ProbStat Forum, Volume 05, October 2012, Pages 133–141
ISSN 0974-3235

ProbStat Forum is an e-journal.
For details please visit www.probstat.org.in

# Nonparametric test for homogeneity of scale parameters against ordered alternatives based on subsample medians

**Anil Gaur, Kalpana K. Mahajan[1], Sangeeta Arora**

*Department of Statistics, Panjab University, Chandigarh - 160014, India*

**Abstract.** A nonparametric test for several sample scale problem is proposed by considering the subsample medians of three observations. The proposed statistic has the advantage of not requiring the several distribution functions to have a common median, but rather any common quantile of order $\alpha$, $0 \leq \alpha \leq 1$ (not necessarily $1/2$) which is assumed to be known. Asymptotic distribution of the test statistic is obtained under the null hypothesis as well as under the sequence of local alternatives. Asymptotic relative efficiencies of this test relative to the existing tests are obtained and it is seen that the proposed test performs better for heavy-tailed distributions.

## 1. Introduction

Let $X_{i1}, X_{i2}, \ldots, X_{in_i}$; $i = 1, 2, \ldots, k$, be independent random samples of size $n_i$ from absolutely continuous cumulative distribution functions $F_i(x) = F(x\theta_i)$, $i = 1, 2, \ldots, k$, for some $F$. We assume that these distribution functions have zero as the common quantile of order $\alpha\,(0 \leq \alpha \leq 1)$, i.e., $F_i(0) = \alpha$ for $i = 1, 2, \ldots, k$. It is also assumed that, $F_i(x)$, $i = 1, 2, \ldots, k$, are identical in all respects except possibly their scale parameters. The hypothesis, which is of interest in this paper, could be formally stated as follows:

$$H_0 : \theta_1 = \theta_2 = \ldots = \theta_k$$

against the ordered alternative

$$H_1 : \theta_1 \leq \theta_2 \leq \ldots \leq \theta_k$$

with at least one strict inequality.

For some earlier work on this problem, see Govindarajulu and Haller [5], Govindarajulu and Gupta [6], Rao [18], Kochar and Gupta [10], Shanubhogue [20] and Kusum and Bagai [13]. In general, most of the tests, except Kusum and Bagai [13] test, require the assumption that the common quantile is of order $\alpha = 1/2$, i.e., the distributions have the same median. When the random variables take non-negative values, i.e., $\alpha = 0$, $H_1$ implies the alternative of ordered stochastic ordering for which tests have been proposed by Jonckheere [8], Chacko [2], Puri [16], Tryon and Hettmansperger [23], Amita and Kochar [1], Kumar, Gill and Mehta [11] and Shetty *et al.* [22] among others. However, none of these tests is adequate when the common quantile is different from median. The asymmetry of the situation is not reflected in the statistics used in the above

*Keywords.* $U$-statistics, ordered alternatives, Pitman asymptotic relative efficiency
Received: 12 March 2012; Revised: 02 September 2012; Accepted: 05 October 2012
*Email address:* `mahajan.kkm@gmail.com` (Kalpana K. Mahajan)
[1] Corresponding Author, Tel. No. : +919888558800 (Mobile), +91-172-2534538 (office)

tests. Deshpande and Kusum [4] and Mahajan *et al.* [15] proposed tests based on $U$-statistics for testing homogeneity of two populations against stochastic ordering, when the assumption of a common median is replaced by that of common quantile of order $\alpha$, $(0 \leq \alpha \leq 1)$. Deshpande and Kusum [4] have discussed with examples the necessity of such tests. Some other relevant references in this context are Deshpande and Kochar [3], Kochar [9], Kusum [12] and Shetty and Govindarajlu [21].

**Remark 1.1.** *It can be considered that the distribution functions have $\mu$ as the common quantile of order $\alpha$ $(0 \leq \alpha \leq 1)$, i.e., $F_i(\mu) = \alpha$ for $i = 1, 2, \ldots, k$. Without loss of generality, we can assume that the common known quantile $\mu = F_1^{-1}(\alpha) = \ldots = F_k^{-1}(\alpha)$ of order $\alpha$ is zero for the pre-specified $\alpha$.*

Consider an example where more than two automatic filling machines are available to fill 500ml milk in packets. It is known that each machine is having 2% under filling (less than 500ml) and a machine is considered as more efficient (to attain the target value of 500ml) if the dispersion from this target value is less. As such smaller the scale parameter more is the efficiency of that machine.

Here we propose a nonparametric distribution free test based on weighted linear combinations of Mahajan *et al.* [15] type $U$-statistic by considering all pairs of consecutive two sample $U$-statistics suggested by the alternative $H_1$. The present paper is organized as follows. In section 2, the test procedure for multi sample test for scale parameters against ordered alternative has been proposed. The distribution of test statistic is discussed in Section 3. Section 4 is devoted to optimal choice of weights along with a simulated illustration. In section 5, we consider the performance of the proposed test statistic against its other nonparametric competitors in terms of Pitman asymptotic relative efficiency.

