

# Relations for moments of generalized order statistics from Marshall-Olkin extended Weibull distribution and its characterization

Haseeb Athar, Nayabuddin, Saba Khalid Khwaja

*Department of Statistics and Operations Research, Aligarh Muslim University, Aligarh-202 002, India*

**Abstract.** Marshall-Olkin extended Weibull distribution was introduced by Marshall and Olkin (1997) and extensively studied by Ghitany et al. (2005). In this article recurrence relations for single and product moments of generalized order statistics (*gos*) for Marshall-Olkin extended Weibull distribution have been driven. Moments of order statistics and  $k$ -upper records are discussed as special cases. Characterization of Marshall-Olkin extended Weibull distribution through conditional expectation is also presented.

## 1. Introduction

Let  $X_1, X_2, \dots, X_n$  be a sequence of independent and identically distributed (*iid*) random variables (*rv*) with absolutely continuous distribution function (*df*)  $F(x)$  and the probability density function (*pdf*)  $f(x)$ ,  $x \in (\alpha, \beta)$ . Let  $n \in \mathbb{N}$ , ( $n \geq 2$ ),  $k \geq 1$ ,  $\tilde{m} = (m_1, m_2, \dots, m_{n-1}) \in \mathbb{R}^{n-1}$ ,  $M_r = \sum_{j=r}^{n-1} m_j$ ,  $1 \leq r \leq n-1$  be the parameters such that  $\gamma_r = k + n - r + M_r \geq 1$ , for all  $r \in \{1, 2, \dots, n-1\}$ . Then  $X(r, n, \tilde{m}, k)$  are called generalized order statistics (*gos*) if their joint *pdf* is given by

$$f_{X(1, n, \tilde{m}, k), \dots, X(n, n, \tilde{m}, k)}(x_1, x_2, \dots, x_n) = k \left( \prod_{j=1}^{n-1} \gamma_j \right) \left( \prod_{i=1}^{n-1} [\bar{F}(x_i)]^{m_i} f(x_i) \right) [\bar{F}(x_n)]^{k-1} f(x_n)$$

on the cone  $F^{-1}(0) < x_1 \leq \dots \leq x_n < F^{-1}(1)$ , where  $\bar{F}(x) = 1 - F(x)$ .

Choosing the parameters appropriately, models such as ordinary order statistics ( $\gamma_i = n - i + 1$ ;  $i = 1, 2, \dots, n$ , i.e.  $m_1 = m_2 = \dots = m_{n-1} = 0$ ,  $k = 1$ ),  $k^{\text{th}}$  record values ( $\gamma_i = k$ , i.e.  $m_1 = m_2 = \dots = m_{n-1} = -1$ ,  $k \in \mathbb{N}$ ), sequential order statistics ( $\gamma_i = (n - i + 1)\alpha_i$ ;  $\alpha_1, \alpha_2, \dots, \alpha_n > 0$ ), order statistics with non-integral sample size ( $\gamma_i = \alpha - i + 1$ ,  $\alpha > 0$ ). Pfeifer's record values ( $\gamma_i = \beta_i$ ;  $\beta_1, \beta_2, \dots, \beta_n > 0$ ) and progressive type II censored order statistics ( $m_i \in \mathbb{N}_0$ ,  $k \in \mathbb{N}$ ) are obtained (Kamps (1995), Kamps and Cramer (2001)).

For  $\gamma_i \neq \gamma_j$ ,  $i \neq j$  for all  $i, j \in \{1, 2, \dots, n\}$ , the *pdf* of  $X(r, n, \tilde{m}, k)$  is given by Kamps and Cramer (2001) in the following way

---

2010 *Mathematics Subject Classification.* Primary: 62G30; Secondary: 62E10

*Keywords.* Marshall-Olkin extended Weibull distribution, generalized order statistics, order statistics, record values, recurrence relations and characterization

Received: 16 January 2012; Revised: 19 July 2012; Accepted: 01 August 2012

*Email address:* [haseebathar@hotmail.com](mailto:haseebathar@hotmail.com) (Haseeb Athar)

$$f_{X(r,n,\tilde{m},k)}(x) = C_{r-1} f(x) \sum_{i=1}^r a_i(r) [\bar{F}(x)]^{\gamma_i-1}. \tag{1}$$

