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Relationships for moments of k^{th} record values from doubly truncated p^{th} order exponential and generalized Weibull distributions

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Abstract. In this paper we establish some recurrence relations satisfied by the single and the product moments of k^{th} upper record values from doubly truncated p^{th} order exponential and generalized Weibull distributions.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. absolutely continuous random variables distributed with cdf $F(x) = P(X \leq x)$ and pdf $f(x)$. An observation X_j will be called an upper record value if its value exceeds all the previous observations $X_j > X_i$ for every $i < j$. The indices at which records occur are called record times. Thus X_j denotes an upper record value at j^{th} record time. Let $k \geq 1$ be fixed integer. Then the sequence $\{U_n^{(k)}, n \geq 1\}$ of k^{th} upper record times for the sequence $\{X_n, n \geq 1\}$ is defined as follows:

$$U_1^{(k)} = 1$$
$$U_{n+1}^{(k)} = \min\{j > U_n^{(k)} : X_{j:j+k-1} > X_{U_n^{(k)}:U_n^{(k)}+k-1}\}.$$

For $k = 1$ and $n = 1, 2, \ldots$, we write $U_n^{(1)} = U_n$. Then $\{U_n, n \geq 1\}$ is a sequence of record times for the sequence $\{X_n, n \geq 1\}$. The sequence $\{Y_n^{(k)}, n \geq 1\}$, where $Y_n^{(k)} = X_{U_n^{(k)}}$, is called a sequence of k^{th} upper record values of $\{X_n, n \geq 1\}$. Assume $Y_0^{(k)} = 0$, $Y_n^{(1)} = X_{U_n}$, $n \geq 1$ and $Y_1^{(k)} = \min(X_1, X_2, \ldots, X_k) = X_{1:k}$ [cf. Ahsanullah (1995)]. The pdf of $Y_n^{(k)} (n \geq 1)$ is given by

$$f_{Y_n^{(k)}}(x) = \frac{k^n}{(n-1)!}(-\log(1 - F(x)))^{n-1}(1 - F(x))^{k-1} f(x), \quad x \geq 0.$$

Further, the joint probability density function of $Y_m^{(k)}$ and $Y_n^{(k)}$, $1 \leq m < n$, $n \geq 2$, is given by

$$f_{Y_m^{(k)}, Y_n^{(k)}}(x, y) = \frac{k^n}{(m-1)!(n-m-1)!}(\log(1 - F(x) - \log(1 - F(y)))^{n-m-1}$$
$$\times (-\log(1-F(x)))^{m-1} \frac{f(x)}{1-F(x)}(1-F(y))^{k-1} f(y), \quad 0 \leq x < y < \infty$$

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(see Dziubdziela and Kopocinski (1976), Grudzien (1982)).

Record values arise naturally in many real life applications involving data related to economics, sports, weather and life testing problems. Many authors have studied statistical methodology of record values, since they were first studied by Chandler (1952). The various developments on record values and related topics are extensively studied by Glick (1978), Nevzorov (1987), Resnick (1987), Arnold and Balakrishnan (1989) and Arnold et al. (1992, 1998). Nain (2010a, b) derived recurrence relations for single and product moments of k^{th} upper record values from doubly truncated generalized Weibull distribution and the similar relations were also found for ordinary order statistics from doubly truncated p^{th} order exponential distribution.

In this paper, we establish some recurrence relations for single and product moments of k^{th} upper record values from doubly truncated p^{th} order exponential and doubly truncated generalized Weibull distributions. These distributions have increasing failure rate for large values of x and have many applications in areas like life testing, reliability analysis, and models related to software reliability analysis, etc. Similar results for linear-exponential and modified Weibull distributions were derived by Saran and Singh (2008) and Sultan (2007), respectively.

