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For calculation the p-value in testing process capability index $C^*_{p,n}(u,v)$

Hsin-Lin Kuo[a], Ting-Yu Leo[b]

[a]*Department of Tourism & Travel Management, Chung-Yu Institute of Technology, 40, Yi-7 Road, Keelung, Taiwan 201, R.O.C.*
[b]*Department of Leisure, Recreation and Tourism, Chung-Yu Institute of Technology, Keelung, Taiwan, R.O.C.*

Abstract. The purpose of this paper is to consider an implemental simplified form of the distribution for an estimate $C^*_{p,n}(u,v)$ of the process yield index $C_p(u,v)$ with asymmetric tolerance. And then, based on the theory of testing hypothesis, we develop a simple step-by-step procedure, and provide an efficient software procedure to calculate the p-values and some tables of the critical values using estimator of $C^*_{p,n}(u,v)$ for the practical use in decision making.

1. Introduction

Most of the research effort concerning the development and statistical study of process capability indices are to provide a measurement of the fit between a process and the demands placed on it. They attempt to measure the probability that the process will be able to meet the design tolerances. Until now, the goal should be a never-ending effort to improve process capability for all processes.

The evolution of capability indices has advance very simple indices to more complicated indices that were designed to correct some of the shortcomings of their predecessors.

Further, Vännman (1995) defined a family of process capability indices as

$$C_p(u,v) = \frac{d - u|\mu - m|}{3\sqrt{\sigma^2 + \nu(\mu - T)^2}}, \tag{1}$$

where USL and LSL represent the upper and lower specification limits, respectively; $d = (USL - LSL)/2$ is the half-length of the specification interval, $m = (USL + LSL)/2$ is the center of the specification interval, T is the target value, based on engineering considerations, μ is the process mean, and σ is the process standard deviation, of the underlying process and under statistical controlled conditions. The capability indices C_p, C_{pk}, C_{pm} and C_{pmk} are produced by letting $(u,v) = (0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$, respectively, which have been defined explicitly as

$$C_p = (USL - LSL)/(6\sigma), \tag{2}$$

$$C_{pk} = \min\{USL - \mu, \mu - LSL\}/(3\sigma) = (d - |\mu - m|)/(3\sigma), \tag{3}$$

$$C_{pm} = (USL - LSL)/(6(\sigma^2 + (\mu - T)^2)^{1/2}), \tag{4}$$

$$C_{pmk} = \frac{\min(USL - \mu, \mu - LSL)}{\sqrt[3]{\sigma^2 + (\mu - T)^2}} = \frac{d - |\mu - m|}{3\sqrt{\sigma^2 + (\mu - T)^2}}. \tag{5}$$

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Received: 22 June 2011; Accepted: 04 May 2012
Email addresses: kuang.lien@msa.hinet.net (Corresponding author) (Hsin-Lin Kuo), kuang.lien@msa.hinet.net (Ting-Yu Leo)

Vännman and Kotz (1995a, 1995b) suggested that the indices with $(u, v) = (0, 4)$, and when the normality assumption is not serious violated and the target value is midway between the LSL and USL.

Furthermore, Vännman (1995a) pointed out that we have a two-sided specification interval for the case $T = m$ is quite common in practical situations. For these reasons, we study the case when $C_p(u, v)$ in (1) is given by

$$C_p(u,v) = \frac{d - u|\mu - T|}{3\sqrt{\sigma^2 + \nu(\mu - T)^2}} = \frac{d/\sigma - |\delta|}{3\sqrt{1+\delta^2}}, \tag{6}$$

where $\delta = (\mu - T)/\sigma$, u, $v \geqq 0$ and the probability of non-conformance is never more than $2\Phi(-3C_p(u, v))$. Moreover we see from (6) that the process mean μ lies within the middle $1/(u + 3v^{1/2}Cp(u, v))$ of the specification range.

$$|\mu - T| < \frac{d}{u + 3\sqrt{v}C_p(u,v)}. \tag{7}$$

Vännman (1997) proposed to take among many values of $(u, v), (u, v) = (0, 4)$ and $(u, v) = (1, 3)$ to generate two indices which are most sensitive to process departure from the target value. The index $C_p(u, v)$ is combining the yield-based and the loss-based, furthermore, it is very sensitive with regard to departures of the process mean μ from the target value T, and then the values of u and v in (6) should be large.

Several authors have provided techniques that allow statistically based inferences to be drawn regarding practical implementation of the process capability (Jiao and Tseng (2004), Garg *et al.* (2006), Chen and Chen (2007), Cherif *et al.* (2008)), and so on. In this work, under the assumption of normality and statistical control, we use only the central χ^2 distribution and the normal distribution and avoiding the non-central χ^2 distribution for determining the decision rules based on the p-value of $C^*_{p,n}(u, v)$, to conclude whether the process is capable or not. To obtain an appropriate decision rule, we develop a simple step-by-step procedure and provide an efficient Maple software procedure to calculate the p-values. We also provide tables of the critical values for using estimator of $C^*_{p,n}(u, v)$ to determine whether their process meets the preset capability requirement and making reliable decisions.

