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Probability distributions associated with Mathieu type series

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Abstract. The main object of this article is to present a systematic study of probability density functions and distributions associated with Mathieu series and their generalizations. Characteristic functions and fractional moments related to the probability density functions of the considered distributions are derived by means of Mathieu type series and Hurwitz–Lerch Zeta function. Special attention will be given to the so–called Planck(p) distribution.

1. Introduction and preliminaries

The following familiar infinite series

$$S\,(r) = \sum_{n=1}^{\infty} \frac{2n}{(n^2+r^2)^2}, \qquad r \in \mathbb{R}^+, \tag{1}$$

is named after Emile Leonard Mathieu (1835–1890), who investigated it in his 1890 work (Mathieu [17]) on the elasticity of solid bodies. Alternative version of (1)

$$\widetilde{S}\,(r) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n}{(n^2+r^2)^2}\,, \qquad r \in \mathbb{R}^+, \tag{2}$$

was introduced in Pogány *et al.* [18]. Its Mellin–Barnes integral representation is recently given by Saxena *et al.* [20]. Integral representations of (1) and (2) are given (Elezović *et al.* [5] and Pogány *et al.* [18]) in the form:

$$\begin{aligned} S(r) &= \frac{1}{r} \int_0^\infty \frac{t \sin(rt)}{e^t - 1}\, \mathrm{d}t, \\ \widetilde{S}(r) &= \frac{1}{r} \int_0^\infty \frac{t \sin(rt)}{e^t + 1}\, \mathrm{d}t\,. \end{aligned} \tag{3}$$

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Several interesting problems and solutions dealing with integral representations and bounds for the following mild generalization of the Mathieu series with a fractional power

$$S_\mu(r) = \sum_{n=1}^{\infty} \frac{2n}{(n^2+r^2)^\mu}, \qquad r \in \mathbb{R}^+,\ \mu > 1, \tag{4}$$

can be found in the recent works by Diananda [3], Tomovski and Trenčevski [27] and Cerone and Lenard [2]. Motivated essentially by the works of Cerone and Lenard [2], in Pogány *et al.* [18] has been defined a family of generalized Mathieu series

$$S_\mu^{(\alpha,\beta)}(r;\boldsymbol{a}) = S_\mu^{(\alpha,\beta)}(r;\{a_k\}_{k=1}^\infty) = \sum_{n=1}^{\infty} \frac{2a_n^\beta}{(a_n^\alpha+r^2)^\mu}, \qquad r,\alpha,\mu \in \mathbb{R}^+,\ \beta \ge 0, \tag{5}$$

where it is tacitly assumed that the monotone increasing, divergent sequence of positive real numbers

$$\boldsymbol{a} = \{a_n\}_{n=1}^\infty, \qquad \lim_{n\to\infty} a_n = +\infty\,,$$

is so chosen that the infinite series in definition (5) converges, that is, the following auxiliary series

$$\sum_{n=1}^{\infty} \frac{1}{a_n^{\mu\alpha-\beta}}$$

is convergent. Comparing the definitions (1), (4) and (5), we see that $S_2(r) = S(r)$ and $S_\mu(r) = S_\mu^{(2,1)}(r,\{k\})$. The interesting special cases like $S_2^{(2,1)}(r;\{a_k\})$, $S_\mu(r) = S_\mu^{(2,1)}(r;\{k\})$, $S_\mu(r) = S_\mu^{(2,1)}(r;\{k^\gamma\})$ and $S_\mu(r) = S_\mu^{(\alpha,\alpha/2)}(r;\{k\})$ were investigated by Diananda [3], Tomovski and Leškovski [26] and Cerone and Lenard [2] among others. In Pogány *et al.* [18] and in Srivastava and Tomovski [25] several integral representations have been obtained for (5) and its alternating variants in terms of the generalized hypergeometric functions and the Bessel function of the first kind.

The generalized Hurwitz–Lerch Zeta function $\Phi(z,s,a)$ is defined e.g. in Erdélyi *et al.* [4, p. 27, Eq. 1.11.1] as the power series

$$\Phi(z,s,a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s},$$

where $a \in \mathbb{C}\backslash\mathbb{Z}_0^-$; $\Re\{s\} > 1$ when $|z| = 1$ and $s \in \mathbb{C}$ when $|z| < 1$ and continues meromorphically to the complex s–plane, except for the simple pole at $s = 1$, with its residue equal to 1. The function $\Phi(z,s,a)$ has many special cases such as Riemann Zeta (Erdélyi *et al.* [4]), Hurwitz–Zeta (Srivastava and Choi [23]) and Lerch Zeta function (Whittaker and Watson [29, p. 280, Example 8]). Some other special cases involve the polylogarithm (or Jonqière's function) and the generalized Zeta function (Whittaker and Watson [29, p. 280, Example 8], Srivastava and Choi [23, p. 122, Eq. 2.5]) discussed for the first time by Lipschitz and Lerch.