## 2. The proposed test

First we consider two sample U-statistic, proposed by Mahajan et al. [15] where the assumption of the common quantile of order $\alpha$ $(0 \leq \alpha \leq 1)$ is made and then extend it to the $k$ sample problem. Define for $i < j$; $i, j = 1, 2, \ldots, k$

$$\phi_{ij}\left(X_{i1}, X_{i2}, X_{i3}; X_{j1}, X_{j2}, X_{j3}\right) = \begin{cases} 1, & \text{if } 0 \leq \operatorname{med}\left(X_{i1}, X_{i2}, X_{i3}\right) \leq \operatorname{med}\left(X_{j1}, X_{j2}, X_{j3}\right) \\ & \text{and } X_{i1}, X_{i2}, X_{i3}, X_{j1}, X_{j2}, X_{j3} \geq 0 \\ & \text{or } \operatorname{med}\left(X_{j1}, X_{j2}, X_{j3}\right) \leq \operatorname{med}\left(X_{i1}, X_{i2}, X_{i3}\right) < 0 \\ & \text{and } X_{i1}, X_{i2}, X_{i3}, X_{j1}, X_{j2}, X_{j3} < 0, \\ -1, & \text{if } 0 \leq \operatorname{med}\left(X_{j1}, X_{j2}, X_{j3}\right) \leq \operatorname{med}\left(X_{i1}, X_{i2}, X_{i3}\right) \\ & \text{and } X_{i1}, X_{i2}, X_{i3}, X_{j1}, X_{j2}, X_{j3} \geq 0 \\ & \text{or } \operatorname{med}\left(X_{i1}, X_{i2}, X_{i3}\right) \leq \operatorname{med}\left(X_{j1}, X_{j2}, X_{j3}\right) < 0 \\ & \text{and } X_{i1}, X_{i2}, X_{i3}, X_{j1}, X_{j2}, X_{j3} < 0, \\ 0, & \text{otherwise.} \end{cases}$$

The two sample $U$-statistic corresponding to the kernel $\phi_{ij}$ is

$$U_{ij}\left(X_{i1}, X_{i2}, X_{i3}; X_{j1}, X_{j2}, X_{j3}\right) = \frac{1}{\binom{n_i}{3}\binom{n_j}{3}} \sum_c \phi_{ij}\left(X_{i1}, X_{i2}, X_{i3}; X_{j1}, X_{j2}, X_{j3}\right)$$

where $c$ denotes the summation extended over all possible $\binom{n_i}{3}\binom{n_j}{3}$ combinations of $X_{i1}, X_{i2}, \ldots, X_{in_i}$ and $X_{j1}, X_{j2}, \ldots, X_{jn_j}$.

The statistic $U_{ij}$ is obviously a $U$-statistic (Lehman [14]) corresponding to the kernel $\phi_{ij}$. It can be seen that the kernel takes non-zero value only when both $X_i$'s and $X_j$'s have the same sign.

For testing $H_0$ against $H_1$, with $F_i(0) = \alpha$ for $i = 1, 2, \ldots, k$, we propose the test statistic $U_S$ based on subsample medians of size three as

$$U_S = \sum_{i=1}^{k-1} a_i U_{i,i+1}$$

where $(a_1, a_2, \ldots, a_{k-1})$ are some positive real constants to be chosen as given in (4) which maximizes (5), the efficacy of the statistics.

For each set of values $(a_1, a_2, \ldots, a_{k-1})$, we get a distinct member of this class of test statistic. Large values of $U_S$ are significant for testing $H_0$ against $H_1$.

Practical implementation of this procedure may require an estimator for the common quantile $\mu$, under the null hypothesis, of order $\alpha = F_1(\mu) = \ldots = F_k(\mu)$. We suggest to use a pooled estimator of $\mu = F_1^{-1}(\alpha) = \ldots = F_k^{-1}(\alpha)$ for a given (predetermined values of) $\alpha$. To achieve this, we pool all the observations $X_{i1}, X_{i2}, \ldots, X_{in_i}$; $i = 1, 2, \ldots, k$ into a single vector $Z$ and estimate $\mu$ by obtaining the $\alpha^{\text{th}}$ quantile of $Z$.