The joint probability density function (*pdf*) of  $X(r, n, \tilde{m}, k)$  and  $X(s, n, \tilde{m}, k)$ ,  $1 \leq r < s \leq n$  is given as

$$f_{X(r,n,\tilde{m},k),X(s,n,\tilde{m},k)}(x,y) = C_{s-1} \left( \sum_{i=r+1}^s a_i^{(r)}(s) \left[ \frac{\bar{F}(y)}{\bar{F}(x)} \right]^{\gamma_i} \right) \left( \sum_{i=1}^r a_i(r) [\bar{F}(x)]^{\gamma_i} \right) \frac{f(x)}{\bar{F}(x)} \frac{f(y)}{\bar{F}(y)}, \tag{2}$$

where  $\alpha \leq x < y \leq \beta$  and

$$a_i(r) = \prod_{\substack{j=1 \\ j \neq i}}^r \frac{1}{(\gamma_j - \gamma_i)}, \quad 1 \leq i \leq r \leq n,$$

$$a_i^{(r)}(s) = \prod_{\substack{j=r+1 \\ j \neq i}}^s \frac{1}{\gamma_j - \gamma_i}, \quad r+1 \leq i \leq s \leq n.$$

It may be noted that for  $m_1 = m_2 = \dots = m \neq -1$ ,

$$a_i(r) = \frac{(-1)^{r-i}}{(m+1)^{r-1}(r-1)!} \binom{r-1}{r-i} \tag{3}$$

$$a_i^{(r)}(s) = \frac{(-1)^{s-i}}{(m+1)^{s-r-1}(s-r-1)!} \binom{s-r-1}{s-i} \tag{4}$$

Therefore (*pdf*) of  $X(r, n, \tilde{m}, k)$  given in (1) reduces to

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!} [\bar{F}(x)]^{\gamma_{r-1}} f(x) g_m^{r-1}(F(x)), \tag{5}$$

and joint (*pdf*) of  $X(r, n, \tilde{m}, k)$  and  $X(s, n, \tilde{m}, k)$  given in (2) reduces to

$$f_{X(r,n,m,k),X(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!} [\bar{F}(x)]^m f(x) g_m^{r-1}(F(x)) \times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [\bar{F}(y)]^{\gamma_s-1} f(y), \quad \alpha \leq x < y \leq \beta \tag{6}$$

where

$$C_{r-1} = \prod_{i=1}^r \gamma_i, \quad \gamma_i = k + (n-i)(m+1),$$

$$h_m(x) = \begin{cases} -\frac{1}{m+1} (1-x)^{m+1} & , \quad m \neq -1 \\ -\log(1-x) & , \quad m = -1 \end{cases}$$

and  $g_m(x) = h_m(x) - h_m(0)$ ,  $x \in (0, 1)$ .

A random variable  $X$  is said to have the Marshall-Olkin extended Weibull distribution if its *pdf* is of the form

$$f(x) = \frac{\lambda \theta x^{\theta-1} e^{-x^\theta}}{[1 - (1-\lambda)e^{-x^\theta}]^2}, \quad x > 0, \lambda > 0, \theta > 0 \tag{7}$$

and corresponding  $df$

$$F(x) = 1 - \frac{\lambda e^{-x^\theta}}{[1 - (1 - \lambda)e^{-x^\theta}]}, \quad x > 0, \lambda > 0, \theta > 0. \tag{8}$$

Now in view of (7) and (8), we get

$$\bar{F}(x) = \frac{x^{1-\theta}}{\theta} [1 - (1 - \lambda)e^{-x^\theta}] f(x) \tag{9}$$

where  $\bar{F}(x) = 1 - F(x)$ .

The relation (9) will be utilized to establish recurrence relations for moments of *gos*.

### 2. Single moments

**Theorem 2.1.** For the Marshall-Olkin extended Weibull distribution as given in (7) and  $n \in N, \tilde{m} \in \mathbb{R}, k > 0, 1 \leq r \leq n, j = 1, 2, \dots$

$$\begin{aligned} E[X^j(r, n, \tilde{m}, k)] - E[X^j(r - 1, n, \tilde{m}, k)] &= \frac{j}{\theta \gamma_r} E[X^{j-\theta}(r, n, \tilde{m}, k)] \\ &\quad - \frac{j(1-\lambda)}{\theta \gamma_r} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} E[X^{j-\theta(1-l)}(r, n, \tilde{m}, k)]. \end{aligned} \tag{10}$$

*Proof.* We have by Lemma 2.3 (Athar and Islam (2004))

$$E[\xi\{X(r, n, \tilde{m}, k)\}] - E[\xi\{X(r - 1, n, \tilde{m}, k)\}] = C_{r-2} \int_{\alpha}^{\beta} \xi'(x) \sum_{i=1}^r a_i(r) [\bar{F}(x)]^{\gamma_i} dx$$

where  $\xi(x)$  is a Borel measurable function of  $x \in (\alpha, \beta)$ .

