) = \frac{g'_{s|r}(t)}{\gamma_{r+1}[g_{s|r+1}(t) - g_{s|r}(t)]}.$$

Proof. We have from (5)

$$g_{s|r}(x)F(x) = \frac{c_{s-1}}{c_{r-1}} \int_\alpha^x \xi(y) G_{s-r}\left(\frac{F(y)}{F(x)}\Big|\gamma_{r+1}, \ldots, \gamma_s\right) dF(y) \tag{8}$$

differentiate both the sides of (8) w.r.t. x to get

$$g'_{s|r}(x)F(x) + g_{s|r}(x)f(x) = -\frac{c_{s-1}}{C_{r-1}} \frac{f(x)}{F(x)} \int_\alpha^x \xi(y)\left[(\gamma_{r+1}-1)G_{s-r}\left(\frac{F(y)}{F(x)}\Big|\gamma_{r+1}, \ldots, \gamma_s\right) - G_{s-r-1}\left(\frac{F(y)}{F(x)}\Big|\gamma_{r+2}, \ldots, \gamma_s\right)\right] dF(y)$$

or,

$$g'_{s|r}(x)F(x) + g_{s|r}(x)f(x) = -(\gamma_{r+1} - 1)g_{s|r}(x)f(x) + \gamma_{r+1} g_{s|r+1}(x)f(x)$$

in view of property (vi) and equation (8).

Therefore,

$$\frac{f(x)}{F(x)} = \frac{g'_{s|r}(x)}{\gamma_{r+1}[g_{s|r+1}(x) - g_{s|r}(x)]} = A(x).$$

Also for $s = r+1$

$$\frac{f(x)}{F(x)} = \frac{g'_{r+1|r}(x)}{\gamma_{r+1}[\xi(x) - g_{s|r+1}(x)]} = A(x)$$

and hence the Theorem. □

Remark 2.2. *If*

$$E[\{X'(s,n,\tilde{m},k)\}|X'(r,n,\tilde{m},k) = x] = a^*_{s|r}x + b^*_{s|r} = g_{s|r}(x)$$

then

$$F(x) = [ax + b]^c$$

and

$$E[\xi\{X'(s,n,\tilde{m},k)\}|X'(r,n,\tilde{m},k) = x] = a^*_{s|r}\xi(x) + b^*_{s|r}$$

if and only if

$$F(x) = [a\xi(x) + b]^c$$

where

$$a^*_{s|r} = \prod_{j=r+1}^{s} \frac{c\gamma_j}{1+c\gamma_j}, \quad b^*_{s|r} = -\frac{b}{a}(1-a^*_{s|r})$$

and $\xi(x)$ be a monotonic and continuous function of x.

A number of distributions can be characterized by the proper choice of a, b, c and $\xi(x)$ and the results can be deduced for order statistics and lower records, Khan et al. (2010a).

Theorem 2.3. *Let $X'(i, n, \tilde{m}, k)$, $i = 1, \ldots, n$ be the dgos from a continuous population with the df $F(x)$ and the pdf $f(x)$ over the support (α, β) and $\xi(x)$ be a monotonic and differentiable function of x. If for two consecutive values $s-1$ and $s, 1 \le r < s-1 < n$*

$$g_{r|l}(y) = E[\xi\{X'(r, n, \tilde{m}, k)\}|X'(l, n, \tilde{m}, k) = y], \qquad l = s-1, s \tag{9}$$

then

$$G_s(F(y)|\gamma_1 - \gamma_s + 1, \ldots, \gamma_s - \gamma_s + 1) = a_{(s)}(s) \exp\left[-\int_\alpha^y D(t)dt\right], \qquad \text{if } \gamma_{1:s-1} > \gamma_s, \tag{10}$$

and

$$G_s(F(y)|\gamma_1 - \gamma_s + 1, \ldots, \gamma_s - \gamma_s + 1) = \exp\left[-\int_p^y D(t)dt\right], \qquad \text{if } \gamma_{1:s-1} \le \gamma_s. \tag{11}$$