## 3. Distribution of $U_S$

We have that $E(U_S) = \sum_{i=1}^{k-1} a_i \mu_{i,i+1}$, where $\mu_{i,i+1} = \pi_{i1} - \pi_{i2}$ and

$$\begin{aligned}
\pi_{i1} =& 6\sum_{j=2}^{3}\binom{3}{j}\Bigg[\int_0^\infty [F_i(x)-\alpha]^j\,[1-F_i(x)]^{3-j}\,[F_{i+1}(x)-\alpha]\,[1-F_{i+1}(x)]\,dF_{i+1}(x) \\
&+ \int_{-\infty}^{0} [\alpha - F_i(x)]^j\,[F_i(x)]^{3-j}\,F_{i+1}(x)\,[\alpha - F_{i+1}(x)]\,dF_{i+1}(x)\Bigg] \\
\pi_{i2} =& 6\sum_{j=2}^{3}\binom{3}{j}\Bigg[\int_0^\infty [F_{i+1}(x)-\alpha]^j\,[1-F_{i+1}(x)]^{3-j}\,[F_i(x)-\alpha]\,[1-F_i(x)]\,dF_i(x) \\
&+ \int_{-\infty}^{0} [\alpha - F_{i+1}(x)]^j\,[F_{i+1}(x)]^{3-j}\,F_i(x)\,[\alpha - F_i(x)]\,dF_i(x)\Bigg]
\end{aligned}$$

Under the hypothesis $H_0$ we have that $E(U_S) = 0$. Let

$$\underline{U'} = (U_{1,2}, U_{2,3}, \ldots, U_{k-1,k}).$$

Since $U_{ij}$'s are two sample $U$-statistics, the joint limiting normality of $\{U_{ij}\}$ follows immediately from the following theorem due to Lehman [14].

**Theorem 3.1.** *The asymptotic distribution of* $\sqrt{N}\,[\underline{U} - E(\underline{U})]$ *as* $N \to \infty$ *in such a way that* $\frac{n_i}{N} \to p_i$, $0 < p_i < 1$, *for* $i = 1, 2, \ldots, k$ *is multivariate normal with mean vector* $\underline{0}$ *and dispersion matrix* $\sum = (\sigma_{ij})$, *where* $N = \sum_{i=1}^{k} n_i$ *and*

$$\sigma_{ij} = \begin{cases} \frac{9}{p_i}\xi^{(i)}_{i,i+1;i,i+1} + \frac{9}{p_{i+1}}\xi^{(i+1)}_{i,i+1;i,i+1} & \text{for } i = j = 1, 2, \ldots, k-1, \\ \frac{9}{p_{i+1}}\xi^{(i+1)}_{i,i+1;i+1,i+2} & \text{for } j = i+1; i = 1, 2, \ldots, k-2, \\ \frac{9}{p_i}\xi^{(i)}_{i-1,i;i,i+1} & \text{for } j = i-1; i = 2, 3, \ldots, k-1, \\ 0 & \text{otherwise,} \end{cases}$$

*where*

$$\begin{aligned}
\xi^{(i)}_{i,i+1:i,i+1} &= E\left[\left\{\psi^{(i)}_{i,i+1}(X)\right\}^2\right] - E^2\left[U_{i,i+1}\right] \\
\xi^{(i+1)}_{i,i+1:i,i+1} &= E\left[\left\{\psi^{(i+1)}_{i,i+1}(X)\right\}^2\right] - E^2\left[U_{i,i+1}\right] \\
\xi^{(i+1)}_{i,i+1:i+1,i+2} &= E\left[\left\{\psi^{(i+1)}_{i,i+1}(X)\,\psi^{(i+1)}_{i+1,i+2}(X)\right\}\right] - E\left[U_{i,i+1}\right]E\left[U_{i+1,i+2}\right] \\
\psi^{(i)}_{i,j}(x) &= E\left[\phi_{ij}\left(x, X_{i2}, X_{i3}; X_{j1}, X_{j2}, X_{j3}\right)\right] \\
\psi^{(j)}_{i,j}(x) &= E\left[\phi_{ij}\left(X_{i1}, X_{i2}, X_{i3}; x, X_{j2}, X_{j3}\right)\right]
\end{aligned}$$

*After involved computations, it can be seen that under $H_0$,*

$$\sigma_{ij} = \begin{cases} \frac{1572}{1925}\left[\alpha^{11}+(1-\alpha)^{11}\right]\left(\frac{1}{p_i}+\frac{1}{p_{i+1}}\right) & \text{for } i=j=1,2,\ldots,k-1, \\ \frac{-1572}{1925*p_{i+1}}\left[\alpha^{11}+(1-\alpha)^{11}\right] & \text{for } j=i+1; i=1,2,\ldots,k-2, \\ \frac{-1572}{1925*p_i}\left[\alpha^{11}+(1-\alpha)^{11}\right] & \text{for } j=i-1; i=2,3,\ldots,k-1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

*In case all the sample sizes are equal i.e., $p_1 = p_2 = \ldots = p_k = \frac{1}{k}$, then the covariance matrix given in* (1) *becomes*

$$\sigma_{ij} = \begin{cases} \frac{2k*1572}{1925}\left[\alpha^{11}+(1-\alpha)^{11}\right] & \text{for } i=j=1,2,\ldots,k-1, \\ \frac{-k*1572}{1925}\left[\alpha^{11}+(1-\alpha)^{11}\right] & \text{for } j=i+1; i=1,2,\ldots,k-2, \\ \frac{-k*1572}{1925}\left[\alpha^{11}+(1-\alpha)^{11}\right] & \text{for } j=i-1; i=2,3,\ldots,k-1, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

*Since $U_S$ is the linear combination of the components of $\underline{U}$, the proof of the following theorem follows from the transformation theorem* (*see Serfling* [19]).