Then for  $\xi(x) = x^j, j > 0$ ,

$$E[X^j(r, n, \tilde{m}, k)] - E[X^j(r - 1, n, \tilde{m}, k)] = C_{r-2} j \int_{\alpha}^{\beta} x^{j-1} \sum_{i=1}^r a_i(r) [\bar{F}(x)]^{\gamma_i} dx. \tag{11}$$

On using (9) in (11), we get

$$\begin{aligned} E[X^j(r, n, \tilde{m}, k)] - E[X^j(r - 1, n, \tilde{m}, k)] &= \frac{j}{\theta \gamma_r} C_{r-1} \int_0^{\infty} x^{j-\theta} \sum_{i=1}^r a_i(r) [\bar{F}(x)]^{\gamma_i-1} \{1 - (1 - \lambda)e^{-x^\theta}\} f(x) dx \\ &= \frac{j}{\theta} \frac{C_{r-1}}{\gamma_r} \int_0^{\infty} x^{j-\theta} \sum_{i=1}^r a_i(r) [\bar{F}(x)]^{\gamma_i-1} f(x) dx \\ &\quad - \frac{j(1-\lambda)}{\theta} \frac{C_{r-1}}{\gamma_r} \int_0^{\infty} x^{j-\theta} e^{-x^\theta} \sum_{i=1}^r a_i(r) [\bar{F}(x)]^{\gamma_i-1} f(x) dx. \end{aligned} \tag{12}$$

Now writing expansion of  $e^{-x^\theta}$  in the integrand of second part of (12), we have

$$\begin{aligned} E[X^j(r, n, \tilde{m}, k)] - E[X^j(r - 1, n, \tilde{m}, k)] &= \frac{j}{\theta} \frac{C_{r-1}}{\gamma_r} \int_0^{\infty} x^{j-\theta} \sum_{i=1}^r a_i(r) [\bar{F}(x)]^{\gamma_i-1} f(x) dx \\ &\quad - \frac{j(1-\lambda)}{\theta} \frac{C_{r-1}}{\gamma_r} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \int_0^{\infty} x^{j-\theta(1-l)} \sum_{i=1}^r a_i(r) [\bar{F}(x)]^{\gamma_i-1} f(x) dx, \end{aligned}$$

which after simplification yields the (10).  $\square$

**Corollary 2.2.** For  $m_1 = m_2 = \dots = m_{n-1} = m \neq -1$ , the recurrence relation for single moments of gos for Marshall-Olkin extended Weibull distribution is give as

$$E[X^j(r, n, m, k)] - E[X^j(r-1, n, m, k)] = \frac{j}{\theta\gamma_r} E[X^{j-\theta}(r, n, m, k)] - \frac{j(1-\lambda)}{\theta\gamma_r} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} E[X^{j-\theta(1-l)}(r, n, m, k)].$$

*Proof.* This can easily be deduced from (10) in view of the relations (3).  $\square$

**Remark 2.3.** Recurrence relation for single moments of order statistics (at  $m = 0, k = 1$ ) is

$$E(X_{r:n}^j) = E(X_{r-1:n}^j) + \frac{j}{\theta(n-r+1)} E(X_{r:n}^{j-\theta}) - \frac{j(1-\lambda)}{\theta(n-r+1)} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} E(X_{r:n}^{j-\theta(1-l)}).$$

**Remark 2.4.** Recurrence relation for single moments of  $k$  – th upper record (at  $m = -1$ ) will be

$$E(X_r^{(k)})^j = E(X_{r-1}^{(k)})^j + \frac{j}{\theta k} E(X_r^{(k)})^{j-\theta} - \frac{j(1-\lambda)}{\theta k} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} E(X_r^{(k)})^{j-\theta(1-l)}.$$

**Remark 2.5.** For  $\theta = 1$ , we get recurrence relation for single moments of generalized order statistics from Marshall-Olkin extended exponential distribution and at  $\theta = 1, \lambda = 1$ , we get the relation for standard exponential distribution.

