

Relations for moments of generalized order statistics from Marshall-Olkin extended Weibull distribution and its characterization

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Abstract. Marshall-Olkin extended Weibull distribution was introduced by Marshall and Olkin (1997) and extensively studied by Ghitany et al. (2005). In this article recurrence relations for single and product moments of generalized order statistics (*gos*) for Marshall-Olkin extended Weibull distribution have been driven. Moments of order statistics and k -upper records are discussed as special cases. Characterization of Marshall-Olkin extended Weibull distribution through conditional expectation is also presented.

1. Introduction

Let X_1, X_2, \dots, X_n be a sequence of independent and identically distributed (*iid*) random variables (*rv*) with absolutely continuous distribution function (*df*) $F(x)$ and the probability density function (*pdf*) $f(x)$, $x \in (\alpha, \beta)$. Let $n \in \mathbb{N}$, ($n \geq 2$), $k \geq 1$, $\tilde{m} = (m_1, m_2, \dots, m_{n-1}) \in \mathbb{R}^{n-1}$, $M_r = \sum_{j=r}^{n-1} m_j$, $1 \leq r \leq n-1$ be the parameters such that $\gamma_r = k + n - r + M_r \geq 1$, for all $r \in \{1, 2, \dots, n-1\}$. Then $X(r, n, \tilde{m}, k)$ are called generalized order statistics (*gos*) if their joint *pdf* is given by

$$f_{X(1, n, \tilde{m}, k), \dots, X(n, n, \tilde{m}, k)}(x_1, x_2, \dots, x_n) = k \left(\prod_{j=1}^{n-1} \gamma_j \right) \left(\prod_{i=1}^{n-1} [\bar{F}(x_i)]^{m_i} f(x_i) \right) [\bar{F}(x_n)]^{k-1} f(x_n)$$

on the cone $F^{-1}(0) < x_1 \leq \dots \leq x_n < F^{-1}(1)$, where $\bar{F}(x) = 1 - F(x)$.

Choosing the parameters appropriately, models such as ordinary order statistics ($\gamma_i = n - i + 1$; $i = 1, 2, \dots, n$, i.e. $m_1 = m_2 = \dots m_{n-1} = 0$, $k = 1$), k^{th} record values ($\gamma_i = k$, i.e. $m_1 = m_2 = \dots = m_{n-1} = -1$, $k \in \mathbb{N}$), sequential order statistics ($\gamma_i = (n - i + 1)\alpha_i$; $\alpha_1, \alpha_2, \dots, \alpha_n > 0$), order statistics with non-integral sample size ($\gamma_i = \alpha - i + 1$, $\alpha > 0$). Pfeifer's record values ($\gamma_i = \beta_i$; $\beta_1, \beta_2, \dots, \beta_n > 0$) and progressive type II censored order statistics ($m_i \in \mathbb{N}_0$, $k \in \mathbb{N}$) are obtained (Kamps (1995), Kamps and Cramer (2001)).

For $\gamma_i \neq \gamma_j$, $i \neq j$ for all $i, j \in (1, 2, \dots, n)$, the *pdf* of $X(r, n, \tilde{m}, k)$ is given by Kamps and Cramer (2001) in the following way

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$$f_{X(r,n,\tilde{m},k)}(x) = C_{r-1} f(x) \sum_{i=1}^r a_i(r) [\bar{F}(x)]^{\gamma_i-1}. \tag{1}$$