**Theorem 3.2.** *The asymptotic distribution of $N^{1/2}(U_S - E(U_S))$ as $N \to \infty$ in such a way that $\frac{n_i}{N} \to p_i$, $0 < p_i < 1$, $i = 1, 2, \ldots, k$, is normal with mean zero and variance $\underline{a}' \sum \underline{a}$, where $\underline{a}' = (a_1, a_2, \ldots, a_k)$. Under $H_0$, $E(U_S) = 0$ and*

$$\underline{a}' \sum \underline{a} = \frac{2k*1572}{1925}\left[\alpha^{11}+(1-\alpha)^{11}\right]\left[\sum_{i=1}^{k-1} a_1^2 - \sum_{i=1}^{k-2} a_i a_{i+1}\right],$$

*where $p_i = 1/k$; $i = 1, 2, \ldots, k$.*

## 4. Optimal choice of weights

Now we consider the problem of obtaining the optimal weights $a_i$'s so that the test $U_S$ has maximum efficacy for the sequence of Pitman type alternatives:

$$H_N : F_i(x) = F\left(\frac{x}{\theta + N^{-1/2}\delta_i}\right), i = 1, 2, \ldots, k,$$

where $\delta_i$ and $\theta$ are some real positive constants. We assume without loss of generality that $\theta = 1$, since all relative orderings and hence $U_S$ remains invariant if all the variables are multiplied by the same positive constant. Further, for efficiency comparisons, we consider the equal sample sizes case, i.e., $p_i = 1/k$; $i = 1, 2, \ldots, k$ and the equally spaced alternatives of the type $\delta_i = i\delta$, $\delta > 0$ for $i = 1, 2, \ldots, k$. Thus alternative $H_N$ becomes

$$H_N : F_i(x) = F\left(\frac{x}{1 + N^{-1/2}i\delta}\right), i = 1, 2, \ldots, k.$$

Such type of alternatives are already considered by Rao [18], Kochar Gupta [10], Kusum Bagai [13] and Shanubhogue [20] for the scale problem. The following theorem gives the asymptotic distribution of under the sequence of local alternatives $\{H_N\}$.

**Theorem 4.1.** *Let $X_{ij}$ be independent random variables with cumulative distribution function $F_i(x)$, $j = 1, 2, \ldots, n_i$; $i = 1, 2, \ldots, k$, where $F_i(x) = F\left(\frac{x}{1+N^{-1/2}i\delta}\right)$. Under the following assumptions, the limiting distribution of $N^{1/2}[\underline{U} - \underline{0}]$ is $(k-1)$ dimensional multivariate normal with mean vector $\mu J_{k-1}$ and variance-covariance matrix $\sum = (\sigma_{ij})$ given by* (2):

*(i) $F$ is absolutely continuous with density $f(x)$.*
*(ii) $\left|\frac{f(x)-f(x+h)}{h}\right| \leq g(x)$, for small $h$ and $\int_{-\infty}^{\infty} x\,[g(x)]^i f(x)dx < \infty$, $i = 1, 2, \ldots, k-1$.*
*Here $J_{k-1} = [1]_{1\times(k-1)}$ and $\mu = 72\,\delta\, I$, where*

$$I = \left\{ \int_{-\infty}^{0} xF^2(x)[\alpha - F(x)]^2 f^2(x)dx - \int_{0}^{\infty} x[F(x) - \alpha]^2[1 - F(x)]^2 f^2(x)dx \right\}. \tag{3}$$

*Proof.* The proof follows using the asymptotic theory of $U$-statistic (see Lehman [14], Kochar and Gupta [10]). Proceeding as in Rao [18], the following theorem identifies optimum weights $a_i$'s in the proposed test in terms of Pitman efficacy. □

**Theorem 4.2.** *Under the assumptions of Theorem 4.1 and under the sequence of alternative $\{H_N\}$, the efficacy of the test $U_S$ is maximized if*

$$a_i = \frac{i(k-i)}{2k},\ i = 1, 2, \ldots, k-1. \tag{4}$$

*Proof.* Since $U_S$ is a linear combination of $U$-statistics, it follows from Theorem 4.1, $N^{1/2}[U_S - 0]$ is asymptotically normally distributed with mean $\mu\left(\sum_{i=1}^{k-1} a_i\right)$ and variance $\underline{a}'\sum\underline{a}$.