2. Estimation of $C_p(u, v)$

Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal distribution with mean μ and variance σ^2 as the measurement of the studied characteristic. Vännman (1995a) considered the maximum likelihood estimator (MLE) of $C_p(u, v)$, which is defined as

$$C^*_{p,n}(u,v) = \frac{d - u|\overline{X} - T|}{3\sqrt{\sigma^{2*} + v(\overline{X} - T)^2}} = \frac{D - u|\eta + g|}{3\sqrt{\xi + v(\eta + g)^2}}, \tag{8}$$

where $\overline{X} = \sum_{1-1}^{2} X_i/n$ and $\sigma^{2*} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2$ are the MLEs of μ and σ^2, respectively.

Under the assumption of normality, Vännman (1997) obtained the r$^{\text{th}}$ moment and the first two moments, as well as the mean and the variance of $C^*_{p,n}(u, v)$ for the common cases with $T = m$, further provided a simplified form for the obtained distribution. Thence, the cumulative distribution function of $C^*_{p,n}(u, v)$, for $x > 0$, $u \geqq 0$ or $v \geqq 0$, can be expressed as

$$F^*_{C^*_{p,n}(u,v)}(x) = 1 - \int_0^{\frac{D}{u+3x\sqrt{v}}} F_\xi\left(\frac{(D-ut)^2}{9x^2} - vt^2\right) h(t)dt, \quad \text{for } x > 0, \tag{9}$$

Under the assumption of normality, we now define
(1)

$$D = \sqrt{n}d/\sigma, a = (\mu - T)/\sigma. \tag{10}$$

(2)

$$\eta = \sqrt{n}(\overline{X} - \mu)/\sigma \sim N(0,1), g = \sqrt{n}(\mu - T)/\sigma = a\sqrt{n} \text{ and } \eta^2 \sim \chi_1^2. \tag{11}$$

(3)

$$\xi = \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \overline{X})^2 \sim \chi_{n-1}^2, \text{ and } \xi \text{ and } \eta \text{ are independent.} \tag{12}$$

(4) Let Φ and f_η denote the cumulative distribution function and the probability density function of η, respectively, and let

$$h(t) = f_\eta(-t-g) + f_n(t-g) = \frac{1}{\sqrt{2\pi}}\left(\exp\left(-\frac{(t+g)^2}{2}\right) + \exp\left(-\frac{(t-g)^2}{2}\right)\right). \tag{13}$$

In practice, since the process capability indices are calculated on sampling observations, a certain amount of uncertainty, due to the sampling error, is necessarily present in the evaluation of the process performances. In the operative context, this fact is rather neglected and conclusions about the capability of the process are often based only on the provided by sampling data. Apparently, this approach is not reliable, since sampling errors are ignored. With regard to the sampling errors, we use the natural estimator of $C_p(u,v)$ and provide an efficient p-value tabulated for making reliable decisions, based on the theory of hypothesis testing.

3. Some applications to hypothesis tests

When using process capability indices, a process is defined to be capable if the capability index exceeds a certain value $c_0 > 0$. Some commonly used values are $c_0 = 1.0, 4/3, 5/3, 2$. To conclude if a process is capable or not, we need a decision ruler based on an estimator, In order to obtain such a decision rule we consider the following hypothesis test

$$H_0 : C_p(u,v) \leqq c_0$$
$$H_1 : C_p(u,v) > c_0.$$

The null hypothesis will be rejected whenever $C_{p,n}^*(u,v) > c_\alpha$, where the constant c_α is determined so that the significance level of the test is α. The decision rule to be used is then that, for given α-risk, c_0 and n, the process will be considered capable if $C_{p,n}^*(u,v) > c_\alpha$, and non-capable if $C_{p,n}^*(u,v) \leqq c_\alpha$.

To find the critical region of $C_{p,n}^*(u,v) > c_\alpha$, we have to consider

$$\alpha = P(C_{p,n}^*(u,v) \geqq c_\alpha \big| C_p(u,v) = c_0), \tag{14}$$

which can be done using the results in (9). We see from (9) that the distribution of $C_{p,n}^*(u,v)$ depends on μ and σ, but not solely side to side $C_p(u,v)$. Given a value of $C_p(u,v) = c_0$ and $D = \sqrt{n}d/\sigma = \sqrt{n}(3c_0(1+va^2)^{1/2} + u|a|)$, so, we will evaluate p-value to make a decision. The p-value corresponding to w, a specific value of $C_{p,n}^*(u,v)$ calculated from the sample data, is the probability of wrongly concluding that an incapable process is capable.

$$\begin{aligned} p\text{-value} &= P(C_{p,n}^*(u,v) \geqq w \big| C_p(u,v) = c_0) \\ &= \int_0^{\frac{D}{u+3w\sqrt{v}}} F_\xi\left(\frac{(D-ut)^2}{9w^2} - vt^2\right) h(t)dt \end{aligned} \tag{15}$$

where $w = C_{p,n}^*(u,v)$ denotes the observed value of the test statistic.