Finally, Fox defined the H–function in his celebrated studies of symmetrical Fourier kernels as the Mellin–Barnes type path integral (Mathai and Saxena [16], Srivastava *et al.* [24])

$$H_{p,q}^{m,n}[z] = H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (a_1,A_1),\cdots,(a_p,A_p) \\ (b_1,B_1),\cdots,(b_q,B_q) \end{matrix}\right] := \frac{1}{2\pi \mathrm{i}} \int_{\mathfrak{L}} \chi(s) z^s \,\mathrm{d}s \tag{6}$$

for all $z \neq 0$, where

$$\chi(s) = \frac{\prod_{j=1}^m \Gamma(b_j - B_j s) \cdot \prod_{j=1}^n \Gamma(1-a_j+A_j s)}{\prod_{j=n+1}^p \Gamma(a_j - A_j s) \cdot \prod_{j=m+1}^q \Gamma(1-b_j+B_j s)}\,.$$

Here $0 \leq n \leq p, 1 \leq m \leq q, (a_\ell, \alpha_\ell), (b_\ell, \beta_\ell) \in \mathbb{C} \times \mathbb{R}^+$ such that $A_j(b_h + \ell) \neq B_h(a_j - k - 1)$, for $\ell, k \in \mathbb{N}_0; h = \overline{1,m}, j = \overline{1,n}$. The contour $\mathfrak{L}$ in the complex s–plane extends from $w - \mathrm{i}\infty$ to $w + \mathrm{i}\infty, w > \max_{1\leq h\leq m} |\Im\{b_h\}|/B_h$ separating the points $B_h^{-1}(b_j + \ell), h = \overline{1,m}, \ell \in \mathbb{N}_0$, which are the poles of $\Gamma(b_j - B_j s), j = \overline{1,m}$, from the points $A_j^{-1}(a_j-k-1), j = \overline{1,n}, k \in \mathbb{N}_0$ which are the poles of $\Gamma(1-a_j+A_j s), j = \overline{1,n}$, (Mathai and Saxena [16], Srivastava *et al.* [24]).

The sufficient conditions for the absolute convergence of the contour integral (6) is given by Mathai and Saxena [16], Srivastava *et al.* [24] as

$$\Lambda = \sum_{j=1}^{m} B_j + \sum_{j=1}^{n} A_j - \sum_{j=m+1}^{q} B_j - \sum_{j=n+1}^{p} A_j > 0\,.$$

The region of absolute convergence of the contour integral (6) is $|\arg z| < \pi\Lambda/2$.

The main objective of this article is to introduce, develop and investigate probability density functions (PDF) and cumulative distribution functions (CDF) associated with the Mathieu series and their generalizations. We consider also separately the expressibility of the related statistical functions connected with the Hurwitz–Lerch Zeta function, together with another related numerical characteristics and basic functions associated with the PDF of the considered distributions, such as general fractional order moments $\mathsf{E}\,X^s$ and related characteristic functions (CHF).

2. Probability distributions associated with Mathieu series and Planck's law

Special functions and integral transforms are useful in the development of the theory of probability density functions (PDF). In this connection, one can refer to the books e.g. by Mathai and Saxena [15, 16] or by Johnson and Kotz [11, 12]. Due to usefulness and popularity of Planck distribution in quantum physics, statistical inference etc. the authors are motivated to study the connections between the Planck's probability distribution law and the Mathieu series, investigating its important properties.

The celebrated Planck's law is related to the radiation of black body, namely it describes the electromagnetic radiation emitted from a black body at absolute temperature T. As a function of frequency ν, Planck's law represents the emitted power per unit area of emitting surface in the normal direction, per unit solid angle, per unit frequency; in other words it gives the specific radiative intensity in the function of the frequency ν. Also, Planck's law is sometimes written in terms of the spectral energy density per unit volume of thermodynamic equilibrium cavity radiation (Brehm and Mullin [1]). Then we have the celebrated frequency spectral density

$$u(\nu) = \frac{8\pi h\nu^3}{c^3}\,\frac{1}{\mathrm{e}^{\frac{h\nu}{kT}} - 1}, \qquad \nu > 0\,,$$

where constants h, c, k possess the physical meanings as Planck constant, speed of light and Boltzmann constant respectively. Further generalizations of the Planck's law were considered recently e.g. by Souza and Tsallis [22, Eqs. (2), (3)].

Bearing in mind the well–known identity

$$\int_0^\infty \frac{x^{z-1}}{e^{\alpha x} - 1}\,\mathrm{d}x = \alpha^{-z}\zeta(z)\Gamma(z), \qquad \alpha, \Re(z) > 0\,,$$

where ζ, Γ stand for the Riemann Zeta and Gamma function respetively, we recognize that the random variable X defined on some fixed standard probability space $(\Omega, \mathfrak{F}, \mathsf{P})$, possessing PDF

$$u(x) = \frac{\alpha^{p+2}}{\zeta(p+2)\Gamma(p+2)} \cdot \frac{x^{p+1}}{\mathrm{e}^{\alpha x} - 1} \cdot \chi_{\mathbb{R}_+}(x),$$

is distributed according to Planck's law with parameter $p > -1$. We write this correspondence as $X \sim \text{Planck}(p)$. Here, and in what follows $\chi_S(x)$ denotes the indicator function of the set S. The related CDF reads

$$U(x) = \begin{cases} \dfrac{\alpha^{p+2}}{\zeta(p+2)\Gamma(p+2)} \displaystyle\int_0^x \frac{t^{p+1}}{\mathrm{e}^{\alpha t}-1}\,\mathrm{d}t, & x>0,\\[2ex] 0, & x \le 0. \end{cases}$$

Now, we are interested in CHF $\phi(r) = \mathsf{E}\,\mathrm{e}^{\mathrm{i}rX}, r \in \mathbb{R}$ for the r.v. X having Planck distribution.