Let $\theta = N^{1/2}\delta$. Then efficacy of $U_S$ is given by

$$e(U_S) = \frac{\left[\frac{\partial}{\partial\theta}E_{H_N}(U_S)\Big|_{\theta=0}\right]^2}{\underline{a}'\sum\underline{a}} = \frac{\left(\sum_{i=1}^{k} a_i\right)^2 831600\ I^2}{131\,(a'\sum a)\,[\alpha^{11} + (1-\alpha)^{11}]} \tag{5}$$

where $I$ is given by (3), also $\sum^* = (\sigma^*_{ij})$ and

$$\sigma^*_{ij} = \begin{cases} 2k & \text{if } j = i; i = 1, 2, \ldots, k-1, \\ -k & \text{if } j = i+1; i = 1, 2, \ldots, k-2, \\ -k & \text{if } j = i-1; i = 2, 3, \ldots, k-1, \\ 0 & \text{otherwise.} \end{cases}$$

Now $e(U_S)$ is maximized when $\frac{\left(\sum_{i=1}^{k-1} a_i\right)^2}{\underline{a}'\sum\underline{a}}$ is maximized with respect to $\underline{a}$. This is maximized when

$$\underline{a} = \sum^{*-1} J'_{k-1} \text{ (see p. 60 of Rao [17]).}$$

Also it is known that $\sum^{*-1} = (\sigma^{*ij})$, where

$$\sigma^{*ij} = \begin{cases} i(k-j)/k^2 & \text{if } i \leq j, \\ j(k-i)/k^2 & \text{if } i \geq j. \end{cases} \quad \text{(see Graybill [7]).}$$

Therefore, it follows that the optimum choice of $a_i$'s is

$$a_i = \frac{i(k-i)}{2k}, i = 1, 2, \ldots, k-1 \quad \text{and} \quad J_{k-1}\sum^{*-1} J'_{k-1} = \underline{a}'\sum^{*}\underline{a} = \frac{k^2-1}{12}$$

and the efficacy of the optimum test $U_S$ with these weighting coefficients is

$$e(U_S) = \frac{(k^2-1) * 69300 I^2}{131\,[\alpha^{11} + (1-\alpha)^{11}]},$$

where

$$I = \left\{ \int_{-\infty}^{0} xF^2(x)[\alpha - F(x)]^2 f^2(x)dx - \int_{0}^{\infty} x[F(x) - \alpha]^2[1 - F(x)]^2 f^2(x)dx \right\}.$$

□

*4.1. Simulated example*

Let there be four samples (i.e., $k = 4$) from Laplace distribution with following parameters: $I \sim$ Laplace $(0, 10)$, $II \sim$ Laplace $(0, 20)$, $III \sim$ Laplace $(0, 30)$ and $IV \sim$ Laplace $(0, 40)$ based on sample of size 10 each, $\alpha = 0.5$ and we want to test the null hypothesis:

$$H_0 : \theta_1 = \theta_2 = \theta_3 = \theta_4$$

against the ordered alternative

$$H_1 : \theta_1 \leq \theta_2 \leq \theta_3 \leq \theta_4$$

with at least one strict inequality.
The test statistics, $U_S$ is computed with the help of following data generated from Laplace distribution:

$$\begin{aligned}
L(0, 10) :\ & -4.4561488, -10.2308858, 18.6193195, 5.1828056, 3.9386745, \\
& -8.9269792, 19.3548134, 16.9797951, -1.5658773, 0.3191516. \\
L(0, 20) :\ & 9.749923, -9.635786, 41.050384, 9.435219, -63.327310, -4.063156, \\
& -2.150284, 3.645873, -5.557018, 31.241888. \\
L(0, 30) :\ & 15.4840725, 24.4739366, 28.3412521, -0.1752271, 13.2166460, \\
& 12.6338077, 2.9113102, 54.7771157, 2.3159054, 82.3187221. \\
L(0, 40) :\ & 37.641418, -2.913782, -18.085598, -88.314433, 1.147270, 32.142406, \\
& -16.624027, -189.899069, 28.834246, 53.03192.
\end{aligned}$$

where, $L(a, b)$ denotes Laplace distribution with location (scale) parameter $a(b)$.

Here one can compute, $\binom{10}{3}\binom{10}{3}U_{12} = 60$, $\binom{10}{3}\binom{10}{3}U_{23} = 280$, $\binom{10}{3}\binom{10}{3}U_{34} = 700$. Also, the optimal weights are: $a_1 = 0.375$, $a_2 = 0.500$ and $a_3 = 0.375$. With these values, we can find $\binom{10}{3}\binom{10}{3}U_S = 425$. Using Theorem 3.2 and 4.2, we can see that $\text{var}(U_S) = 0.0009968545$ and calculated standard normal deviate of $U_S$ is 5.912083 which exceeds 1.645 (at 5% level of significance). Therefore, we reject null hypothesis of homogeneity of scale parameters of Laplace distribution for the above data against simple ordered alternative.