### 3. Product moments

**Theorem 3.1.** For distribution as given in (7). Fix a positive integer  $k$  and for  $n \in \mathbb{N}, \tilde{m} \in \mathbb{R}, 1 \leq r < s \leq n$ ,

$$\begin{aligned} & E[X^i(r, n, \tilde{m}, k)X^j(s, n, \tilde{m}, k)] - E[X^i(r, n, \tilde{m}, k)X^j(s-1, n, \tilde{m}, k)] \\ &= \frac{j}{\theta\gamma_s} E[X^i(r, n, \tilde{m}, k)X^{j-\theta}(s, n, \tilde{m}, k)] \\ & - \frac{j(1-\lambda)}{\theta\gamma_s} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} E[X^i(r, n, \tilde{m}, k)X^{j-\theta(1-l)}(s, n, \tilde{m}, k)]. \end{aligned} \tag{13}$$

*Proof.* We have from Lemma 3.2 (Athar and Islam (2004)) that

$$\begin{aligned} & E[\xi\{X(r, n, \tilde{m}, k), X(s, n, \tilde{m}, k)\}] - E[\xi\{X(r, n, \tilde{m}, k), X(s-1, n, \tilde{m}, k)\}] \\ &= C_{s-2} \int_{\alpha}^{\beta} \int_x^{\beta} \frac{\partial}{\partial y} \xi(x, y) \sum_{i=r+1}^s a_i^{(r)}(s) \left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_i} \sum_{i=1}^r a_i(r) (\bar{F}(x))^{\gamma_i} \frac{f(x)}{\bar{F}(x)} dy dx. \end{aligned}$$

If we let  $\xi(x, y) = x^i y^j$ , then

$$\begin{aligned} & E[X^i(r, n, \tilde{m}, k)X^j(s, n, \tilde{m}, k)] - E[X^i(r, n, \tilde{m}, k)X^j(s-1, n, \tilde{m}, k)] \\ &= \frac{jC_{s-1}}{\gamma_s} \int_0^{\infty} \int_x^{\infty} x^i y^{j-1} \sum_{i=r+1}^s a_i^{(r)}(s) \left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_i} \sum_{i=1}^r a_i(r) (\bar{F}(x))^{\gamma_i} \frac{f(x)}{\bar{F}(x)} dy dx. \end{aligned}$$

In view of (9), note that

$$\frac{\bar{F}(y)}{f(y)} = \frac{y^{1-\theta}}{\theta} [1 - (1-\lambda)e^{-y^\theta}].$$

Therefore,

$$\begin{aligned} & E[X^i(r, n, \tilde{m}, k)X^j(s, n, \tilde{m}, k)] - E[X^i(r, n, \tilde{m}, k)X^j(s - 1, n, \tilde{m}, k)] \\ &= \frac{jC_{s-1}}{\theta\gamma_s} \int_0^\infty \int_x^\infty x^i y^{j-\theta} \{1 - (1 - \lambda)e^{-y^\theta}\} \sum_{i=r+1}^s a_i^{(r)}(s) \left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_i} \sum_{i=1}^r a_i(r) (\bar{F}(x))^{\gamma_i} \frac{f(x)}{\bar{F}(x)} \frac{f(y)}{\bar{F}(y)} dy dx \\ &= \frac{jC_{s-1}}{\theta\gamma_s} \int_0^\infty \int_x^\infty x^i y^{j-\theta} \sum_{i=r+1}^s a_i^{(r)}(s) \left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_i} \sum_{i=1}^r a_i(r) (\bar{F}(x))^{\gamma_i} \frac{f(x)}{\bar{F}(x)} \frac{f(y)}{\bar{F}(y)} dy dx \\ &\quad - \frac{j(1 - \lambda)C_{s-1}}{\theta\gamma_s} \int_0^\infty \int_x^\infty x^i y^{j-\theta} e^{-y^\theta} \sum_{i=r+1}^s a_i^{(r)}(s) \left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_i} \sum_{i=1}^r a_i(r) (\bar{F}(x))^{\gamma_i} \frac{f(x)}{\bar{F}(x)} \frac{f(y)}{\bar{F}(y)} dy dx \end{aligned}$$