Notations

1. $\mu^r_{(n):k} = E(Y_n^{(k)})^r$; $r, n = 1, 2, \ldots$
2. $\mu^{r,s}_{(m,n):k} = E\big((Y_m^{(k)})^r (Y_n^{(k)})^s\big)$; $1 \le m \le n-1$ and $r, s, m, n = 1, 2, \ldots$

2. Recurrence relations for doubly truncated p^{th} order exponential distribution

Consider a family of exponential distributions defined by the function

$$f(x) = \Psi'(x)e^{-\Psi(x)}, \quad 0 \le x < \infty,$$

where $\Psi(x)$ is some function of x satisfying $\Psi(0) = 0$. Expanding $\Psi(x)$ by using Maclaurin's theorem, we get:

$$\Psi(x) = \Psi(0) + x\Psi^1(0) + \frac{x^2}{2!}\Psi^2(0) + \ldots .$$

Putting $\Psi^r(0) = \dfrac{a_{r-1}}{(r-1)!}$, $r = 1, 2, \ldots$ and assuming $(p+1)^{\text{th}}$ derivative of $\Psi(x)$ to be constant, we have

$$\Psi(x) = a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + \ldots + a_p \frac{x^{p+1}}{p+1}, \quad a_p > 0,$$

and the pdf of the distribution takes the form

$$f(x) = \left(\sum_{j=0}^{p} a_j x^j\right) e^{-\sum\limits_{j=0}^{p} a_j \frac{x^{j+1}}{j+1}}, \quad 0 \le x < \infty. \tag{1}$$

A random variable X is said to have p^{th} order exponential distribution [cf. Nain (2010b)] if its probability density function (pdf) is of the form (1), where $a_p > 0$ and p is some positive integer. The cumulative distribution function (cdf) and pdf of random variable X, respectively, takes the form

$$F(x) = 1 - e^{-\left(a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + \ldots + a_p \frac{x^{p+1}}{p+1}\right)}$$

and

$$f(x) = \left(\sum_{j=0}^{p} a_j x^j\right)(1 - F(x)).$$

The table given below demonstrates a few standard distributions obtained from (1) by choosing appropriate values of parameters p and a_j, $j = 0, 1, 2, \ldots$.

S. No.	Choice of parameters	p^{th} order exponential distribution becomes
1	$a_0 > 0$ and $a_j = 0, j \geq 1$	Exponential
2	$a_p > 0$ and $a_j = 0, 0 \leq j \leq p-1$	Weibull
3	$a_0 \neq 0, a_1 > 0$ and $a_j = 0, j \geq 2$	Linear-exponential
4	$a_j = 1, j \geq 0$ and $p = \infty$	Power series

The doubly truncated p^{th} order exponential distribution has pdf

$$f(x) = \frac{\Psi'(x)e^{-\Psi(x)}}{P-Q}, \quad Q_1 \leq x \leq P_1,$$

where Q and $1-P$ $(Q<P)$ are, respectively, the proportions of truncation on the left and right of the pdf $f(x)$, and $P = 1-e^{-\Psi(P_1)}$ and $Q = 1-e^{-\Psi(Q_1)}$.

Assuming $P_2 = \dfrac{1-P}{P-Q}$ and $Q_2 = \dfrac{1-Q}{P-Q}$, the cdf of the doubly truncated p^{th} order exponential distribution takes the form

$$F(x) = \int_{Q_1}^{x} f(\theta)d\theta = \frac{1-Q}{P-Q} - \frac{e^{-\Psi(x)}}{P-Q} = Q_2 - \frac{f(x)}{\Psi'(x)}.$$

The pdf and cdf of the doubly truncated p^{th} order exponential distribution satisfy the relation:

$$f(x) = (P_2 + (1-F(x)))\Psi'(x), \quad Q_1 \leq x \leq P_1,$$

or

$$f(x) = (P_2 + (1-F(x)))\sum_{i=0}^{p} a_i x^i, a_p > 0, Q_1 \leq x \leq P_1. \tag{2}$$

By letting $Q \to 0$ $(P \to 1)$, this distribution reduces to the right (left) truncated distribution and by letting $Q \to 0$, $P \to 1$, it becomes the original non-truncated distribution.