The joint probability density function (*pdf*) of $X(r, n, \tilde{m}, k)$ and $X(s, n, \tilde{m}, k)$, $1 \leq r < s \leq n$ is given as

$$f_{X(r,n,\tilde{m},k),X(s,n,\tilde{m},k)}(x,y) = C_{s-1} \left(\sum_{i=r+1}^s a_i^{(r)}(s) \left[\frac{\bar{F}(y)}{\bar{F}(x)} \right]^{\gamma_i} \right) \left(\sum_{i=1}^r a_i(r) [\bar{F}(x)]^{\gamma_i} \right) \frac{f(x)}{\bar{F}(x)} \frac{f(y)}{\bar{F}(y)}, \tag{2}$$

where $\alpha \leq x < y \leq \beta$ and

$$a_i(r) = \prod_{\substack{j=1 \\ j \neq i}}^r \frac{1}{(\gamma_j - \gamma_i)}, \quad 1 \leq i \leq r \leq n,$$

$$a_i^{(r)}(s) = \prod_{\substack{j=r+1 \\ j \neq i}}^s \frac{1}{\gamma_j - \gamma_i}, \quad r+1 \leq i \leq s \leq n.$$

It may be noted that for $m_1 = m_2 = \dots = m \neq -1$,

$$a_i(r) = \frac{(-1)^{r-i}}{(m+1)^{r-1}(r-1)!} \binom{r-1}{r-i} \tag{3}$$

$$a_i^{(r)}(s) = \frac{(-1)^{s-i}}{(m+1)^{s-r-1}(s-r-1)!} \binom{s-r-1}{s-i} \tag{4}$$

Therefore (*pdf*) of $X(r, n, \tilde{m}, k)$ given in (1) reduces to

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!} [\bar{F}(x)]^{\gamma_{r-1}} f(x) g_m^{r-1}(F(x)), \tag{5}$$

and joint (*pdf*) of $X(r, n, \tilde{m}, k)$ and $X(s, n, \tilde{m}, k)$ given in (2) reduces to

$$f_{X(r,n,m,k),X(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!} [\bar{F}(x)]^m f(x) g_m^{r-1}(F(x)) \times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [\bar{F}(y)]^{\gamma_s-1} f(y), \quad \alpha \leq x < y \leq \beta \tag{6}$$

where

$$C_{r-1} = \prod_{i=1}^r \gamma_i, \quad \gamma_i = k + (n-i)(m+1),$$

$$h_m(x) = \begin{cases} -\frac{1}{m+1} (1-x)^{m+1} & , \quad m \neq -1 \\ -\log(1-x) & , \quad m = -1 \end{cases}$$

and $g_m(x) = h_m(x) - h_m(0)$, $x \in (0, 1)$.

A random variable X is said to have the Marshall-Olkin extended Weibull distribution if its *pdf* is of the form

$$f(x) = \frac{\lambda \theta x^{\theta-1} e^{-x^\theta}}{[1 - (1-\lambda)e^{-x^\theta}]^2}, \quad x > 0, \lambda > 0, \theta > 0 \tag{7}$$

and corresponding df

$$F(x) = 1 - \frac{\lambda e^{-x^\theta}}{[1 - (1 - \lambda)e^{-x^\theta}]}, \quad x > 0, \lambda > 0, \theta > 0. \tag{8}$$

Now in view of (7) and (8), we get

$$\bar{F}(x) = \frac{x^{1-\theta}}{\theta} [1 - (1 - \lambda)e^{-x^\theta}] f(x) \tag{9}$$

where $\bar{F}(x) = 1 - F(x)$.

The relation (9) will be utilized to establish recurrence relations for moments of *gos*.

2. Single moments

Theorem 2.1. For the Marshall-Olkin extended Weibull distribution as given in (7) and $n \in N, \tilde{m} \in \mathbb{R}, k > 0, 1 \leq r \leq n, j = 1, 2, \dots$

$$\begin{aligned} E[X^j(r, n, \tilde{m}, k)] - E[X^j(r - 1, n, \tilde{m}, k)] &= \frac{j}{\theta \gamma_r} E[X^{j-\theta}(r, n, \tilde{m}, k)] \\ &\quad - \frac{j(1-\lambda)}{\theta \gamma_r} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} E[X^{j-\theta(1-l)}(r, n, \tilde{m}, k)]. \end{aligned} \tag{10}$$

Proof. We have by Lemma 2.3 (Athar and Islam (2004))

$$E[\xi\{X(r, n, \tilde{m}, k)\}] - E[\xi\{X(r - 1, n, \tilde{m}, k)\}] = C_{r-2} \int_{\alpha}^{\beta} \xi'(x) \sum_{i=1}^r a_i(r) [\bar{F}(x)]^{\gamma_i} dx$$

where $\xi(x)$ is a Borel measurable function of $x \in (\alpha, \beta)$.