which after simplification leads to (13).  $\square$

**Corollary 3.2.** For  $m_1 = m_2 = \dots = m_{n-1} = m \neq -1$ , the relation for product moment is given as

$$\begin{aligned} E[X^i(r, n, m, k)X^j(s, n, m, k)] &= E[X^i(r, n, m, k)X^j(s - 1, n, m, k)] + \frac{j}{\theta\gamma_s} E[X^i(r, n, m, k)X^{j-\theta}(s, n, m, k)] \\ &\quad - \frac{j(1 - \lambda)}{\theta\gamma_s} \sum_{l=0}^\infty \frac{(-1)^l}{l!} E[X^i(r, n, m, k)X^{j-\theta(1-l)}(s, n, m, k)]. \end{aligned} \tag{14}$$

*Proof.* Equation (14) can be reduced from (13) in view of (3) and (4) or by replacing  $\tilde{m}$  with  $m$  in (13).  $\square$

**Remark 3.3.** Recurrence relation for product moments of order statistics (at  $m = 0, k = 1$ ) is

$$E(X_{r,s;n}^{i,j}) = E(X_{r,s-1;n}^{i,j}) + \frac{j}{\theta(n - s + 1)} \left[ E(X_{r,s;n}^{i,j-\theta}) - (1 - \lambda) \sum_{l=0}^\infty \frac{(-1)^l}{l!} E(X_{r,s;n}^{i,j-\theta(1-l)}) \right].$$

**Remark 3.4.** Recurrence relation for product moments of  $k - th$  record values will be

$$E[(X_r^{(k)})^i (X_s^{(k)})^j] = E[(X_r^{(k)})^i (X_{s-1}^{(k)})^j] + \frac{j}{\theta k} \left[ E[(X_r^{(k)})^i (X_s^{(k)})^{j-\theta}] - (1 - \lambda) \sum_{l=0}^\infty \frac{(-1)^l}{l!} E[(X_r^{(k)})^i (X_s^{(k)})^{j-\theta(1-l)}] \right].$$

**Remark 3.5.** At  $i = 0$ , we obtain recurrence relation for single moments as given in (10).

**Remark 3.6.** At  $\theta = 1$ , we get recurrence relation for product moments of generalized order statistics from Marshall-Olkin extended exponential distribution.

#### 4. Characterization

Let  $X(r, n, m, k), r = 1, 2, \dots$  be gos, then the conditional pdf of  $X(s, n, m, k)$  given  $X(r, n, m, k) = x, 1 \leq r < s \leq n$ , in view of (5) and (6) is

$$f_{s|r}(y|x) = \frac{C_{s-1}}{(s - r - 1)!C_{r-1}} [\bar{F}(x)]^{m-\gamma_r+1} \times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [\bar{F}(y)]^{\gamma_s-1} f(y).$$

**Theorem 4.1.** Let  $X$  be an absolutely continuous rv with df  $F(x)$  and pdf  $f(x)$  with  $F(x) < 1$ , for all  $x \in (0, \infty)$ . Then for two consecutive values  $r$  and  $r + 1, 2 \leq r + 1 \leq s \leq n$ ,

$$E[X^\theta(s, n, m, k)|X(l, n, m, k) = x] = A(x) + B, \quad l = r, r + 1 \tag{15}$$

where  $A(x) = \log\{e^{x^\theta} - (1 - \lambda)\}$  and  $B = \log(1 - \lambda) + \sum_{j=r+1}^s \frac{1}{\gamma_j}$  if and only if

$$F(x) = 1 - \frac{\lambda e^{-x^\theta}}{[1 - (1 - \lambda)e^{-x^\theta}]}, \quad x > 0, \lambda > 0, \theta > 0 \tag{16}$$

*Proof.* We have for  $s \geq r + 1$ ,

$$\begin{aligned}
 g_{s|r} &= E[X^\theta(s, n, m, k)|X(r, n, m, k) = x] \\
 &= \frac{C_{s-1}}{C_{r-1}(s-r-1)!} \frac{1}{(m+1)^{s-r-1}} \int_x^\infty y^\theta \left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_s-1} \left[1 - \frac{(\bar{F}(y))^{m+1}}{(\bar{F}(x))^{m+1}}\right]^{s-r-1} \frac{f(y)}{\bar{F}(x)} dy. \quad (17)
 \end{aligned}$$