Theorem 2.1. *The* CHF *associated with the r.v.* $X \sim \mathrm{Planck}(p)$ *reads, as follows*

$$\phi(r) = \frac{1}{\zeta(p+2)}\,\Phi\left(1, p+2, 1-\frac{\mathrm{i}r}{\alpha}\right), \qquad r \in \mathbb{R}\,. \tag{7}$$

Moreover, there holds the estimate

$$\big|\phi(r)\big| \le \frac{1}{2\,\zeta(p+2)}\,S^{(2,0)}_{p/2+1}\Big(\frac{r}{\alpha};\{k\}\Big), \qquad r \in \mathbb{R}\,. \tag{8}$$

Proof. By direct calculations we have

$$\begin{aligned} \phi(r) = \mathsf{E}\,\mathrm{e}^{\mathrm{i}rX} &= \frac{\alpha^{p+2}}{\zeta(p+2)\Gamma(p+2)} \int_0^\infty \mathrm{e}^{\mathrm{i}rx}\,\frac{x^{p+1}}{\mathrm{e}^{\alpha x}-1}\,\mathrm{d}x \\ &= \frac{\alpha^{p+2}}{\zeta(p+2)\Gamma(p+2)} \sum_{n=0}^\infty \int_0^\infty x^{p+1}\mathrm{e}^{-((n+1)\alpha-\mathrm{i}r)x}\,\mathrm{d}x \\ &= \frac{1}{\zeta(p+2)} \sum_{n=0}^\infty \frac{1}{\big(n+1-\frac{\mathrm{i}r}{\alpha}\big)^{p+2}}\,, \end{aligned}$$

which proves (7) being the Hurwitz–Lerch series convergent $(p+1>0)$. Now, evaluating the last expression we deduce

$$\big|\phi(r)\big| \le \frac{1}{\zeta(p+2)} \sum_{n=0}^\infty \frac{1}{\big|n+1-\frac{\mathrm{i}r}{\alpha}\big|^{p+2}} = \frac{1}{2\,\zeta(p+2)}\,\lim_{\beta\to 0} S^{(2,\beta)}_{p/2+1}\Big(\frac{r}{\alpha};\{k\}\Big)\,,$$

which proves (8). □

We point out that $\phi(0)=1$ such that is one of the main properties of CHFs. Another kind of expression for CHF $\phi(r)$ we obtain in terms of derivatives of Mathieu series $S(r)$.

Differentiating $p \in \mathbb{N}_0$ times with respect to r both sides of the formula [19]

$$\int_0^\infty \frac{x\,\cos(rx)}{e^{\alpha x}-1}\,\mathrm{d}x = \frac{1}{2r^2} - \frac{\pi^2}{2\alpha^2}\,\frac{1}{\sinh^2\left(\frac{r\pi}{\alpha}\right)}\,,$$

we get

$$\int_0^\infty \frac{x^{p+1}}{e^{\alpha x}-1}\,\cos\Big(rx+p\frac{\pi}{2}\Big)\,\mathrm{d}x = \frac{(-1)^p(p+1)!}{r^{p+2}} - \frac{\pi^2}{2\alpha^2}\,\Big(\sinh^{-2}\Big(\frac{r\pi}{\alpha}\Big)\Big)^{(p)}\,,$$

which one expands into

$$\begin{aligned} \cos\Big(p\frac{\pi}{2}\Big) \int_0^\infty \frac{x^{p+1}\,\cos(rx)}{e^{\alpha x}-1}\,\mathrm{d}x - \sin\Big(p\frac{\pi}{2}\Big) \int_0^\infty \frac{x^{p+1}\,\sin(rx)}{e^{\alpha x}-1}\,\mathrm{d}x \\ = \frac{(-1)^p(p+1)!}{2\,r^{2+p}} - \frac{\pi^2}{2\alpha^2}\,\Big(\sinh^{-2}\Big(\frac{r\pi}{\alpha}\Big)\Big)^{(p)}\,. \end{aligned} \tag{9}$$

Differentiating now p times both sides of (3) with respect to r, we have

$$\int_0^\infty \frac{x^{p+1}}{e^{\alpha x}-1}\,\sin\Big(rx+p\frac{\pi}{2}\Big)\,\mathrm{d}x = \frac{1}{\alpha^3}\,\Big(rS\Big(\frac{r}{\alpha}\Big)\Big)^{(p)}\,.$$

By the Leibnitz rule for differentiating the product on the right we have

$$\begin{aligned}\sin\Big(p\frac{\pi}{2}\Big)\int_0^\infty \frac{x^{p+1}\,\cos(rx)}{e^{\alpha x}-1}\,\mathrm{d}x + \cos\Big(p\frac{\pi}{2}\Big)\int_0^\infty \frac{x^{p+1}\,\sin(rx)}{e^{\alpha x}-1}\,\mathrm{d}x \\ = \frac{1}{\alpha^3}\,\Big\{rS^{(p)}\Big(\frac{r}{\alpha}\Big)+pS^{(p-1)}\Big(\frac{r}{\alpha}\Big)\Big\}\,.\end{aligned} \tag{10}$$