Then for $\xi(x) = x^j, j > 0$,

$$E[X^j(r, n, \tilde{m}, k)] - E[X^j(r - 1, n, \tilde{m}, k)] = C_{r-2} j \int_{\alpha}^{\beta} x^{j-1} \sum_{i=1}^r a_i(r) [\bar{F}(x)]^{\gamma_i} dx. \tag{11}$$

On using (9) in (11), we get

$$\begin{aligned} E[X^j(r, n, \tilde{m}, k)] - E[X^j(r - 1, n, \tilde{m}, k)] &= \frac{j}{\theta \gamma_r} C_{r-1} \int_0^{\infty} x^{j-\theta} \sum_{i=1}^r a_i(r) [\bar{F}(x)]^{\gamma_i-1} \{1 - (1 - \lambda)e^{-x^\theta}\} f(x) dx \\ &= \frac{j}{\theta} \frac{C_{r-1}}{\gamma_r} \int_0^{\infty} x^{j-\theta} \sum_{i=1}^r a_i(r) [\bar{F}(x)]^{\gamma_i-1} f(x) dx \\ &\quad - \frac{j(1-\lambda)}{\theta} \frac{C_{r-1}}{\gamma_r} \int_0^{\infty} x^{j-\theta} e^{-x^\theta} \sum_{i=1}^r a_i(r) [\bar{F}(x)]^{\gamma_i-1} f(x) dx. \end{aligned} \tag{12}$$

Now writing expansion of e^{-x^θ} in the integrand of second part of (12), we have

$$\begin{aligned} E[X^j(r, n, \tilde{m}, k)] - E[X^j(r - 1, n, \tilde{m}, k)] &= \frac{j}{\theta} \frac{C_{r-1}}{\gamma_r} \int_0^{\infty} x^{j-\theta} \sum_{i=1}^r a_i(r) [\bar{F}(x)]^{\gamma_i-1} f(x) dx \\ &\quad - \frac{j(1-\lambda)}{\theta} \frac{C_{r-1}}{\gamma_r} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \int_0^{\infty} x^{j-\theta(1-l)} \sum_{i=1}^r a_i(r) [\bar{F}(x)]^{\gamma_i-1} f(x) dx, \end{aligned}$$

which after simplification yields the (10). \square

Corollary 2.2. For $m_1 = m_2 = \dots = m_{n-1} = m \neq -1$, the recurrence relation for single moments of gos for Marshall-Olkin extended Weibull distribution is give as

$$E[X^j(r, n, m, k)] - E[X^j(r-1, n, m, k)] = \frac{j}{\theta\gamma_r} E[X^{j-\theta}(r, n, m, k)] - \frac{j(1-\lambda)}{\theta\gamma_r} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} E[X^{j-\theta(1-l)}(r, n, m, k)].$$

Proof. This can easily be deduced from (10) in view of the relations (3). \square

Remark 2.3. Recurrence relation for single moments of order statistics (at $m = 0, k = 1$) is

$$E(X_{r:n}^j) = E(X_{r-1:n}^j) + \frac{j}{\theta(n-r+1)} E(X_{r:n}^{j-\theta}) - \frac{j(1-\lambda)}{\theta(n-r+1)} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} E(X_{r:n}^{j-\theta(1-l)}).$$

Remark 2.4. Recurrence relation for single moments of k -th upper record (at $m = -1$) will be

$$E(X_r^{(k)})^j = E(X_{r-1}^{(k)})^j + \frac{j}{\theta k} E(X_r^{(k)})^{j-\theta} - \frac{j(1-\lambda)}{\theta k} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} E(X_r^{(k)})^{j-\theta(1-l)}.$$

Remark 2.5. For $\theta = 1$, we get recurrence relation for single moments of generalized order statistics from Marshall-Olkin extended exponential distribution and at $\theta = 1, \lambda = 1$, we get the relation for standard exponential distribution.