Moreover, given values of the capability requirement c_0, the parameter a, the sample size n, and risk α, the critical value c_α can be obtained by solving the following equation

$$\alpha = \int_0^{\frac{D}{u+3c_\alpha\sqrt{v}}} F_\xi\left(\frac{(D-ut)^2}{9c_\alpha^2} - vt^2\right) h(t)dt. \tag{16}$$

Given values of n, c_0, and risk α, the critical value c_α for $|a|$ is the same, since form (16) is an even function of a. Vännman and Kotz (1995a) have shown that when μ moves away from T in either direction, then $P(C^*_{p,n}(u,v) \geqq x|C_p(u,v) = c_0)$ decreases for any given values of x and a.

In general, it does not seem feasible to derive c_α analytically. Instead, one can use a symbolic software, such as "Mathematica", "IMSL" or "Maple" etc., to numerically obtain the critical values of c_α for given values of n, a, c_0 and α-risk. The critical values c_α for $a = 0(0.1)1$, $n = 30(10)210$, $c_0 = 1$ and $\alpha = 0.05$, $(u,v) = (0,4)$ are calculated. We find that the critical value c_α can attain its maximum, this shift should range from $a = 0.48$ to $a = 0.55$, correspond to the sample size n from 210 decreasingly to 30. But, the difference critical values, between $a = 0.50$ and the others, is less than 10^{-3} or 10^{-4}. In practical situations, for these reasons we restrict our attention in this paper to the case when $a = 0, 0.50$.

We summarize the p-values for various sample sizes n and a range of values, $w = 1(0.1)2$, $a = 0, 0.5$, $(u,v) = (0,4)$, for $c_0 = 1, 1.33, 1.5, 1.67, 2$ in Tables 1–10.

Table 1: The p-value for $C_p(0,4) = 1.0$, $w = 1(0.1)2$, $a(= (\mu - T)/\sigma) = 0$, and $n = 30(5)200$

n	$w = 1$	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
80	0.4382	0.0998	0.0118	0.0009	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
90	0.4408	0.0889	0.0084	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
95	0.4420	0.0839	0.0071	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
100	0.4432	0.0792	0.0060	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
105	0.4442	0.0749	0.0051	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
110	0.4452	0.0708	0.0043	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
115	0.4462	0.0670	0.0037	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
120	0.4471	0.0634	0.0031	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
125	0.4480	0.0600	0.0026	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
130	0.4488	0.0568	0.0022	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
135	0.4496	0.0538	0.0019	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
140	0.4503	0.0510	0.0016	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
145	0.4510	0.0483	0.0014	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
150	0.4517	0.0458	0.0012	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
155	0.4523	0.0434	0.0010	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
160	0.4530	0.0412	0.0008	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 2: The p-value for $C_p(0,4) = 1.00$, $w = 1(0.1)2$, $a(= (\mu - T)/\sigma) = 0.5$, and $n = 30(5)200$

n	$w = 1$	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
85	0.5064	0.2119	0.0632	0.0141	0.0025	0.0004	0.0000	0.0000	0.0000	0.0000	0.0000
90	0.5062	0.2050	0.0578	0.0120	0.0019	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000
95	0.5060	0.1984	0.0529	0.0102	0.0015	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000
100	0.5059	0.1922	0.0484	0.0086	0.0011	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
105	0.5058	0.1862	0.0444	0.0073	0.0009	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
110	0.5056	0.1804	0.0407	0.0051	0.0007	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
115	0.5050	0.1749	0.0374	0.0053	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
120	0.5054	0.1697	0.0343	0.0045	0.0004	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
125	0.5053	0.1646	0.0315	0.0039	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
130	0.5052	0.1597	0.0290	0.0033	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
135	0.5051	0.1551	0.0266	0.0028	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
140	0.5050	0.1519	0.0245	0.0024	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
145	0.5049	0.1462	0.0226	0.0020	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
150	0.5048	0.1420	0.0208	0.0018	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
155	0.5047	0.1380	0.0191	0.0015	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
160	0.5047	0.1341	0.0176	0.0013	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

We obtain that p-value is decreasing for various w when n and a are fixed. The p-value is quite small, when w is very largely; in addition, the p-value is quite great when c_0 is increasing.

Tables 11 $\sim$ 12 display the critical values c_α for various $d = 3\sigma \sim 6\sigma$, with sample size $n = 30(10)100$, and α-risk $= 0.01, 0.025, 0.05, 0.1$. From these tabulates, we found that the critical value is decreasing, as α-risk is increasing and n is fixed. The critical value is decreasing, when n is increasing for any α-risk and a.

Table 3: The p-value for $C_p(0,4) = 1.3333$, $w = 1(0.1)2$, $a(= (\mu - T)/\sigma) = 0$, and $n = 30(5)200$