## 5. Asymptotic relative efficiencies

In this section, we compare the asymptotic relative efficiency of proposed test relative to the tests proposed by Kochar and Gupta [10], Kusum and Bagai [13] and Shanubhogue [20].

Kochar and Gupta [10] developed a class of distribution free tests based on $W_{c,d}$, where the statistic $W_{c,d}$ is based on maxima and minima of $(X_{i1}, X_{i2}, \ldots, X_{ic}, X_{j1}, X_{j2}, \ldots, X_{jd})$ for two fixed integers $c$ and $d$. In their case, $F_i(0) = 1/2$, $i = 1, 2, \ldots, k$. However, with $F_i(0) = \alpha$ and $c = d$, the efficacy of the Kochar and Gupta's test $W^*_{c,c}$ with optimum weights is given by Kusum and Bagai [13]

$$e\left(W^*_{c,c}\right) = \frac{(k^2 - 1)G^2_{2c}}{24\rho_{2c}},$$

where

$$G_{2c} = \int_{-\infty}^{\infty} x \left[F^{2c-2}(x) - \overline{F}^{2c-2}(x)\right] f^2(x)dx$$

and

$$\rho_{2c} = \frac{1}{(2c-1)^2}\left[\frac{1}{4c-1} - \frac{2}{4c^2} + \frac{((2c-1)!)^2}{(4c-1)!}\right].$$

The efficacies of Kusum and Bagai [13] and Shanubhogue [20] tests are given by

$$e(T_k) = \frac{(k^2-1)\left[\int_{-\infty}^{\infty} |x| f(x) dF(x)\right]^2}{3\alpha^2 - 3\alpha + 1},$$

$$e(A_{c,c}) = \frac{1}{3}(k^2-1)(4c-1)c^2\left[\int_{-\infty}^{\infty} x f^2(x) F^{2c-2}(x) dx\right]^2.$$

For different distributions viz., uniform distribution, exponential distribution, Cauchy distribution, Laplace distribution and logistic distribution, the values of ARE of $U_S$ with respect to $W^*_{c,c}$, $A_{c,c}$ and $T_k$ test are given in Table 1, Table 2 and Table 3 respectively.

Table 1: ARE of $U_S$ with respect to $W^*_{c,c}$ for uniform, exponential, Cauchy, Laplace and logistic distributions

| Distribution | $c$ | $\alpha$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| Uniform | 2 | 0.32 | 0.29 | 0.26 | 0.25 | 0.36 | 0.25 | 0.26 | 0.29 | 0.32 |
| | 3 | 0.26 | 0.23 | 0.21 | 0.2 | 0.29 | 0.2 | 0.21 | 0.23 | 0.26 |
| | 4 | 0.21 | 0.18 | 0.16 | 0.16 | 0.23 | 0.16 | 0.16 | 0.18 | 0.21 |
| | 5 | 0.17 | 0.15 | 0.13 | 0.13 | 0.19 | 0.13 | 0.13 | 0.15 | 0.17 |
| Exponential | 2 | 1.67 | 0.72 | 0.36 | 0.24 | 0.42 | 0.36 | 0.3 | 0.27 | 0.24 |
| | 3 | 1.48 | 0.62 | 0.3 | 0.2 | 0.35 | 0.29 | 0.24 | 0.22 | 0.2 |
| | 4 | 1.29 | 0.53 | 0.25 | 0.17 | 0.28 | 0.23 | 0.19 | 0.17 | 0.16 |
| | 5 | 1.15 | 0.46 | 0.21 | 0.14 | 0.24 | 0.19 | 0.16 | 0.14 | 0.13 |
| Cauchy | 2 | 10.33 | 2.95 | 1.44 | 0.97 | 1.11 | 0.97 | 1.44 | 2.95 | 10.33 |
| | 3 | 10.96 | 3.12 | 1.53 | 1.02 | 1.17 | 1.02 | 1.53 | 3.12 | 10.96 |
| | 4 | 15.08 | 4.31 | 2.11 | 1.42 | 1.62 | 1.42 | 2.11 | 4.31 | 15.08 |
| | 5 | 15.26 | 4.36 | 2.14 | 1.43 | 1.64 | 1.43 | 2.14 | 4.36 | 15.26 |
| Laplace | 2 | 3.6 | 1.2 | 1.2 | 1.12 | 1.57 | 1.12 | 1.2 | 1.2 | 3.6 |
| | 3 | 3.46 | 1.76 | 1.16 | 1.08 | 1.51 | 1.08 | 1.16 | 1.76 | 3.46 |
| | 4 | 3.37 | 1.71 | 1.12 | 1.05 | 1.47 | 1.05 | 1.12 | 1.71 | 3.37 |
| | 5 | 3.34 | 1.69 | 1.11 | 1.04 | 1.46 | 1.04 | 1.11 | 1.69 | 3.34 |
| Logistic | 2 | 1.72 | 1.03 | 0.73 | 0.6 | 0.78 | 0.6 | 0.73 | 1.03 | 1.72 |
| | 3 | 1.65 | 0.99 | 0.69 | 0.58 | 0.74 | 0.58 | 0.69 | 0.99 | 1.65 |
| | 4 | 1.59 | 0.96 | 0.67 | 0.56 | 0.72 | 0.56 | 0.67 | 0.96 | 1.59 |
| | 5 | 1.57 | 0.94 | 0.66 | 0.55 | 0.71 | 0.55 | 0.66 | 0.94 | 1.57 |