Solving the system of equations (9), (10) with respect to cosine and sine integrals we obtain

$$\begin{aligned}\int_0^\infty \frac{x^{p+1}\,\cos(rx)}{e^{\alpha x}-1}\,\mathrm{d}x &= \cos\Big(p\frac{\pi}{2}\Big)\,\Big\{\frac{(-1)^p(p+1)!}{2\,r^{2+p}} - \frac{\pi^2}{2\alpha^2}\,\Big(\sinh^{-2}\Big(\frac{r\pi}{\alpha}\Big)\Big)^{(p)}\Big\} \\ &\quad + \frac{\sin\Big(p\frac{\pi}{2}\Big)}{\alpha^3}\,\Big\{rS^{(p)}\Big(\frac{r}{\alpha}\Big)+pS^{(p-1)}\Big(\frac{r}{\alpha}\Big)\Big\} \\ \int_0^\infty \frac{x^{p+1}\,\sin(rx)}{e^{\alpha x}-1}\,\mathrm{d}x &= -\sin\Big(p\frac{\pi}{2}\Big)\,\Big\{\frac{(-1)^p(p+1)!}{2\,r^{2+p}} - \frac{\pi^2}{2\alpha^2}\,\Big(\sinh^{-2}\Big(\frac{r\pi}{\alpha}\Big)\Big)^{(p)}\Big\} \\ &\quad + \frac{\cos\Big(p\frac{\pi}{2}\Big)}{\alpha^3}\,\Big\{rS^{(p)}\Big(\frac{r}{\alpha}\Big)+pS^{(p-1)}\Big(\frac{r}{\alpha}\Big)\Big\}\,.\end{aligned}$$

Hence the following result.

Theorem 2.2. *The* CHF *related to r.v.* $X\sim \mathrm{Planck}(p), p\in\mathbb{N}_0$ *is*

$$\begin{aligned}\phi(r) = \frac{\alpha^{p+2}(-\mathrm{i})^p}{\zeta(p+2)\,(p+1)!}\,\Bigg\{&\frac{(-1)^p(p+1)!}{2\,r^{2+p}} - \frac{\pi^2}{2\alpha^2}\Big(\sinh^{-2}\Big(\frac{r\pi}{\alpha}\Big)\Big)^{(p)} \\ &+ \frac{1}{\alpha^3}\,\Big(rS^{(p)}\Big(\frac{r}{\alpha}\Big)+pS^{(p-1)}\Big(\frac{r}{\alpha}\Big)\Big)\Bigg\}\,.\end{aligned} \tag{11}$$

Now, obvious calculations lead us to the next result.

Theorem 2.3. *Let* $X\sim \mathrm{Planck}(p), p>-1$. *Then the fractional moment* $\mathsf{E}\,X^s$ *of order* s *is given by*

$$\mathsf{E}\,X^s = \frac{\Gamma(s+p+2)\zeta(s+p+2)}{\alpha^s\Gamma(p+2)\zeta(p+2)}\,, \qquad s>-p-1\,. \tag{12}$$

Example 2.4. Consider the r.v. $X_0\sim \mathrm{Planck}(0)$, when parameter $\alpha=1$. The associated PDF takes the form

$$u_0(x) = \frac{6x}{\pi^2(\mathrm{e}^x-1)}\,\chi_{\mathbb{R}_+}(x)\,,$$

and the related CDF will be

$$U_0(x) = \frac{6}{\pi^2}\,\chi_{\mathbb{R}_+}(x)\int_0^x \frac{t}{\mathrm{e}^t-1}\,\mathrm{d}t\,.$$

According to Theorem 2.1 we express the CHF as

$$\phi_0(r) = \frac{6}{\pi^2}\,\Phi\big(1,2,1-\mathrm{i}r\big), \qquad r\in\mathbb{R}\,.$$

Further, by virtue of Theorems 2.2 and 2.3, the another form CHF and the moment of general fractional order $s > -1$ are respectively

$$\begin{aligned}\phi_0(r) &= \frac{6}{\pi^2}\left(\frac{1}{2r^2} - \frac{\pi^2}{2\sinh^2(\pi r)} + \mathrm{i}r\, S(r)\right)\\ \mathsf{E}\, X_0^s &= \frac{6}{\pi^2}\Gamma(s+2)\zeta(s+2)\,.\end{aligned}$$

Example 2.5. Now, we consider the r.v. $X_2 \sim$ Planck(2), when $\alpha = \frac{h}{kT}$, which corresponds to the celebrated Planck's spectral distribution. The related PDF is

$$u_2(x) = \frac{15\alpha^4}{\pi^4}\,\frac{x^3}{\mathrm{e}^x - 1}\,\chi_{\mathbb{R}_+}(x)\,,$$

and formula (12) one reduces to

$$\mathsf{E}\, X_2^s = \frac{15}{\alpha^4\pi^4}\,\Gamma(s+4)\zeta(s+4), \qquad s > -3\,.$$

The characteristic function take, *via* Theorem 2.1, the form:

$$\phi_2(r) = \frac{90}{\pi^4}\,\Phi\left(1, 4, 1 - \mathrm{i}\frac{r}{\alpha}\right), \qquad r \in \mathbb{R}\,.$$