n	$w = 1$	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
80	0.9996	0.9869	0.8733	0.5604	0.2298	0.0617	0.0118	0.0018	0.0002	0.0000	0.0000
85	0.9997	0.9894	0.8827	0.5657	0.2251	0.0568	0.0100	0.0013	0.0001	0.0000	0.0000
90	0.9998	0.9914	0.8913	0.5709	0.2206	0.0524	0.0084	0.0010	0.0001	0.0000	0.0000
95	0.9999	0.9931	0.8992	0.5758	0.2162	0.0483	0.0071	0.0008	0.0001	0.0000	0.0000
100	0.9999	0.9944	0.9065	0.5805	0.2119	0.0446	0.0060	0.0006	0.0001	0.0000	0.0000
105	0.9999	0.9955	0.9132	0.5851	0.2078	0.0411	0.0051	0.0004	0.0000	0.0000	0.0000
110	1.0000	0.9963	0.9194	0.5896	0.2038	0.0380	0.0043	0.0003	0.0000	0.0000	0.0000
115	1.0000	0.9970	0.9251	0.5939	0.1999	0.0351	0.0037	0.0003	0.0000	0.0000	0.0000
120	1.0000	0.9976	0.9303	0.5980	0.1961	0.0325	0.0031	0.0002	0.0000	0.0000	0.0000
125	1.0000	0.9980	0.9352	0.6021	0.1924	0.0300	0.0026	0.0002	0.0000	0.0000	0.0000
130	1.0000	0.9984	0.9397	0.6060	0.1888	0.0278	0.0022	0.0001	0.0000	0.0000	0.0000
135	1.0000	0.9987	0.9439	0.6098	0.1853	0.0257	0.0019	0.0001	0.0000	0.0000	0.0000
140	1.0000	0.9990	0.9478	0.6135	0.1819	0.0238	0.0016	0.0001	0.0000	0.0000	0.0000
145	1.0000	0.9992	0.9513	0.6171	0.1786	0.0221	0.0014	0.0001	0.0000	0.0000	0.0000
150	1.0000	0.9993	0.9547	0.6207	0.1753	0.0205	0.0012	0.0000	0.0000	0.0000	0.0000
155	1.0000	0.9994	0.9578	0.6241	0.1722	0.0190	0.0010	0.0000	0.0000	0.0000	0.0000
160	1.0000	0.9995	0.9606	0.6275	0.1691	0.0176	0.0009	0.0000	0.0000	0.0000	0.0000

Table 4: The p-value for $C_p(0,4) = 1.3333$, $w = 1(0.1)2$, $a(= (\mu - T)/\sigma) = 0.5$, and $n = 30(5)200$

n	$w = 1$	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
80	0.9956	0.9566	0.8225	0.5913	0.3476	0.1688	0.0691	0.0243	0.0075	0.0021	0.0005
85	0.9966	0.9612	0.8296	0.5937	0.3428	0.1613	0.0632	0.0210	0.0061	0.0015	0.0004
90	0.9973	0.9652	0.8363	0.5960	0.3382	0.1541	0.0578	0.0182	0.0049	0.0012	0.0002
95	0.9979	0.9688	0.8426	0.5982	0.3337	0.1474	0.0529	0.0158	0.0040	0.0009	0.0002
100	0.9983	0.9720	0.8486	0.6004	0.3294	0.1411	0.0484	0.0137	0.0033	0.0007	0.0001
105	0.9987	0.9749	0.8544	0.6026	0.3252	0.1351	0.0444	0.0119	0.0026	0.0005	0.0001
110	0.9989	0.9774	0.8598	0.6047	0.3211	0.1294	0.0407	0.0103	0.0022	0.0004	0.0001
115	0.9992	0.9797	0.8650	0.6067	0.3172	0.1240	0.0374	0.0090	0.0018	0.0003	0.0000
120	0.9993	0.9817	0.8700	0.6087	0.3134	0.1189	0.0343	0.0078	0.0014	0.0002	0.0000
125	0.9995	0.9836	0.8747	0.6107	0.3096	0.1140	0.0315	0.0068	0.0012	0.0002	0.0000
130	0.9996	0.9852	0.8792	0.6127	0.3060	0.1094	0.0290	0.0059	0.0010	0.0001	0.0000
135	0.9997	0.9867	0.8836	0.6146	0.3024	0.1050	0.0266	0.0051	0.0008	0.0001	0.0000
140	0.9997	0.9880	0.8877	0.6164	0.2990	0.1008	0.0245	0.0045	0.0006	0.0001	0.0000
145	0.9998	0.9892	0.8917	0.6183	0.2956	0.0968	0.0225	0.0039	0.0005	0.0001	0.0000
150	0.9998	0.9902	0.8955	0.6201	0.2923	0.0930	0.0207	0.0034	0.0004	0.0000	0.0000
155	0.9999	0.9912	0.8991	0.6218	0.2890	0.0893	0.0191	0.0030	0.0003	0.0000	0.0000
160	0.9999	0.9920	0.9026	0.6236	0.2859	0.0858	0.0176	0.0026	0.0003	0.0000	0.0000

Table 5: The p-value for $C_p(0,4) = 1.5$, $w = 1(0.1)2$, $a(= (\mu - T)/\sigma) = 0$, and $n = 30(5)200$