## 6. Conclusion

It is interesting to note that the proposed test based on subsample medians performs better than all the above three tests for heavy-tailed distributions like the Laplace distribution and Cauchy distribution while the ARE of the proposed test with respect to Shanubhogue test is greater than one for all the values of $\alpha \geq 0.5$ even for light-tailed distributions like logistic and exponential. Also, the proposed test is better than Kochar Gupta test at tails i.e. when $\alpha$ is 0.1 or 0.9 for light-tailed distributions.

## Acknowledgment

The authors are thankful to the anonymous referee for valuable comments and suggestions which have led to an improvement of the paper.

Table 2: ARE of $U_S$ with respect to $A_{c,c}$ for uniform, exponential, Cauchy, Laplace and logistic distributions

| Distribution | $c$ | $\alpha$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| Uniform | 2 | 0.31 | 0.39 | 0.51 | 0.83 | 2.33 | 4.55 | 41.34 | 46.07 | 5.83 |
| | 3 | 0.19 | 0.22 | 0.28 | 0.43 | 1.03 | 1.48 | 4.57 | 82.55 | 23.21 |
| | 4 | 0.14 | 0.16 | 0.19 | 0.28 | 0.66 | 0.86 | 2.14 | 13.18 | 18.42 |
| | 5 | 0.11 | 0.13 | 0.15 | 0.22 | 0.48 | 0.60 | 1.37 | 6.19 | 9.33 |
| Exponential | 2 | 0.84 | 0.96 | 1.2 | 2.19 | 13.22 | 95.96 | 245 | 19.42 | 6.49 |
| | 3 | 0.78 | 0.83 | 0.94 | 1.44 | 6.21 | 16.61 | 104 | 373 | 23.49 |
| | 4 | 0.78 | 0.81 | 0.87 | 1.25 | 4.86 | 10.59 | 34.37 | 416 | 71.78 |
| | 5 | 0.79 | 0.81 | 0.85 | 1.18 | 4.37 | 8.66 | 22.54 | 174 | 229 |
| Cauchy | 2 | 1.59 | 1.59 | 1.77 | 2.5 | 7.14 | 34.81 | 110 | 9.07 | 3.32 |
| | 3 | 2.22 | 1.89 | 1.78 | 2.04 | 4.17 | 8.21 | 85.65 | 63.45 | 8.23 |
| | 4 | 3.4 | 2.68 | 2.35 | 2.52 | 4.67 | 7.68 | 39.55 | 544 | 17.55 |
| | 5 | 4.86 | 3.39 | 2.69 | 2.62 | 4.29 | 5.8 | 17.81 | 407 | 53.66 |
| Laplace | 2 | 4.08 | 13.77 | 1458 | 25.57 | 10.09 | 25.57 | 1458 | 13.77 | 4.08 |
| | 3 | 11.16 | 226.6 | 33.3 | 7.45 | 5.34 | 7.45 | 33.3 | 226.6 | 11.16 |
| | 4 | 31.3 | 382 | 12.65 | 4.93 | 4.23 | 4.93 | 12.65 | 382 | 31.3 |
| | 5 | 97.15 | 65.49 | 8.42 | 4.04 | 3.8 | 4.04 | 8.42 | 65.49 | 97.15 |
| Logistic | 2 | 0.83 | 0.94 | 1.15 | 1.76 | 4.99 | 14.74 | 166 | 15.12 | 4.13 |
| | 3 | 1.65 | 1.45 | 1.37 | 1.51 | 2.62 | 2.91 | 5.6 | 17.51 | 379 |
| | 4 | 0.81 | 0.79 | 0.82 | 1.02 | 2.06 | 2.89 | 8.82 | 163 | 31.85 |
| | 5 | 0.85 | 0.8 | 0.81 | 0.96 | 1.85 | 2.37 | 6.01 | 42.07 | 101.19 |