Further, by Theorem 2.2 we obtain another form for the CHF which reads as follows

$$\phi_2(r) = \frac{15\big(3 + 2\sinh^2\left(\frac{r\pi}{\alpha}\right)\big)}{\sinh^4\left(\frac{r\pi}{\alpha}\right)} - \frac{45\alpha^4}{\pi^4 r^4} - \frac{15\alpha}{\pi^4}\left(r\, S''\left(\frac{r}{\alpha}\right) + 2\, S'\left(\frac{r}{\alpha}\right)\right)\,.$$

Example 2.6. Consider the r.v. X_g having PDF

$$g(x) = \frac{12}{\pi^2}\,\frac{x}{e^x + 1}\,\chi_{\mathbb{R}_+}(x)\,;$$

the corresponding CDF will be

$$G(x) = \frac{12}{\pi^2}\,\chi_{\mathbb{R}_+}(x)\int_0^x \frac{t}{e^t + 1}\,\mathrm{d}t\,.$$

The characteristic function expressed via Hurwitz–Lerch Zeta becomes

$$\phi_g(r) = \frac{12}{\pi^2}\,\Phi(-1, 2, 1 - \mathrm{i}r)\,.$$

Indeed, repeating this procedure of Theorem 2.1 we easily confirm this result. Furtheron, the CHF via alternating Mathieu series $\widetilde{S}(r)$ reads

$$\begin{aligned}\phi_g(r) &= \frac{12}{\pi^2}\left(\frac{\pi^2\,\cosh(\pi r)}{2\,\sinh^2(\pi r)} - \frac{1}{2r^2}\right) + \mathrm{i}\,\frac{12}{\pi^2}\,r\widetilde{S}(r)\\ &= \frac{6\,\cosh(\pi r)}{\sinh^2(\pi r)} - \frac{6}{\pi^2 r^2} + \mathrm{i}\,\frac{12}{\pi^2}\,r\widetilde{S}(r)\,,\end{aligned}$$

while

$$\mathsf{E}\, X_g^s = \frac{12}{\pi^2}\,\eta(s+2)\,\Gamma(s+2)\,,$$

where

$$\eta(q) = \sum_{n=1}^{\infty} = \frac{(-1)^{n-1}}{n^q} = \left(1 - 2^{1-q}\right)\,\zeta(q), \qquad \Re(q) > 0\,,$$

denotes the Dirichlet Eta–function.

Example 2.7. Using the result [8]

$$\int_0^\infty S(r)\,\mathrm{d}x = \frac{\pi^3}{12}$$

we define a r.v. X possessing PDF

$$p(x) = \frac{12}{\pi^3}\,S(x)\,\chi_{\mathbb{R}_+}(x)\,.$$

Applying the Fourier sine and cosine transforms of $S(x)$ [5], we get

$$\phi(r) = \frac{6}{\pi^2}\int_r^\infty \frac{x\,\mathrm{d}x}{e^x-1} + \frac{6\mathrm{i}}{\pi^3}\,\mathrm{PV}\int_0^\infty \frac{x}{e^x-1}\ln\left|\frac{x+r}{x-r}\right|\mathrm{d}x\,,$$

where the Cauchy Principal Value PV of the last integral is assumed to exist.

It is well–known that

$$\int_0^\infty \frac{\sin x}{x^s}\,\mathrm{d}x = \cos\left(\frac{\pi s}{2}\right)\Gamma(1-s), \qquad 0 < \Re(s) < 2\,,$$

accordingly

$$\int_0^\infty \frac{\sin(qx)}{x^{1-s}}\,\mathrm{d}x = \frac{\Gamma(s)}{q^s}\sin\left(\frac{\pi s}{2}\right), \qquad |\Re(s)| < 1\,.$$

Hence, we get

$$\begin{aligned}\mathsf{E}\,X^s &= \int_0^\infty x^s\,p(x)\,\mathrm{d}x = \frac{12}{\pi^3}\int_0^\infty x^{s-1}\left(\int_0^\infty \frac{t\sin(tx)}{e^t-1}\,\mathrm{d}t\right)\mathrm{d}x \\ &= \frac{12}{\pi^3}\int_0^\infty \frac{t}{e^t-1}\left(\int_0^\infty \frac{\sin(tx)}{x^{1-s}}\,\mathrm{d}x\right)\mathrm{d}t = \frac{12}{\pi^3}\Gamma(s)\sin\left(\frac{\pi s}{2}\right)\int_0^\infty \frac{t^{1-s}}{e^t-1}\mathrm{d}t \\ &= \frac{12}{\pi^3}\Gamma(s)\sin\left(\frac{\pi s}{2}\right)\Gamma(2-s)\zeta(2-s)\,.\end{aligned}$$

Finally, reductions give us

$$\mathsf{E}\,X^s = \frac{6}{\pi^2}\,\frac{(1-s)\zeta(2-s)}{\cos\left(\frac{\pi s}{2}\right)}\,.$$

3. Distributions associated with generalized Mathieu series

Special functions play a highly significant roles in the study of probability distribution and further statistical inference see, for example the monographs (Lebedev [13], Mathai and Saxena [15, 16], Mathai [14], Johnson and Kotz [11, 12]). Bivariate distributions are studied by Gupta *et al.* [9]. Hurwitz–Lerch Zeta distribution is introduced and applied in reliability theory by Gupta *et al.* [10], Garg *et al.* [7] and others. Very recently, Saxena *et al.* [21] proposed and studied two new statistical distributions named as, generalized Hurwitz–Lerch Zeta Beta prime distribution and generalized Hurwitz–Lerch Zeta Gamma distribution and investigate their statistical functions, such as moments, distribution and survivor function, characteristic function, the hazard rate function and the mean residue life functions, see [21] and the references therein.