n	w =1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
80	1.0000	0.9998	0.9953	0.9467	0.7549	0.4382	0.1776	0.0522	0.0118	0.0022	0.0004
85	1.0000	0.9999	0.9964	0.9531	0.7647	0.4395	0.1721	0.0476	0.0100	0.0017	0.0002
90	1.0000	0.9999	0.9973	0.9587	0.7739	0.4408	0.1667	0.0435	0.0084	0.0013	0.0002
95	1.0000	1.0000	0.9979	0.9636	0.7826	0.4420	0.1617	0.0398	0.0071	0.0010	0.0001
100	1.0000	1.0000	0.9984	0.9679	0.7909	0.4432	0.1568	0.0364	0.0060	0.0008	0.0001
105	1.0000	1.0000	0.9988	0.9717	0.7988	0.4442	0.1521	0.0333	0.0051	0.0006	0.0001
110	1.0000	1.0000	0.9991	0.9750	0.8063	0.4452	0.1476	0.0305	0.0043	0.0005	0.0000
115	1.0000	1.0000	0.9993	0.9779	0.8134	0.4462	0.1432	0.0280	0.0037	0.0004	0.0000
120	1.0000	1.0000	0.9995	0.9805	0.8202	0.4471	0.1391	0.0257	0.0031	0.0003	0.0000
125	1.0000	1.0000	0.9996	0.9828	0.8267	0.4480	0.1350	0.0235	0.0026	0.0002	0.0000
130	1.0000	1.0000	0.9997	0.9848	0.8329	0.4488	0.1312	0.0216	0.0022	0.0002	0.0000
135	1.0000	1.0000	0.9998	0.9865	0.8389	0.4496	0.1274	0.0198	0.0019	0.0001	0.0000
140	1.0000	1.0000	0.9998	0.9881	0.8445	0.4503	0.1238	0.0182	0.0016	0.0001	0.0000
145	1.0000	1.0000	0.9999	0.9895	0.8500	0.4510	0.1203	0.0167	0.0014	0.0001	0.0000
150	1.0000	1.0000	0.9999	0.9907	0.8552	0.4517	0.1170	0.0154	0.0012	0.0001	0.0000
155	1.0000	1.0000	0.9999	0.9917	0.8602	0.4523	0.1137	0.0141	0.0010	0.0000	0.0000
160	1.0000	1.0000	0.9999	0.9927	0.8649	0.4530	0.1106	0.0130	0.0008	0.0000	0.0000

Table 6: The p-value for $C_p(0,4) = 1.5$, $w = 1(0.1)2$, $a(= (\mu - T)/\sigma) = 0.5$, and $n = 30(5)200$

n	w =1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
80	0.9999	0.9978	0.9773	0.8966	0.7280	0.5066	0.3009	0.1542	0.0691	0.0275	0.0099
85	1.0000	0.9983	0.9804	0.9032	0.7338	0.5064	0.2950	0.1466	0.0632	0.0240	0.0081
90	1.0000	0.9987	0.9830	0.9094	0.7394	0.5062	0.2893	0.1395	0.0578	0.0209	0.0067
95	1.0000	0.9990	0.9853	0.9151	0.7448	0.5060	0.2838	0.1329	0.0529	0.0182	0.0055
100	1.0000	0.9993	0.9872	0.9204	0.7500	0.5059	0.2786	0.1266	0.0484	0.0159	0.0045
105	1.0000	0.9994	0.9889	0.9253	0.7551	0.5058	0.2735	0.1207	0.0444	0.0139	0.0038
110	1.0000	0.9996	0.9904	0.9299	0.7600	0.5056	0.2686	0.1152	0.0407	0.0121	0.0031
115	1.0000	0.9997	0.9916	0.9341	0.7647	0.5055	0.2638	0.1099	0.0374	0.0106	0.0026
120	1.0000	0.9997	0.9927	0.9381	0.7692	0.5054	0.2592	0.1049	0.0343	0.0093	0.0021
125	1.0000	0.9998	0.9937	0.9418	0.7737	0.5053	0.2548	0.1002	0.0315	0.0081	0.0018
130	1.0000	0.9999	0.9945	0.9453	0.7780	0.5052	0.2504	0.0958	0.0290	0.0071	0.0015
135	1.0000	0.9999	0.9952	0.9486	0.7821	0.5051	0.2462	0.0915	0.0266	0.0062	0.0012
140	1.0000	0.9999	0.9958	0.9516	0.7862	0.5050	0.2421	0.0875	0.0245	0.0055	0.0010
145	1.0000	0.9999	0.9964	0.9544	0.7901	0.5049	0.2381	0.0837	0.0226	0.0048	0.0008
150	1.0000	0.9999	0.9968	0.9571	0.7940	0.5048	0.2343	0.0801	0.0208	0.0042	0.0007
155	1.0000	1.0000	0.9972	0.9596	0.7977	0.5047	0.2305	0.0766	0.0191	0.0037	0.0006
160	1.0000	1.0000	0.9976	0.9619	0.8013	0.5047	0.2268	0.0733	0.0176	0.0033	0.0005

Table 7: The p-value for $C_p(0,4) = 1.6667$, $w = 1(0.1)2$, $a(= (\mu - T)/\sigma) = 0$, and $n = 30(5)200$