3. Product moments

Theorem 3.1. For distribution as given in (7). Fix a positive integer k and for $n \in \mathbb{N}, \tilde{m} \in \mathbb{R}, 1 \leq r < s \leq n$,

$$\begin{aligned} & E[X^i(r, n, \tilde{m}, k)X^j(s, n, \tilde{m}, k)] - E[X^i(r, n, \tilde{m}, k)X^j(s-1, n, \tilde{m}, k)] \\ &= \frac{j}{\theta\gamma_s} E[X^i(r, n, \tilde{m}, k)X^{j-\theta}(s, n, \tilde{m}, k)] \\ & - \frac{j(1-\lambda)}{\theta\gamma_s} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} E[X^i(r, n, \tilde{m}, k)X^{j-\theta(1-l)}(s, n, \tilde{m}, k)]. \end{aligned} \tag{13}$$

Proof. We have from Lemma 3.2 (Athar and Islam (2004)) that

$$\begin{aligned} & E[\xi\{X(r, n, \tilde{m}, k), X(s, n, \tilde{m}, k)\}] - E[\xi\{X(r, n, \tilde{m}, k), X(s-1, n, \tilde{m}, k)\}] \\ &= C_{s-2} \int_{\alpha}^{\beta} \int_x^{\beta} \frac{\partial}{\partial y} \xi(x, y) \sum_{i=r+1}^s a_i^{(r)}(s) \left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_i} \sum_{i=1}^r a_i(r) (\bar{F}(x))^{\gamma_i} \frac{f(x)}{\bar{F}(x)} dy dx. \end{aligned}$$

If we let $\xi(x, y) = x^i y^j$, then

$$\begin{aligned} & E[X^i(r, n, \tilde{m}, k)X^j(s, n, \tilde{m}, k)] - E[X^i(r, n, \tilde{m}, k)X^j(s-1, n, \tilde{m}, k)] \\ &= \frac{jC_{s-1}}{\gamma_s} \int_0^{\infty} \int_x^{\infty} x^i y^{j-1} \sum_{i=r+1}^s a_i^{(r)}(s) \left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_i} \sum_{i=1}^r a_i(r) (\bar{F}(x))^{\gamma_i} \frac{f(x)}{\bar{F}(x)} dy dx. \end{aligned}$$

In view of (9), note that

$$\frac{\bar{F}(y)}{f(y)} = \frac{y^{1-\theta}}{\theta} [1 - (1-\lambda)e^{-y^\theta}].$$

Therefore,

$$\begin{aligned} & E[X^i(r, n, \tilde{m}, k)X^j(s, n, \tilde{m}, k)] - E[X^i(r, n, \tilde{m}, k)X^j(s - 1, n, \tilde{m}, k)] \\ &= \frac{jC_{s-1}}{\theta\gamma_s} \int_0^\infty \int_x^\infty x^i y^{j-\theta} \{1 - (1 - \lambda)e^{-y^\theta}\} \sum_{i=r+1}^s a_i^{(r)}(s) \left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_i} \sum_{i=1}^r a_i(r) (\bar{F}(x))^{\gamma_i} \frac{f(x)}{\bar{F}(x)} \frac{f(y)}{\bar{F}(y)} dy dx \\ &= \frac{jC_{s-1}}{\theta\gamma_s} \int_0^\infty \int_x^\infty x^i y^{j-\theta} \sum_{i=r+1}^s a_i^{(r)}(s) \left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_i} \sum_{i=1}^r a_i(r) (\bar{F}(x))^{\gamma_i} \frac{f(x)}{\bar{F}(x)} \frac{f(y)}{\bar{F}(y)} dy dx \\ &\quad - \frac{j(1 - \lambda)C_{s-1}}{\theta\gamma_s} \int_0^\infty \int_x^\infty x^i y^{j-\theta} e^{-y^\theta} \sum_{i=r+1}^s a_i^{(r)}(s) \left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_i} \sum_{i=1}^r a_i(r) (\bar{F}(x))^{\gamma_i} \frac{f(x)}{\bar{F}(x)} \frac{f(y)}{\bar{F}(y)} dy dx \end{aligned}$$