In this chapter we consider Fourier sine and cosine and Mellin integral transform formulæ for few kind of generalized Mathieu type series belonging to Cerone and Lenard [2] and to Srivastava and Tomovski [25]. Making use of these formulæ, one defines various type PDFs and in the same time one introduces appropriate probability distributions.

3.1.

Cerone and Lenard reported [2] that

$$\int_0^\infty S_{\mu+1}(x)\,\mathrm{d}x = \frac{\sqrt{\pi}\,\Gamma(\mu+\frac{1}{2})}{\Gamma(\mu+1)}\zeta(2\mu), \qquad \mu > \frac{1}{2}\,.$$

So, we define a new r.v. X_μ associated with the PDF

$$f_\mu(x) = \frac{\Gamma(\mu+1)\,S_{\mu+1}(x)}{\sqrt{\pi}\,\Gamma(\mu+\frac{1}{2})\zeta(2\mu)}\,\chi_{\mathbb{R}_+}(x)\,.$$

Obviously, the related CDF becomes

$$F_\mu(x) = \frac{\Gamma(\mu+1)}{\sqrt{\pi}\,\Gamma(\mu+\frac{1}{2})\zeta(2\mu)}\,\chi_{\mathbb{R}_+}(x)\int_0^x S_{\mu+1}(t)\,\mathrm{d}t\,.$$

By virtue of the Fourier sine and cosine transforms of $S_{\mu+1}(x)$ (Elezović *et al.* [5]), we obtain the related CHF. Also, making use of the Mellin–transform formula for $S_{\mu+1}(x)$ (Elezović *et al.* [5]) we derive the sth moment.

Theorem 3.1. *Let* $\Re(\mu) > \frac{1}{2}$. *Then the* CHF *of the r.v.* X_μ *equals*

$$\phi_\mu(r) = \frac{2^{1/2-\mu}}{\Gamma(\mu+\frac{1}{2})\zeta(2\mu)}\left(\int_r^\infty \frac{t^{\mu+\frac{1}{2}}}{e^t-1}\,\Theta_c(\mu;r,t)\,\mathrm{d}t + \mathrm{i}\int_0^r \frac{t^{\mu+\frac{1}{2}}}{e^t-1}\,\Theta_s^{(1)}(\mu;r,t)\,\mathrm{d}t + \mathrm{i}\int_r^\infty \frac{t^{\mu+\frac{1}{2}}}{e^t-1}\,\Theta_s^{(2)}(\mu;r,t)\,\mathrm{d}r\right),$$

where

$$\begin{aligned}
\Theta_c(\mu;r,t) &= \frac{\sqrt{\pi}(t^2-r^2)}{2^{\mu-1/2}t^{\mu-1}\,\Gamma(\mu)}, && t > r,\\
\Theta_s^{(1)}(\mu;r,t) &= \frac{t^{\mu-1/2}}{2^{\mu-1/2}\,r\,\Gamma(\mu+1/2)}\;{}_2F_1\left[\begin{matrix}1,\ \frac{1}{2}\\ \mu+\frac{1}{2}\end{matrix}\;\middle|\;\frac{t^2}{r^2}\right], && 0 < t < r,\\
\Theta_s^{(2)}(\mu;r,t) &= \frac{rt^{\mu-5/2}}{2^{\mu-3/2}\Gamma(\mu-1/2)}\;{}_2F_1\left[\begin{matrix}1,\ \frac{3}{2}-\mu\\ \frac{3}{2}\end{matrix}\;\middle|\;\frac{r^2}{t^2}\right], && t > r\,.
\end{aligned}$$

Moreover, for all $0 < \Re(s) < 2\mu-1$ *the* s*th moment of* X_μ *is*

$$\mathsf{E}\,X_\mu^s = \frac{\Gamma(\mu+1)}{\sqrt{\pi}\,\Gamma(\mu+\frac{1}{2})\zeta(2\mu)}\,\mathrm{B}\Big(\frac{s+1}{2},\mu-\frac{s-1}{2}\Big)\zeta(2\mu-s)\,,$$

where $\mathrm{B}(\cdot,\cdot)$ *stands for the familiar Euler's Beta function.*

3.2.

Recall another result reported in (Elezović *et al.* [5]), that is

$$\int_0^\infty S_\mu^{(\alpha,\beta)}(r;\{k^{2/\alpha}\})\,\mathrm{d}r = \frac{\sqrt{\pi}\,\Gamma(\mu-\frac{1}{2})}{\Gamma(\mu)}\zeta\Big(2\Big[\mu-\frac{\beta}{\alpha}\Big]-1\Big)$$

defined for the parameter space $\alpha,\beta\in\mathbb{R}_+$, $\mu-\frac{\beta}{\alpha} > \frac{1}{2}$.