n	$w = 1$	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
80	1.0000	1.0000	0.9999	0.9981	0.9773	0.8734	0.6340	0.3469	0.1424	0.0455	0.0118
85	1.0000	1.0000	1.0000	0.9986	0.9811	0.8828	0.6414	0.3453	0.1366	0.0412	0.0100
90	1.0000	1.0000	1.0000	0.9990	0.9842	0.8914	0.6486	0.3437	0.1311	0.0374	0.0084
95	1.0000	1.0000	1.0000	0.9993	0.9868	0.8993	0.6554	0.3421	0.1258	0.0340	0.0071
100	1.0000	1.0000	1.0000	0.9995	0.9889	0.9066	0.6620	0.3405	0.1208	0.0308	0.0060
105	1.0000	1.0000	1.0000	0.9996	0.9907	0.9133	0.6683	0.3390	0.1161	0.0280	0.0051
110	1.0000	1.0000	1.0000	0.9997	0.9922	0.9195	0.6745	0.3374	0.1116	0.0255	0.0043
115	1.0000	1.0000	1.0000	0.9998	0.9935	0.9252	0.6803	0.3359	0.1073	0.0232	0.0037
120	1.0000	1.0000	1.0000	0.9999	0.9945	0.9304	0.6860	0.3343	0.1032	0.0211	0.0031
125	1.0000	1.0000	1.0000	0.9999	0.9954	0.9353	0.6916	0.3328	0.0992	0.0193	0.0026
130	1.0000	1.0000	1.0000	0.9999	0.9962	0.9398	0.6969	0.3313	0.0955	0.0176	0.0022
135	1.0000	1.0000	1.0000	0.9999	0.9968	0.9440	0.7021	0.3298	0.0919	0.0160	0.0019
140	1.0000	1.0000	1.0000	1.0000	0.9973	0.9478	0.7071	0.3283	0.0885	0.0146	0.0016
145	1.0000	1.0000	1.0000	1.0000	0.9977	0.9514	0.7120	0.3268	0.0852	0.0133	0.0014
150	1.0000	1.0000	1.0000	1.0000	0.9981	0.9547	0.7167	0.3253	0.0820	0.0122	0.0012
155	1.0000	1.0000	1.0000	1.0000	0.9984	0.9578	0.7213	0.3238	0.0790	0.0111	0.0010
160	1.0000	1.0000	1.0000	1.0000	0.9987	0.9607	0.7258	0.3224	0.0761	0.0101	0.0008

Table 8: The p-value for $C_p(0,4) = 1.6667$, $w = 1(0.1)2$, $a(= (\mu - T)/\sigma) = 0.5$, and $n = 30(5)200$

n	$w = 1$	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
80	1.0000	1.0000	0.9988	0.9875	0.9390	0.8226	0.6420	0.4405	0.2663	0.1431	0.0692
85	1.0000	1.0000	0.9991	0.9896	0.9445	0.8297	0.6458	0.4383	0.2597	0.1356	0.0632
90	1.0000	1.0000	0.9993	0.9913	0.9494	0.8364	0.6495	0.4361	0.2534	0.1286	0.0578
95	1.0000	1.0000	0.9995	0.9927	0.9538	0.8427	0.6531	0.4341	0.2474	0.1221	0.0529
100	1.0000	1.0000	0.9996	0.9939	0.9578	0.8487	0.6565	0.4321	0.2415	0.1159	0.0485
105	1.0000	1.0000	0.9997	0.9948	0.9614	0.8545	0.6599	0.4301	0.2360	0.1102	0.0444
110	1.0000	1.0000	0.9998	0.9957	0.9647	0.8599	0.6632	0.4282	0.2306	0.1047	0.0407
115	1.0000	1.0000	0.9999	0.9963	0.9677	0.8651	0.6665	0.4264	0.2254	0.0996	0.0374
120	1.0000	1.0000	0.9999	0.9969	0.9705	0.8701	0.6696	0.4246	0.2204	0.0947	0.0343
125	1.0000	1.0000	0.9999	0.9974	0.9730	0.8748	0.6727	0.4229	0.2156	0.0902	0.0315
130	1.0000	1.0000	0.9999	0.9978	0.9752	0.8793	0.6757	0.4212	0.2109	0.0859	0.0290
135	1.0000	1.0000	1.0000	0.9982	0.9773	0.8837	0.6786	0.4195	0.2063	0.0818	0.0267
140	1.0000	1.0000	1.0000	0.9984	0.9792	0.8878	0.6815	0.4178	0.2020	0.0779	0.0245
145	1.0000	1.0000	1.0000	0.9987	0.9809	0.8918	0.6843	0.4162	0.1977	0.0743	0.0226
150	1.0000	1.0000	1.0000	0.9989	0.9825	0.8956	0.6871	0.4147	0.1936	0.0708	0.0208
155	1.0000	1.0000	1.0000	0.9991	0.9839	0.8992	0.6898	0.4131	0.1896	0.0675	0.0191
160	1.0000	1.0000	1.0000	0.9992	0.9852	0.9027	0.6925	0.4116	0.1857	0.0644	0.0176

Table 9: The p-value for $C_p(0,4) = 2$, $w = 1(0.1)2$, $a(= (\mu - T)/\sigma) = 0$, and $n = 30(5)200$

n	$w = 1$	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
80	1.0000	1.0000	1.0000	1.0000	1.0000	0.9996	0.9953	0.9682	0.8734	0.6812	0.4382
85	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9964	0.9729	0.8828	0.6898	0.4395
90	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9973	0.9769	0.8914	0.6980	0.4408
95	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9979	0.9802	0.8993	0.7058	0.4420
100	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9984	0.9831	0.9066	0.7133	0.4432
105	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9988	0.9856	0.9133	0.7204	0.4442
110	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9991	0.9877	0.9194	0.7273	0.4452
115	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9993	0.9894	0.9251	0.7340	0.4462
120	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9995	0.9910	0.9304	0.7403	0.4471
125	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9996	0.9923	0.9353	0.7465	0.4480
130	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9934	0.9398	0.7524	0.4488
135	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9943	0.9439	0.7582	0.4496
140	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9951	0.9478	0.7637	0.4503
145	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9958	0.9514	0.7691	0.4510
150	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9964	0.9547	0.7742	0.4517
155	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9969	0.9578	0.7793	0.4523
160	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9974	0.9607	0.7841	0.4530