The mathematical form of pdf, as given in (2), is very useful for deriving recurrence relations for single and product moments of k^{th} upper record values from doubly truncated p^{th} order exponential distribution.

Theorem 2.1. *For $k > 1$, $n = 1, 2, \ldots$ and $r = 0, 1, \ldots$*

$$\mu^{r}_{(n):k} = k\sum_{i=0}^{p} \frac{a_i}{i+r+1}\left\{\left(\frac{k}{k-1}\right)^{n-1} P_2(\mu^{i+r+1}_{(n):k-1} - \mu^{i+r+1}_{(n-1):k-1}) + (\mu^{i+r+1}_{(n):k} - \mu^{i+r+1}_{(n-1):k})\right\}. \tag{3}$$

Proof. The r^{th} order moment of $Y_n^{(k)}$ is given by

$$\mu^{r}_{(n):k} = \frac{k^n}{(n-1)!}\int_{Q_1}^{P_1} x^r(-\log(1-F(x)))^{n-1}(1-F(x))^{k-1}f(x)dx.$$

Substituting $f(x)$ as given in (2), we have for all positive integral values of n and $r = 0, 1, 2, \ldots$

$$\mu^{r}_{(n):k} = \frac{k^n}{(n-1)!}\{P_2\Delta(k) + \Delta(k+1)\}, \tag{4}$$

where

$$\Delta(k) = \sum_{i=0}^{p} a_i \int_{Q_1}^{P_1} x^{r+i}(-\log(1-F(x)))^{n-1}(1-F(x))^{k-1}dx.$$

Integrating the RHS by parts and then after little simplification, we get

$$\Delta(k) = \frac{(n-1)!}{(k-1)^{n-1}} \sum_{i=0}^{p} \frac{a_i}{i+r+1} (\mu^{i+r+1}_{(n):k-1} - \mu^{i+r+1}_{(n-1):k-1}).$$

Substituting this expression in (4) and simplifying, it leads to (3). □

Remark 2.2. *By putting $Q = 0$, $P = 1$, $p = 1$, $a_0 = \lambda$ and $a_1 = \nu$ in* (3), *we get Theorem* 1 *of Saran and Singh* (2008) *for linear-exponential distribution.*

Theorem 2.3. *For $k > 1$, $1 \leq m \leq n-1$ and $r, s, m, n = 0, 1, 2, \ldots$*

$$\mu^{r,s}_{(m,n):k} = k \sum_{i=0}^{p} \frac{a_i}{i+s+1} \left\{ \left(\frac{k}{k-1} \right)^{n-1} P_2 (\mu^{r,i+s+1}_{(m,n):k-1} - \mu^{r,i+s+1}_{(m,n-1):k-1}) + (\mu^{r,i+s+1}_{(m,n):k} - \mu^{r,i+s+1}_{(m,n-1):k}) \right\} \quad (5)$$

and for $m \geq 1, n = m+1$ and $r, s = 0, 1, 2, \ldots$

$$\mu^{r,s}_{(m,m+1):k} = k \sum_{i=0}^{p} \frac{a_i}{i+s+1} \left\{ \left(\frac{k}{k-1} \right)^{m} P_2 (\mu^{r,i+s+1}_{(m,m+1):k-1} - \mu^{i+r+s+1}_{(m-1):k-1}) + (\mu^{r,i+s+1}_{(m,m+1):k} - \mu^{i+r+s+1}_{(m):k}) \right\}. \quad (6)$$

Proof. The $(r, s)^{\text{th}}$ product moment of k^{th} upper record values is given as

$$\mu^{r,s}_{(m,n):k} = \frac{k^n}{(m-1)!(n-m-1)!} \int_{Q_1}^{P_1} x^r (-\log(1-F(x)))^{m-1} \frac{f(x)}{1-F(x)} J(x) dx, \quad (7)$$

where

$$J(x) = \int_{x}^{P_1} \left(y^s (\log(1-F(x)) - \log(1-F(y)))^{n-m-1} (1-F(y))^{k-1} (P_2 + (1-F(y))) \sum_{i=0}^{p} a_i y^i \right) dy.$$