Having in mind this result we can define a r.v. X_σ with the related PDF of the form

$$f_\sigma(x) = \frac{\Gamma(\mu)\,S_\mu^{(\alpha,\beta)}(x;\{k^{2/\alpha}\})}{\sqrt{\pi}\,\Gamma(\mu-\frac{1}{2})\zeta\big(2\big[\mu-\frac{\beta}{\alpha}\big]-1\big)}\,\chi_{\mathbb{R}_+}(x)\,.$$

Applying the Fourier cosine and sine transforms and separately the Mellin transform of $S_\mu^{(\alpha,\beta)}(x;\{k^{2/\alpha}\})$ [5] we obtain the related CHF and the sth moment respectively.

Theorem 3.2. *Assume that parameters of the distribution of r.v. X_σ satisfy condition $\mu > \max\left\{\frac{2\beta}{\alpha}, \frac{\beta}{\alpha} + \frac{1}{2}, 0\right\}$. Then the related* CHF *is of the form*

$$\phi_\sigma(r) = \frac{\Gamma(\mu)}{\sqrt{\pi}\,\Gamma(\mu-\frac12)\zeta\big(2\big[\mu-\frac{\beta}{\alpha}\big]-1\big)\Gamma\big(2\big[\mu-\frac{\beta}{\alpha}\big]\big)} \int_r^\infty \frac{t^{\mu-\frac12}}{e^t-1}\,\Phi_c\Big[\begin{array}{c}\mu, \mu-\frac{\beta}{\alpha}\\ \mu-\frac{\beta}{\alpha}+\frac12\end{array}; \frac{r}{2}, \frac{t}{2}\Big]\,\mathrm{d}t$$
$$+ \frac{2\Gamma(\mu)\mathrm{i}}{\sqrt{\pi}\,\Gamma(\mu-\frac12)\zeta\big(2\big[\mu-\frac{\beta}{\alpha}\big]-1\big)\Gamma\big(2\big[\mu-\frac{\beta}{\alpha}\big]\big)} \Bigg\{\int_0^r \frac{t^{2(\mu-\frac{\beta}{\alpha})-1}}{e^t-1}\,\Phi_s^{(1)}\Big[\begin{array}{c}\mu, \mu-\frac{\beta}{\alpha}\\ \mu-\frac{\beta}{\alpha}+\frac12\end{array}; \frac{r}{2}, \frac{t}{2}\Big]\,\mathrm{d}t$$
$$+ \int_r^\infty \frac{t^{2(\mu-\frac{\beta}{\alpha})-1}}{e^t-1}\,\Phi_s^{(1)}\Big[\begin{array}{c}\mu, \mu-\frac{\beta}{\alpha}\\ \mu-\frac{\beta}{\alpha}+\frac12\end{array}; \frac{r}{2}, \frac{t}{2}\Big]\,\mathrm{d}t\Bigg\},$$

where, for $0 < \Re(\rho) < \Re(\sigma+\tau) - \frac12$, it is

$$\Phi_c\Big[\begin{array}{c}\rho, \sigma\\ \tau\end{array}; a, b\Big] = \frac{\sqrt{\pi}}{2b}\,\frac{\Gamma(\rho-\frac12)\Gamma(\sigma)\Gamma(\tau)}{\Gamma(\rho)\Gamma(\sigma-\frac12)\Gamma(\tau-\frac12)}\,{}_2F_1\Big[\begin{array}{c}\frac32-\sigma,\ \frac32-\tau\\ \frac32-\rho\end{array}\Big|\frac{a^2}{b^2}\Big]$$
$$+ \frac{\sqrt{\pi}}{2a}\Big(\frac{a^2}{b^2}\Big)^\alpha \frac{\Gamma(\frac12-\rho)\Gamma(\sigma)\Gamma(\tau)}{\Gamma(\rho)\Gamma(\sigma-\rho)\Gamma(\tau-\rho)}\,{}_2F_1\Big[\begin{array}{c}1+\rho-\sigma,\ 1+\rho-\tau\\ \frac12+\rho\end{array}\Big|\frac{a^2}{b^2}\Big],$$

which holds for $0 < a < b$.

Also

$$\Phi_s^{(1)}\Big[\begin{array}{c}\rho, \sigma\\ \tau\end{array}; a, b\Big] = \frac{1}{2a}\,{}_3F_2\Big[\begin{array}{c}\frac12,\ 1,\ \rho\\ \sigma,\ \tau\end{array}\Big|\frac{b^2}{a^2}\Big]$$

for $0 < b < a$.

Finally

$$\Phi_s^{(2)}\Big[\begin{array}{c}\rho, \sigma\\ \tau\end{array}; a, b\Big] = \frac{a}{b^2}\,\frac{(\sigma-1)(\tau-1)}{\rho-1}\,{}_3F_2\Big[\begin{array}{c}1,\ 2-\sigma,\ 2-\tau\\ \frac32,\ 2-\rho\end{array}\Big|\frac{a^2}{b^2}\Big]$$
$$+ \frac{\sqrt{\pi}}{2a}\Big(\frac{a^2}{b^2}\Big)^\alpha \frac{\Gamma(1-\rho)\Gamma(\sigma)\Gamma(\tau)}{\Gamma(\frac12+\rho)\Gamma(\sigma-\rho)\Gamma(\tau-\rho)}\,{}_2F_1\Big[\begin{array}{c}1+\rho-\sigma, 1+\rho-\tau\\ \frac12+\rho\end{array}\Big|\frac{a^2}{b^2}\Big],$$

under $0 < a < b$.