which after simplification leads to (13). \square

Corollary 3.2. For $m_1 = m_2 = \dots = m_{n-1} = m \neq -1$, the relation for product moment is given as

$$\begin{aligned} E[X^i(r, n, m, k)X^j(s, n, m, k)] &= E[X^i(r, n, m, k)X^j(s - 1, n, m, k)] + \frac{j}{\theta\gamma_s} E[X^i(r, n, m, k)X^{j-\theta}(s, n, m, k)] \\ &\quad - \frac{j(1 - \lambda)}{\theta\gamma_s} \sum_{l=0}^\infty \frac{(-1)^l}{l!} E[X^i(r, n, m, k)X^{j-\theta(1-l)}(s, n, m, k)]. \end{aligned} \tag{14}$$

Proof. Equation (14) can be reduced from (13) in view of (3) and (4) or by replacing \tilde{m} with m in (13). \square

Remark 3.3. Recurrence relation for product moments of order statistics (at $m = 0, k = 1$) is

$$E(X_{r,s;n}^{i,j}) = E(X_{r,s-1;n}^{i,j}) + \frac{j}{\theta(n - s + 1)} \left[E(X_{r,s;n}^{i,j-\theta}) - (1 - \lambda) \sum_{l=0}^\infty \frac{(-1)^l}{l!} E(X_{r,s;n}^{i,j-\theta(1-l)}) \right].$$

Remark 3.4. Recurrence relation for product moments of $k - th$ record values will be

$$E[(X_r^{(k)})^i (X_s^{(k)})^j] = E[(X_r^{(k)})^i (X_{s-1}^{(k)})^j] + \frac{j}{\theta k} \left[E[(X_r^{(k)})^i (X_s^{(k)})^{j-\theta}] - (1 - \lambda) \sum_{l=0}^\infty \frac{(-1)^l}{l!} E[(X_r^{(k)})^i (X_s^{(k)})^{j-\theta(1-l)}] \right].$$

Remark 3.5. At $i = 0$, we obtain recurrence relation for single moments as given in (10).

Remark 3.6. At $\theta = 1$, we get recurrence relation for product moments of generalized order statistics from Marshall-Olkin extended exponential distribution.

4. Characterization

Let $X(r, n, m, k), r = 1, 2, \dots$ be gos, then the conditional pdf of $X(s, n, m, k)$ given $X(r, n, m, k) = x, 1 \leq r < s \leq n$, in view of (5) and (6) is

$$f_{s|r}(y|x) = \frac{C_{s-1}}{(s - r - 1)!C_{r-1}} [\bar{F}(x)]^{m-\gamma_r+1} \times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [\bar{F}(y)]^{\gamma_s-1} f(y).$$