Table 3: ARE of $U_S$ with respect to $T_k$ for uniform, exponential, Cauchy, Laplace and logistic distributions

| Distribution | $\alpha$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| Uniform | 0.59 | 0.54 | 0.47 | 0.44 | 0.61 | 0.44 | 0.47 | 0.54 | 0.59 |
| Exponential | 0.88 | 0.73 | 0.51 | 0.38 | 0.66 | 0.59 | 0.54 | 0.55 | 0.56 |
| Cauchy | 1.19 | 1.08 | 0.95 | 0.84 | 1.04 | 0.84 | 0.95 | 1.08 | 1.19 |
| Laplace | 1.15 | 1.11 | 1.08 | 1.23 | 1.82 | 1.23 | 1.08 | 1.11 | 1.15 |
| Logistic | 0.95 | 0.88 | 0.77 | 0.71 | 0.93 | 0.71 | 0.77 | 0.88 | 0.95 |

## References

[1] Amita, Kochar, S.C. (1989) Some distribution free tests for testing homogeneity of location parameters against ordered alternatives, Journal of Indian Statistical Association 27, 1–8.
[2] Chacko, V.J. (1963) Testing homogeneity against ordered alternatives, Annals of Mathematical Statistics 34, 945–956.
[3] Deshpande, J.V., Kochar, S.C. (1982) Some competitors of Wolcoxon-Mann-Whitney test for location alternative, Journal of Indian Statistical Association 19, 9–18.
[4] Deshpande, J.V., Kusum, K. (1984) A test for the non-parametric two sample scale problem, Australian Journal of Statistics 26, 1, 16–24.
[5] Govindarajulu, Z., Haller, H.S. (1977) $c$-sample tests of homogeneity against ordered alternatives, Proceedings of the symposium to honour Jerzy Neyman (ed. R. Bartoszynski et al. ), Polish Scientific Publishers, Warszawa, 91–102.
[6] Govindarajulu, Z., Gupta, G.D. (1978) Tests for homogeneity of scale against ordered alternatives, Trans. 8th Prague conference on Information Theory, statistical decision Functions, Random Processes (eds. J. Kozesnik et al.), academia Publication House, Prague, Vol. A, 235–245.
[7] Graybill, F.A. (1969) Introduction to matrices with applications in Statistics, Wordsworth Publication Corporation, California.
[8] Jonckheere, A.R. (1954) A distribution-free k-samples test against ordered alternatives, Biometrika 41, 133–145.
[9] Kochar, S.C. (1978) A class of distribution free test for the two sample slippage problem Communication in Statistics Theory and Methods 7, 1243–1252.
[10] Kochar, S.C., Gupta, R.P. (1985) A class of distribution free tests for testing homogeneity of variance against ordered alternatives, Proceedings of the conference on advances in order restricted statistical inference at Iowa city (eds. R. Dyksha et al. ), Publications Springer, Verlag, 169–183.
[11] Kumar, N., Gill, A.N., Mehta, G.P. (1994) Distribution-free test for homogeneity against ordered alternatives, Communications in Statistics Theory and Methods 23, 1247–1256.
[12] Kusum, K. (1985) A new distribution free test for two sample scale problem, Journal of Indian Statistical association 23, 97–107.
[13] Kusum, K., Bagai, I. (1988) A new class of distribution free procedures for testing homogeneity of scale parameters against ordered alternatives, Communication in Statistics Theory and Methods 17, 1365–1376.
[14] Lehmann, E.L. (1963) Robust estimation in analysis of variance, Annals of Mathematical Statistics 34, 957–966.
[15] Mahajan, K., Gaur, A., Arora, S. (2011) A nonparametric test for a two sample scale problem based on subsample medians, Statistics and Probability Letters 81, 983–988.
[16] Puri, M.L. (1965) Some distribution-free $k$-sample rank order test of homogeneity against ordered alternatives, Communication Pure Applied Mathematics 18, 51–63.
[17] Rao, C.R. (1973) Linear Statistical Inference and its applications, Wiley Eastern Ltd. New York.
[18] Rao, K.S.M. (1982) Non-parametric tests for homogeneity of scale against ordered alternatives, Annals of Institute of statistical and Mathematics 34, 327–334.
[19] Serfling, R.J. (1980) Approximation theorems of mathematical statistics, John Wiley, New York.
[20] Shanubhogue, A. (1988) Distribution-free test for homogeneity against stochastic ordering, Metrika 35, 109–119.
[21] Shetty, I.D., Govindarajlu, Z. (1988) A two-sample test for location, Communications in Statistics Theory and Methods 17, 2389–2401.
[22] Shetty, I.D., Govindarajulu, Z., Bhat, S.V. (1998) A class of distribution free tests for testing homogeneity against ordered location alternatives, Communications in Statistics Theory and Methods 27, 747–760.
[23] Tryon, P.V., Hettmansperger, T.P. (1973) A class of non-parametric tests for homogeneity against ordered alternatives, Annals of Statistics 1, 1061–1070.