Let

$$\Delta_i(k) = \int_{Q_1}^{P_1} x^r (-\log(1-F(x)))^{m-1} \frac{f(x)}{1-F(x)} \times \int_{x}^{P_1} y^{s+i} (\log(1-F(x)) - \log(1-F(y)))^{n-m-1} (1-F(y))^k dy dx. \quad (8)$$

Then (7) takes the form

$$\mu^{r,s}_{(m,n):k} = \frac{k^n}{(m-1)!(n-m-1)!} \sum_{i=0}^{p} a_i (P_2 \Delta_i(k-1) + \Delta_i(k)). \quad (9)$$

Integrating the inner integral in (8) by parts and then simplifying, we get

$$\Delta_i(k) = \frac{(m-1)!, (n-m-1)!}{k^{n-1}} \frac{1}{(i+s+1)} (\mu^{r,i+s+1}_{(m,n):k} - \mu^{r,i+s+1}_{(m,n-1):k}).$$

Substituting it in (9), we get (5).

Proceeding in a like manner, for the case $n = m+1$, one can easily obtain equation (6). □

Remark 2.4. *By putting $Q = 0$, $P = 1$, $p = 1$, $a_0 = \lambda$ and $a_1 = \nu$ in* (5) *and* (6), *we get Theorem* 2 *of Saran and Singh* (2008) *for linear-exponential distribution.*

3. Recurrence relations for doubly truncated generalized Weibull distribution

Let a random variable X belong to a class of distributions, whose cdf is of the form

$$F(x) = 1 - e^{-\psi_1(x)\cdot e^{\lambda x}}, \quad x > 0, \tag{10}$$

where $\psi_1(x)$ is some function of x, differentiable in the interval $(0, \infty)$ and satisfying $\psi_1(0) = 0$.

From (10), we have

$$-\log(1 - F(x)) = \psi_1(x) \cdot e^{\lambda x}.$$

Differentiating with respect to x, we get

$$\begin{aligned}\frac{f(x)}{1-F(x)} &= (\lambda\psi_1(x) + \psi_1'(x))e^{\lambda x}\\ &= \left(\frac{\lambda\psi_1(x) + \psi_1'(x)}{\psi_1(x)}\right)(-\log(1-F(x)))\\ &= \left(\lambda + \frac{\psi_1'(x)}{\psi_1(x)}\right)(-\log(1-F(x))). \end{aligned}\tag{11}$$

Now putting

$$\left(\lambda + \frac{\psi_1'(x)}{\psi_1(x)}\right) = \frac{\sum_{i=0}^{p} a_i x^i}{x}, \qquad a_0 > 0,\ a_i \in R,\ i = 1, 2, \ldots, p \tag{12}$$

and then simplifying yields

$$\psi_1(x) = c \cdot x^{a_0} \exp(e^{(a_1-\lambda)x + \frac{a_2}{2}x^2 + \ldots + \frac{a_p}{p}x^p}).$$

If $a_0 = b$, $a_1 = \lambda$, $c = a$ and $a_2 = a_3 = \ldots = a_p = 0$, then it implies $\psi_1(x) = ax^b$ and from (10), we have

$$F(x) = 1 - \exp(-ax^b e^{\lambda x}),$$

which is modified Weibull distribution (MWD) defined in Sultan (2007).

Equations (11) and (12) taken together imply

$$f(x) = (1 - F(x))(-\log(1 - F(x)))\ \frac{\sum_{i=0}^{p} a_i x^i}{x}, \quad a_0 > 0,\ x > 0.$$