Moreover, the sth moment of the r.v. X_σ becomes

$$\mathsf{E}\,X_\sigma^s = \mathrm{B}\Big(\frac{s+1}{2}, \mu-\frac{s+1}{2}\Big)\,\zeta\Big(2\Big[\mu-\frac{\beta}{\alpha}\Big]-s-1\Big), \quad -1 < s < 2\Big(\mu-\frac{\beta}{\alpha}-1\Big).$$

3.3.

Finally, bearing in mind the formula (Elezović *et al.*, [5])

$$\int_0^\infty S_\mu^{(\alpha,\beta)}(x;\{k^\gamma\})\,\mathrm{d}x = \frac{\sqrt{\pi}\,\Gamma(\mu-\frac12)}{\Gamma(\mu)}\zeta\Big(\gamma(\mu\alpha-\beta)-\frac{\gamma\alpha}{2}\Big)$$

such that holds true for all

$$\alpha, \beta, \gamma \in \mathbb{R}_+,\ \mu > \tfrac12 \quad \text{and} \quad \gamma\alpha(2\mu-1) > 2(1+\beta\gamma),$$

we define the r.v. X_γ possessing PDF

$$h(x) = \frac{\Gamma(\mu)\,S_\mu^{(\alpha,\beta)}(x;\{k^\gamma\})}{\sqrt{\pi}\,\Gamma(\mu-\frac12)\,\zeta(\gamma(\mu\alpha-\beta)-\frac{\gamma\alpha}{2})}\,\chi_{\mathbb{R}_+}(x).$$

The Fourier sine and cosine transform of $S_\mu^{(\alpha,\beta)}(x;\{k^\gamma\})$ have been established in terms of the Fox's $H_{3,2}^{1,2}$–function (Tomovski and Tuan, [28]):

$$\int_0^\infty \left\{ \begin{array}{c} \sin(rx) \\ \cos(rx) \end{array} \right\} S_\mu^{(\alpha,\beta)}(r;\{k^\gamma\})\,\mathrm{d}r$$
$$= \frac{2\sqrt{\pi}}{x\Gamma(\mu)} \int_0^\infty \frac{t^{\gamma(\mu\alpha-\beta)-1}}{e^t-1} \left\{ \begin{array}{c} H_{3,2}^{1,2}\Big[\frac{4t^{\gamma\alpha}}{x^2}\Big| \begin{array}{c} (0,1),(1-\mu,1),(\frac{1}{2},1) \\ (0,1),\,(1-\gamma(\mu\alpha-\beta),\gamma\alpha) \end{array}\Big] \\ \\ H_{3,2}^{1,2}\Big[\frac{4t^{\gamma\alpha}}{x^2}\Big| \begin{array}{c} (\frac{1}{2},1),(1-\mu,1),(0,1) \\ (0,1),\,(1-\gamma(\mu\alpha-\beta),\gamma\alpha) \end{array}\Big] \end{array} \right\} \mathrm{d}t\,,$$

which holds true for all $\alpha,\beta,\gamma \in \mathbb{R}_+$; $1<\gamma(\mu\alpha-\beta)$.

Calculating now the CHF with the aid of previous formulæ, and deriving the fractional order moments by the Mellin transform of the generalized Mathieu series $S_\mu^{(\alpha,\beta)}(r;\{k^\gamma\})$, we arrive at the following results.

Theorem 3.3. *Let $\alpha,\beta,\gamma \in \mathbb{R}_+$; $1<\gamma(\mu\alpha-\beta)$. Then the* CHF *for the r.v. X_γ having* PDF *$h(x)$, equals*

$$\phi_\gamma(r) = C_\mu^{(\alpha,\beta)} \int_0^\infty \frac{t^{\gamma(\mu\alpha-\beta)-1}}{e^t-1} \Big\{ H_{3,2}^{1,2}\Big[\frac{4t^{\gamma\alpha}}{x^2}\Big| \begin{array}{c} (\frac{1}{2},1),(1-\mu,1),(0,1) \\ (0,1),\,(1-\gamma(\mu\alpha-\beta),\gamma\alpha) \end{array}\Big]$$
$$+\,\mathrm{i}\,H_{3,2}^{1,2}\Big[\frac{4t^{\gamma\alpha}}{x^2}\Big| \begin{array}{c} (0,1),(1-\mu,1),(\frac{1}{2},1) \\ (0,1),\,(1-\gamma(\mu\alpha-\beta),\gamma\alpha) \end{array}\Big]\Big\}\,\mathrm{d}t\,,$$

where

$$C_\mu^{(\alpha,\beta)} = \frac{2}{\Gamma(\mu-\frac{1}{2})\zeta(\gamma(\mu\alpha-\beta)-\frac{\gamma\alpha}{2})}\,.$$

Moreover, for all $\alpha,\beta,\gamma,\mu-\frac{1}{2}>0$ and

$$-1<s<\min\Big\{2\mu-1,\,\frac{2}{\alpha\gamma}\big(\gamma(\mu\alpha-\beta)-1\big)-1\Big\}\,,$$

there holds true

$$\mathsf{E}\,X_\gamma^s = \mathrm{B}\Big(\frac{s+1}{2},\mu-\frac{s+1}{2}\Big)\,\zeta\left(\gamma\,(\mu\alpha-\beta)-\frac{\gamma\alpha\,(s+1)}{2}\right)\,.$$

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