Theorem 4.1. Let X be an absolutely continuous rv with df $F(x)$ and pdf $f(x)$ with $F(x) < 1$, for all $x \in (0, \infty)$. Then for two consecutive values r and $r + 1, 2 \leq r + 1 \leq s \leq n$,

$$E[X^\theta(s, n, m, k)|X(l, n, m, k) = x] = A(x) + B, \quad l = r, r + 1 \tag{15}$$

where $A(x) = \log\{e^{x^\theta} - (1 - \lambda)\}$ and $B = \log(1 - \lambda) + \sum_{j=r+1}^s \frac{1}{\gamma_j}$ if and only if

$$F(x) = 1 - \frac{\lambda e^{-x^\theta}}{[1 - (1 - \lambda)e^{-x^\theta}]}, \quad x > 0, \lambda > 0, \theta > 0 \tag{16}$$

Proof. We have for $s \geq r + 1$,

$$\begin{aligned}
 g_{s|r} &= E[X^\theta(s, n, m, k)|X(r, n, m, k) = x] \\
 &= \frac{C_{s-1}}{C_{r-1}(s-r-1)!} \frac{1}{(m+1)^{s-r-1}} \int_x^\infty y^\theta \left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_s-1} \left[1 - \frac{(\bar{F}(y))^{m+1}}{(\bar{F}(x))^{m+1}}\right]^{s-r-1} \frac{f(y)}{\bar{F}(x)} dy. \quad (17)
 \end{aligned}$$

Let $u = \frac{\bar{F}(y)}{\bar{F}(x)}$, which implies $y^\theta = [\log(e^{x^\theta} - (1-\lambda)) + \log(1-\lambda) - \log u]$. Thus, (17) becomes

$$g_{s|r} = \frac{C_{s-1}}{C_{r-1}(s-r-1)!} \frac{1}{(m+1)^{s-r-1}} \int_0^1 [\log(e^{x^\theta} - (1-\lambda)) + \log(1-\lambda) - \log u] u^{\gamma_s-1} [1-u^{(m+1)}]^{s-r-1} du.$$

Set $u^{m+1} = t$ to get

$$\begin{aligned}
 &E[X^\theta(s, n, m, k)|X(r, n, m, k) = x] \\
 &= \log\{e^{x^\theta} - (1-\lambda)\} + \log(1-\lambda) - \frac{1}{(m+1)(s-r-1)!(m+1)^{s-r}} \int_0^1 \log t \ t^{\frac{\gamma_s}{m+1}-1} (1-t)^{s-r-1} dt \\
 &= \log\{e^{x^\theta} - (1-\lambda)\} + \log(1-\lambda) - \frac{1}{(m+1)(s-r-1)!(m+1)^{s-r}} B\left(\frac{\gamma_s}{m+1}, s-r\right) \left[\psi\left(\frac{\gamma_s}{m+1}\right) - \psi\left(\frac{\gamma_r}{m+1}\right)\right]
 \end{aligned}$$

where $\psi(x) = \frac{d}{dx} \ln \Gamma(x)$ (Gradshteyn and Ryzhik (2007), pp. 540). Since $\psi(x-n) - \psi(x) = -\sum_{k=1}^n \frac{1}{x-k}$ (Gradshteyn and Ryzhik (2007), pp. 905), we have that

$$E[X^\theta(s, n, m, k)|X(r, n, m, k) = x] = \log\{e^{x^\theta} - (1-\lambda)\} + \log(1-\lambda) + \sum_{j=r+1}^s \frac{1}{\gamma_j}.$$

Hence (15).

To show (15) implies (16), we have $g_{s|r+1}(x) - g_{s|r}(x) = -\frac{1}{\gamma_{r+1}}$. Therefore in view of Theorem 2.1 of Khan et al. (2006), we have

$$\frac{1}{\gamma_{r+1}} \frac{g'_{s|r}(x)}{g_{s|r+1}(x) - g_{s|r}(x)} = \frac{\theta x^{\theta-1}}{[1 - (1-\lambda)e^{-x^\theta}]},$$

which implies that

$$\bar{F}(x) = \frac{\lambda e^{-x^\theta}}{[1 - (1-\lambda)e^{-x^\theta}]}, \quad x > 0, \lambda > 0, \theta > 0,$$

and hence the result. \square

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