To prove (11), we note that $\log F(y)$, $\alpha < y < \beta$ is a non-decreasing function in $(-\infty, 0)$, therefore exists a $p, \alpha < p < \beta$, such that

$$-logF(p) = 1 \tag{12}$$

and

$$D(t) = \frac{g'_{r|s}(t)}{[g_{r|s}(t) - g_{r|s-1}(t)]}.$$

Proof. We have,

$$g_{r|s}(y)G_s(F(y)|\gamma_1, \ldots, \gamma_s) = \int_y^\beta \frac{\xi(x)}{F(x)} G_{s-r}\left(\frac{F(y)}{F(x)}\Big|\gamma_{r+1}, \ldots, \gamma_s\right) G_r(F(x)|\gamma_1, \ldots, \gamma_r)dF(x) \tag{13}$$

differentiate both the sides of (13) w.r .t. y, to get

$$g'_{r|s}(y)G_s(F(y)|\gamma_1, \ldots, \gamma_s) = \frac{f(y)}{F(y)} G_{s-1}(F(y)|\gamma_1, \ldots, \gamma_{s-1})g_{r|s}(y) - \frac{f(y)}{F(y)} G_{s-1}(F(y)\big|\gamma_1, \ldots, \gamma_{s-1})g_{r|s-1}(y)$$

$$g'_{r|s}(y)G_s(F(y)|\gamma_1, \ldots, \gamma_s) = \frac{f(y)}{F(y)} G_{s-1}(F(y)|\gamma_1, \ldots, \gamma_{s-1})[g_{r|s}(y) - g_{r|s-1}(y)]$$

$$\frac{g'_{r|s}(y)}{[g_{r|s}(y) - g_{r|s-1}(y)]} = \frac{f(y)}{F(y)} \frac{G_{s-1}(F(y)|\gamma_1, \ldots, \gamma_{s-1})}{G_s(F(y)|\gamma_1, \ldots, \gamma_s)}$$

and $s = r + 1$

$$\frac{f(y)}{F(y)} \frac{G_r(F(y)|\gamma_1, \ldots, \gamma_{s-1})}{G_{r+1}(F(y)|\gamma_1, \ldots, \gamma_s)} = \frac{g'_{r|r+1}(y)}{[g_{r|r+1}(y) - \xi(y)]}.$$

Now using the property (vi), we have

$$(\gamma_{s-1}-1)\frac{f(y)}{F(y)}-\frac{\frac{d}{dy}G_s(F(y)|\gamma_1,\ldots,\gamma_s)}{G_s(F(y)|\gamma_1,\ldots,\gamma_s)}=\frac{g'_{r|s}(t)}{[g_{r|s}(t)-g_{r|s-1}(t)]}$$

integrating both the sides w.r.t. y over (α, y), to get

$$\left[\frac{F(y)}{F(\alpha)}\right]^{1-\gamma_s}\frac{G_s(F(y)|\gamma_1,\ldots,\gamma_s)}{G_s(F(\alpha)|\gamma_1,\ldots,\gamma_s)}=\exp\left[-\int_\alpha^y D(t)dt\right]$$

or

$$G_s(F(y)|\gamma_1-\gamma_s+1,\ldots,\gamma_s-\gamma_s+1)=a_{(s)}\exp\left[-\int_\alpha^y D(t)dt\right],\qquad \text{if } \gamma_{1:s-1}>\gamma_s,$$

or

$$G_s(F(\alpha)|\gamma_1-\gamma_s+1,\ldots,\gamma_s-\gamma_s+1)\to a_{(s)}(s)$$

when $\gamma_{1:s-1}\le\gamma_s$, then exists a p which satisfied (12). Hence

$$G_s(F(y)|\gamma_1-\gamma_s+1,\ldots,\gamma_s-\gamma_s+1)=\exp\left[-\int_p^y D(t)dt\right],\qquad \text{if } \gamma_{1:s-1}\le\gamma_s.$$