Let  $u = \frac{\bar{F}(y)}{\bar{F}(x)}$ , which implies  $y^\theta = [\log(e^{x^\theta} - (1-\lambda)) + \log(1-\lambda) - \log u]$ . Thus, (17) becomes

$$g_{s|r} = \frac{C_{s-1}}{C_{r-1}(s-r-1)!} \frac{1}{(m+1)^{s-r-1}} \int_0^1 [\log(e^{x^\theta} - (1-\lambda)) + \log(1-\lambda) - \log u] u^{\gamma_s-1} [1-u^{(m+1)}]^{s-r-1} du.$$

Set  $u^{m+1} = t$  to get

$$\begin{aligned}
 &E[X^\theta(s, n, m, k)|X(r, n, m, k) = x] \\
 &= \log\{e^{x^\theta} - (1-\lambda)\} + \log(1-\lambda) - \frac{1}{(m+1)(s-r-1)!(m+1)^{s-r}} \int_0^1 \log t \ t^{\frac{\gamma_s}{m+1}-1} (1-t)^{s-r-1} dt \\
 &= \log\{e^{x^\theta} - (1-\lambda)\} + \log(1-\lambda) - \frac{1}{(m+1)(s-r-1)!(m+1)^{s-r}} B\left(\frac{\gamma_s}{m+1}, s-r\right) \left[\psi\left(\frac{\gamma_s}{m+1}\right) - \psi\left(\frac{\gamma_r}{m+1}\right)\right]
 \end{aligned}$$

where  $\psi(x) = \frac{d}{dx} \ln \Gamma(x)$  (Gradshteyn and Ryzhik (2007), pp. 540). Since  $\psi(x-n) - \psi(x) = -\sum_{k=1}^n \frac{1}{x-k}$  (Gradshteyn and Ryzhik (2007), pp. 905), we have that

$$E[X^\theta(s, n, m, k)|X(r, n, m, k) = x] = \log\{e^{x^\theta} - (1-\lambda)\} + \log(1-\lambda) + \sum_{j=r+1}^s \frac{1}{\gamma_j}.$$

Hence (15).

To show (15) implies (16), we have  $g_{s|r+1}(x) - g_{s|r}(x) = -\frac{1}{\gamma_{r+1}}$ . Therefore in view of Theorem 2.1 of Khan et al. (2006), we have

$$\frac{1}{\gamma_{r+1}} \frac{g'_{s|r}(x)}{g_{s|r+1}(x) - g_{s|r}(x)} = \frac{\theta x^{\theta-1}}{[1 - (1-\lambda)e^{-x^\theta}]},$$

which implies that

$$\bar{F}(x) = \frac{\lambda e^{-x^\theta}}{[1 - (1-\lambda)e^{-x^\theta}]}, \quad x > 0, \lambda > 0, \theta > 0,$$

and hence the result.  $\square$

### Acknowledgement

The authors acknowledge with thanks to referee for his/her comments which lead to improvement in the manuscript.

### References

Athar, H., Islam, H.M. (2004) Recurrence relations between single and product moments of generalized order statistics from a general class of distributions, *Metron* LXII, 327-337.  
 Ghitany, M.E., Al-Hussaini, E.K., Al-Jarallah, R.A. (2005) Marshal-Olkin extended Weibull Distribution, *Journal of Applied Statistics* 32(10), 1025-1034.  
 Gradshteyn, I.S., Ryzhik, I.M. (2007) *Tables of Integrals, Series and Products*, Edited by Jeffrey, A. and Zwillinger, D., 7th Ed., Academic Press, New York.  
 Kamps, U. (1995) *A concept of generalized order statistics*, B.G. Teubner Stuttgart, Germany.  
 Kamps, U., Cramer, E. (2001) On distributions of generalized order statistics, *Statistics* 35, 269-280.  
 Khan, A. H., Khan, R. U., Yaqub, M. (2006) Characterization of continuous distributions through conditional expectation of generalized order statistics, *J. Appl. Prob. Statist.* 1, 115-131.  
 Marshall, A.W., Olkin, I. (1997) A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families, *Biometrika* 84(3), 641-652.