Table 10: The p-value for $C_p(0,4) = 2$, $w = 1(0.1)2$, $a(= (\mu - T)/\sigma) = 0.5$, and $n = 30(5)200$

n	$w = 1$	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
80	1.0000	1.0000	1.0000	1.0000	0.9995	0.9956	0.9773	0.9248	0.8226	0.6749	0.5066
85	1.0000	1.0000	1.0000	1.0000	0.9997	0.9966	0.9804	0.9308	0.8296	0.6796	0.5064
90	1.0000	1.0000	1.0000	1.0000	0.9998	0.9973	0.9830	0.9362	0.8363	0.6841	0.5062
95	1.0000	1.0000	1.0000	1.0000	0.9999	0.9979	0.9853	0.9412	0.8427	0.6885	0.5060
100	1.0000	1.0000	1.0000	1.0000	0.9999	0.9983	0.9872	0.9457	0.8487	0.6927	0.5059
105	1.0000	1.0000	1.0000	1.0000	0.9999	0.9987	0.9889	0.9498	0.8544	0.6968	0.5058
110	1.0000	1.0000	1.0000	1.0000	0.9999	0.9989	0.9904	0.9537	0.8599	0.7008	0.5056
115	1.0000	1.0000	1.0000	1.0000	1.0000	0.9992	0.9916	0.9572	0.8651	0.7047	0.5055
120	1.0000	1.0000	1.0000	1.0000	1.0000	0.9993	0.9927	0.9604	0.8700	0.7084	0.5054
125	1.0000	1.0000	1.0000	1.0000	1.0000	0.9995	0.9937	0.9633	0.8748	0.7121	0.5053
130	1.0000	1.0000	1.0000	1.0000	1.0000	0.9996	0.9945	0.9660	0.8793	0.7157	0.5052
135	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9952	0.9685	0.8836	0.7192	0.5051
140	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9958	0.9709	0.8878	0.7227	0.5050
145	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9964	0.9730	0.8917	0.7260	0.5049
150	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9968	0.9750	0.8955	0.7293	0.5048
155	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9972	0.9768	0.8992	0.7325	0.5047
160	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9976	0.9784	0.9027	0.7356	0.5047

4. Testing procedure and conclusion

As mentioned before, our goal is to provide a step-by-step testing procedure to determine whether the process meets the capability requirement. Firstly, we determine the process capability value, c_0, of $C_p(u,v)$, say, $1, 1.33, 1.67$ or 2 etc., and the α-risk. Secondly, we calculate the value of the estimator $w = C^*_{p,n}(u,v)$ from the sample data. Thirdly, from the appropriate Table, we find the corresponding p-value based on w, c_0 and the sample size n. Finally, if the estimated of the p-value is less than or equal to α, then we conclude that the process is capable; Otherwise, we do not have sufficient information to conclude that the process meets the present capability requirement. We summarize this approach in the Table 13.

For example, assume that $LSL = 1.7$, $USL = 2.3$, with a target value of $T = 2$, $C_p(0,4) = c_0 = 1.0$, $n = 120$, α-risk $= 0.025$, $\overline{x} = 2.013$, and $\sigma^* = 0.0728$. Here we set $a = \hat{a} = 0.178571$ and find $w = C^*_{p,n}(0,4) = 1.293602$, then we can find the corresponding p-value $= 0.000427$, and the critical value $c_{0.025} = 1.161771$.

We can conclude that the process is "capable", because p-value $= 0.000427 < \alpha$-risk $= 0.025$ and $w = C^*_{p,n}(0,4) = 1.293602 > c_{0.025} = 1.161771$.

When a decision has been made about which index to use the result of the p-value is easy to apply to conclude whether the process is capable or not. On the other hand, the numerical calculations are made by use of the mathematic software, as IMSL, Maple etc.. We only have to integrate a function based on the central X^2 distribution and the normal distribution. Also, we may obtain the critical value c_α for given values α-risk and n. The process will be considered to be capable if $C^*_{p,n}(u,v) > c_\alpha$ and non-capable if $C^*_{p,n}(u,v) \leqq c_\alpha$.