□

Corollary 2.4. *For $m_1=m_2=\ldots=m_{n-1}=m$*

$$G_s(1-x|\gamma_1-\gamma_s+1,\ldots,\gamma_s-\gamma_s+1)=\frac{1}{(s-1)!}[g_m(x)]^{s-1}$$

where

$$g_m(x)=\begin{cases}\frac{1}{m+1}[1-(1-x)^{m+1}] & \textit{if } m\ne-1\\ -\log(1-x) & \textit{if } m=-1\end{cases}$$

and hence (10) *and* (11) *can be rewritten as follows:*

$$F(y)=\left[1-\exp\left\{-\int_\alpha^y D(t)dt\right\}\right]^{\frac{1}{m+1}},\quad m>-1$$
$$F(x)=\exp\left[-\exp\left\{-\int_p^y D(t)dt\right\}\right],\quad m=-1$$

as obtained by Khan et al. (2010a).

Remark 2.5.

$$g_{r|s}(y)=E[\{X'(r,n,\tilde{m},k)\}|X'(s,n,\tilde{m},k)=y]=a^*_{r|s}y+b^*_{r|s}$$

if and only if

$$1-[F(y)]^{m+1}=[ay+b]^c,\qquad \alpha<y<\beta, m>-1,$$

where $F(y)$ *is* df *over* (α, β) *and*

$$a^*_{r|s} = \prod_{j=1}^{s-r} \frac{c(s-j)}{1+c(s-j)}, \quad b^*_{r|s} = -\frac{b}{a}(1 - a^*_{r|s})$$

and $-\log F(y) = [ay+b]^c$ *at* $m = -1$ *with* $-\log F(p) = 1$.

For the proof and the related results for order statistics and lower records one may refer to Khan *et al.* (2010a).

Acknowledgement

It is a pleasure to record our sincere thanks to Prof. A.H. Khan, Aligarh Muslim University, Aligarh under whose supervision this work was done. The authors also thank the referees.

References

[1] Ahsanullah, M. (2004) A characterization of the uniform distribution by dual generalized order statistics, Comm. Statist. Theory Methods 33, 2921–2928.
[2] Burkschat, M., Cramer, E., Kamps, U. (2003) Dual generalized order statistics, Metron LXI (1), 13–26.
[3] Cramer, E., Kamps, U. (2003) Marginal distributions of sequential and generalized order statistics, Metrika 58, 293–310.
[4] Cramer, E., Kamps, U., Rychlik, T. (2004b) Unimodality of uniform generalized order statistics with application to mean bound, Ann. Inst. Statist. Math. 56, 183–192.
[5] Faizan, M., Khan, M.I. (2011) Characterization of some probability distributions by conditional expectation of dual generalized order statistics, Journal of Statistics Sciences 3, 143–150.
[6] Khan, A.H., Anwar, Z., Chishti, S. (2010a) Characterization of continuous distributions through conditional expectation of functions of dual generalized order statistics, Pak. J .Statist. 26, 615–628.
[7] Khan, A.H., Faizan, M., Haque, Z. (2010b) Characterization of probability distributions by conditional variance of generalized order statistics and dual generalized order statistics, J. Statist. Theory Appl. 9, 375–385.
[8] Khan, M.J.S., Haque, Z., Faizan, M. (2009) On characterization of continuous distributions conditioned on a pair of non-adjacent dual generalized order statistics, Aligarh J. Statist. 29, 107–119.
[9] Mathai, A.M. (1993) A Handbook of Generalized Special Functions for Statistical and Physical Sciences, Oxford Science Publications, New York.
[10] Mbah, A.K., Ahsanullah, M. (2007) Some characterization of the power function distribution based on lower generalized order statistics, Pak. J. Statist. 23, 139–146.