Let $\phi(x) = \Psi_1(x)e^{\lambda x} = -\log(1-F(x))$. The probability density function of doubly truncated generalized Weibull distribution (DTGWD) is defined as

$$f(x) = \frac{\phi'(x)e^{-\phi(x)}}{P - Q}, \quad Q_1 \le x \le P_1, \tag{13}$$

where $P = 1 - e^{-\phi(P_1)}$ and $Q = 1 - e^{-\phi(Q_1)}$. On letting $P_2 = \dfrac{1-P}{P-Q}$ and $Q_2 = \dfrac{1-Q}{P-Q}$, which implies $Q_2 - P_2 = 1$, we get the cdf of the DTGWD given by

$$F(x) = \int_{Q_1}^{x} f(\theta)d\theta = \frac{1-P}{P-Q} - \frac{e^{-\phi(x)}}{P-Q} = Q_2 - \frac{f(x)}{\phi'(x)}.$$

And hence

$$f(x) = (Q_2 - F(x))\phi'(x), \quad Q_1 \le x \le P_1$$

or

$$f(x) = (P_2 + (1 - F(x)))\phi'(x), \quad Q_1 \le x \le P_1, \tag{14}$$

where

$$\phi'(x) = \phi(x)\left(\lambda + \frac{\psi_1'(x)}{\psi_1(x)}\right) = \phi(x)\left(\frac{\sum_{i=0}^{p} a_i x^i}{x}\right) = (-\log(1 - F(x)))\left(\frac{\sum_{i=0}^{p} a_i x^i}{x}\right).$$

On substituting the value of $\phi'(x)$, so obtained, in (14), we get

$$f(x) = (P_2 + (1 - F(x)))(-\log(1 - F(x)))\frac{\sum_{i=0}^{p} a_i x^i}{x}, \quad a_0 > 0, Q_1 \le x \le P_1. \tag{15}$$

The form of pdf as given in (15) is very useful for deriving recurrence relations between moments of k^{th} upper record values from DTGWD.

Theorem 3.1. *For $k > 1$, $n = 1, 2, \ldots$ and $r = 0, 1, \ldots$*

$$\mu^r_{(n):k} = n\left\{\sum_{i=0}^{p} \frac{a_i}{i + r}\left[P_2\left(\frac{k}{k-1}\right)^n (\mu^{i+r}_{(n-1):k-1} - \mu^{i+r}_{(n):k-1}) + (\mu^{i+r}_{(n+1):k} - \mu^{i+r}_{(n):k})\right]\right\}. \tag{16}$$

Proof. Let $\{Y_n^{(k)}, n \ge 1\}$, where $Y_n^{(k)} = X_{U_n^{(k)}}$, be a sequence of k^{th} upper record values from the distribution given in (13). Then the r^{th} order moment of $Y_n^{(k)}$ is given by

$$\mu^r_{(n):k} = \frac{k^n}{(n-1)!}\int_{Q_1}^{P_1} x^r(-\log(1 - F(x)))^{n-1}(1 - F(x))^{k-1} f(x)dx\,.$$

Substituting for $f(x)$, as given in (15), we have for all positive integral values of n and $r = 0, 1, 2, \ldots$

$$\mu^r_{(n):k} = \frac{k^n}{(n-1)!}\sum_{i=0}^{p} a_i(P_2\Delta_i(k-1) + \Delta_i(k)), \tag{17}$$

where

$$\Delta_i(k) = \int_{Q_1}^{P_1} x^{i+r-1}(-\log(1 - F(x)))^n(1 - F(x))^k dx\,.$$

Integrating by parts, treating $(-\log(1 - F(x)))^n(1 - F(x))^k$ for differentiation, we get

$$\Delta_i(k) = \frac{n!}{(i + r)\,k^n}(\mu^{i+r}_{n+1:k} - \mu^{i+r}_{n:k}),$$

which on substituting in (17) gives (16). □

Remark 3.2. *By putting $P_2 = 0$, $p = 1$, $a_0 = b$, $a_1 = \lambda$ in* (16)*, we get Relation* 2.1 *of Sultan* (2007) *for modified Weibull distribution.*