Table 11: The critical value c_α for various n and α, as $\mu = m = T$, $d = 3\sigma \sim 6\sigma$, $(u, v) = (0, 4)$

	$n\backslash\alpha$	0.01	0.025	0.05	0.1
$a = 0$ $f = e = 3$ $d = 3\sigma$ $u = 0$ $v = 4$	30	1.374927	1.296098	1.233659	1.167141
	40	1.312554	1.249051	1.198114	1.143204
	50	1.272809	1.218702	1.174943	1.127394
	60	1.244680	1.196937	1.158223	1.115778
	70	1.223540	1.180529	1.145494	1.106922
	80	1.206918	1.167621	1.135334	1.099845
	90	1.193388	1.157001	1.127069	1.094021
	100	1.182178	1.148180	1.120144	1.089168
$a = 0$ $f = e = 4$ $d = 4\sigma$ $u = 0$ $v = 4$	30	1.833375	1.728087	1.644838	1.556149
	40	1.750071	1.665401	1.645012	1.524407
	50	1.697023	1.624884	1.566540	1.503155
	60	1.659527	1.595969	1.544259	1.487765
	70	1.631335	1.574080	1.527288	1.475947
	80	1.609184	1.556789	1.513815	1.466502
	90	1.591209	1.542695	1.502788	1.458730
	100	1.576255	1.530928	1.452188	1.452188
$a = 0$ $f = e = 5$ $d = 5\sigma$ $u = 0$ $v = 4$	30	2.291823	2.160429	2.056357	1.945490
	40	2.187804	2.081961	1.997064	1.905547
	50	2.121375	2.031196	1.958263	1.879028
	60	2.074503	1.995051	1.930411	1.859790
	70	2.039261	1.967689	1.909196	1.845016
	80	2.011571	1.946074	1.892354	1.833210
	90	1.989100	1.928456	1.878570	1.823494
	100	1.970408	1.913746	1.867022	1.815316
$a = 0$ $f = e = 6$ $d = 6\sigma$ $u = 0$ $v = 4$	30	2.750132	2.592463	2.467580	2.334541
	40	2.625313	2.498303	2.396429	2.286611
	50	2.545599	2.437386	2.349869	2.254789
	60	2.489353	2.394013	2.316447	2.231704
	70	2.447064	2.361179	2.290989	2.213975
	80	2.413837	2.335242	2.270779	2.199808
	90	2.386872	2.314101	2.254239	2.188149
	100	2.364442	2.296449	2.240382	2.178336

Table 12: The critical value c_α for various n and α, as $m = T$,$|\mu - T| = 0.5\sigma$, $(u, v) = (0, 4)$

	$n\backslash\alpha$	0.01	0.025	0.05	0.1
$a = 0.5$, $f = e = 3$, $d = 3\sigma$, $u = 0$, $v = 4$	30	1.595284	1.483903	1.393963	1.296487
	40	1.500480	1.408508	1.333551	1.251771
	50	1.438184	1.358537	1.293323	1.221911
	60	1.393424	1.322490	1.264242	1.200286
	70	1.359381	1.295012	1.242041	1.183749
	80	1.332435	1.273226	1.224417	1.170601
	90	1.310463	1.255440	1.210012	1.159837
	100	1.292130	1.240582	1.197965	1.150823
$a = 0.5$, $f = e = 4$, $d = 4\sigma$, $u = 0$, $v = 4$	30	2.126991	1.978488	1.858571	1.728579
	40	2.000590	1.877964	1.778023	1.668987
	50	1.917530	1.811338	1.724388	1.629172
	60	1.857853	1.763276	1.685614	1.600341
	70	1.812463	1.726639	1.656013	1.578292
	80	1.776535	1.697593	1.632515	1.560762
	90	1.747240	1.673878	1.613309	1.546410
	100	1.722796	1.654068	1.597247	1.534392
$a = 0.5$, $f = e = 5$, $d = 5\sigma$, $u = 0$, $v = 4$	30	2.658859	2.473222	2.323318	2.160855
	40	2.500850	2.347561	2.222629	2.086327
	50	2.397021	2.264274	2.155582	2.036559
	60	2.322420	2.204195	2.107112	2.000516
	70	2.265681	2.158396	2.070109	1.972954
	80	2.220769	2.122086	2.040736	1.951040
	90	2.184148	2.092441	2.016727	1.933100
	100	2.153592	2.067678	1.996649	1.918076
$a = 0.5$, $f = e = 6$, $d = 6\sigma$, $u = 0$, $v = 4$	30	3.190567	2.967807	2.787926	2.592974
	40	3.000960	2.817016	2.667102	2.503543
	50	2.876367	2.717075	2.586646	2.443822
	60	2.786849	2.644981	2.528484	2.400571
	70	2.718762	2.590023	2.484081	2.367497
	80	2.664869	2.546453	2.448834	2.341201
	90	2.620925	2.493978	2.408537	2.313012
	100	2.562758	2.466466	2.385994	2.295922

Table 13: The procedure of the p-value for $C^*_{p,n}(u, v)$

Step	p-value
1.	Determine the value of c_0 (set to $1, 1.33, 1.5, 1.67$ or 2) and the α-risk (set to $0.01, 0.025, 0.05$), the chance of incorrectly accepting an incapable process as capable.
2.	Calculate the estimator of $C_p(u, v)$ from the sample, $C^*_{p,n}(u, v)$.
3.	(a). To obtain w. (b). Calculate the value $a = \hat{a} = (\overline{x} - T)/\sigma^*$.
4.	To find the corresponding p-value based on w, a, c_0 and n.
5.	If the p-value $\leqq \alpha$, we can conclude that the process is capable; Otherwise, we do not have enough information to conclude that the process is capable.

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