Theorem 3.3. *For $k > 1$, $1 \le m \le n-1$ and $r, s, m, n = 0, 1, 2, \ldots$*

$$\mu^{r,s}_{(m,n):k} = \Bigg\{ \sum_{i=0}^{p} \frac{a_i}{i+s} \Bigg[P_2 \Big(\frac{k}{k-1}\Big)^n ((n-m) \cdot (\mu^{r,i+s}_{(m,n+1):k-1} - \mu^{r,i+s}_{(m,n):k-1}) + m.(\mu^{r,i+s}_{(m+1,n+1):k-1} - \mu^{r,i+s}_{(m+1,n):k-1})) + (n-m) \cdot (\mu^{r,i+s}_{(m,n+1):k} - \mu^{r,i+s}_{(m,n):k}) + m.(\mu^{r,i+s}_{(m+1,n+1):k} - \mu^{r,i+s}_{(m+1,n):k}) \Bigg] \Bigg\} \tag{18}$$

and for $m \ge 1, n = m+1$ and $r, s = 0, 1, 2, \ldots$

$$\mu^{r,s}_{(m,m+1):k} = \Bigg\{ \sum_{i=0}^{p} \frac{a_i}{i+s} \Bigg[P_2 \Big(\frac{k}{k-1}\Big)^{m+1} (m.(\mu^{r+s+i}_{(m+1):k-1} + \mu^{r,i+s}_{(m+1,m+2):k-1}) + (\mu^{r,i+s}_{(m,m+2):k-1} - \mu^{r,i+s}_{(m,m+1):k-1})) - m(\mu^{r+s+i}_{(m+1):k} + \mu^{r,i+s}_{(m+1,m+2):k}) + (\mu^{r,i+s}_{(m,m+2):k} - \mu^{r,i+s}_{(m,m+1):k}) \Bigg] \Bigg\}. \tag{19}$$

Proof. The $(r,s)^{\text{th}}$ product moment of k^{th} upper record values is given as

$$\mu^{r,s}_{(m,n):k} = \frac{k^n}{(m-1)!(n-m-1)!} \int_{Q_1}^{P_1} x^r (-\log(1-F(x))^{m-1} \frac{f(x)}{1-F(x)} L(x) dx, \tag{20}$$

where

$$L(x) = \int_x^{P_1} \Bigg(y^s (\log(1-F(x)) - \log(1-F(y)))^{n-m-1} (1-F(y))^{k-1} \cdot (P_2 + (1-F(y)))(-\log(1-F(y))) \frac{\sum_{i=0}^{p} a_i y^i}{y} \Bigg) dy. \tag{21}$$

Putting $-\log(1-F(y)) = \{(\log(1-F(x)) - \log(1-F(y))) + (-\log(1-F(x)))\}$ in (21), we get from (20)

$$\mu^{r,s}_{(m,n):k} = \frac{k^n}{(m-1)!(n-m-1)!} \sum_{i=0}^{p} a_i \{P_2(\Delta_i(m,k-1) + \Delta_i(m+1,k-1)) + (\Delta_i(m,k) + \Delta_i(m+1,\, k))\}, \tag{22}$$

where

$$\Delta_i(m,k) = \int_{Q_1}^{P_1} x^r (-\log(1-F(x)))^{m-1} \frac{f(x)}{1-F(x)} \Bigg(\int_x^{P_1} y^{s+i-1} (\log(1-F(x)) - \log(1-F(y)))^{n-m} (1-F(y))^k dy \Bigg) dx.$$

Integrating the inner integral by parts, treating $(\log(1-F(x)) - \log(1-F(y)))^{n-m}(1-F(y))^k$ for differentiation, we get after simplification

$$\Delta_i(m,k) = \frac{(m-1)!(n-m)!}{k^n} (\mu^{r,i+s}_{(m,n+1):k} - \mu^{r,i+s}_{(m,n):k}).$$

Substituting it in (22), it is easy to see that (18) holds.

Proceeding in a similar manner for the case $n = m+1$, one can easily establish (19). □

4. Conclusion

In the study presented above, we demonstrate the recurrence relations for the single and the product moments of k^{th} upper record values from doubly truncated p^{th} order exponential distribution and generalized Weibull distribution. These results generalize the corresponding results of Saran and Singh (2008), Saran and Pushkarna (2000) and Sultan (2007).

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