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Bounds on weighted discrete Čebyšev functional

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Abstract. Two sets of inequalities are derived for the weighted discrete Čebyšev functional, considering the probabilistic setting and also the Lipschitz class of functions.

1. Introduction and motivation

The weighted discrete Čebyšev functional is defined as the sum

$$\mathfrak{T}_{\boldsymbol{p}}(\boldsymbol{a},\boldsymbol{b}) := \sum_{i=1}^n p_i a_i b_i - \sum_{i=1}^n p_i a_i \sum_{i=1}^n p_i b_i \,, \tag{1}$$

where $\boldsymbol{a} = (a_1,\ldots,a_n)$, $\boldsymbol{b} = (b_1,\ldots,b_n) \in \mathbb{R}^n$ and the weight–vector $\boldsymbol{p} = (p_1,\ldots,p_n) \in \mathbb{R}_+^n$, satisfying $\sum_{i=1}^n p_i = 1$. Replacing the uniform weights n^{-1} by $p_j, j = \overline{1,n}$ in the the celebrated Korkin's identity (Korkin, 1882, 1883), (Mitrinović and Vasić, 1974, pp. 6–7)

$$\frac{1}{n}\sum_{j=1}^n a_j b_j = \Big(\frac{1}{n}\sum_{j=1}^n a_j\Big)\Big(\frac{1}{n}\sum_{j=1}^n b_j\Big) + \frac{1}{n^2}\sum_{1\le j<k\le n}(a_j-a_k)(b_j-b_k)\,,$$

we express $\mathfrak{T}_{\boldsymbol{p}}(\boldsymbol{a},\boldsymbol{b})$ in the forms

$$\begin{aligned}\mathfrak{T}_{\boldsymbol{p}}(\boldsymbol{a},\boldsymbol{b}) &= \frac{1}{2}\sum_{i,j=1}^n p_i p_j (a_j-a_i)(b_j-b_i) \\ &= \sum_{1\le i<j\le n} p_i p_j (a_j-a_i)(b_j-b_i)\,.\end{aligned} \tag{2}$$

For further details about Korkin's identity and related topics consult Jankov and Pogány (2012); Mitrinović and Vasić (1974); Mitrinović et al. (1993).

Using Sonin's identity (Sonin, 1898), we redefine (1) into the equivalent form

$$\mathfrak{T}_{\boldsymbol{p}}(\boldsymbol{a},\boldsymbol{b}) = \sum_{i=1}^n \Big(p_i a_i - \sum_{i=1}^n p_i a_i\Big)\Big(b_i - \sum_{i=1}^n p_i b_i\Big)\,.$$

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Remark 1.1. *Čebyšev proved* (Čebyšev, 1982, 1983) *that if n–tuples $\boldsymbol{a}$ and $\boldsymbol{b}$ are synchronous (asynchronous), that is:*

$$(a_j - a_k)(b_j - b_k) \geq (\leq)\, 0, \qquad j, k \in \{1, \ldots, n\}\,,$$

then

$$\mathfrak{T}_{\boldsymbol{p}}(\boldsymbol{a}, \boldsymbol{b}) \geq (\leq)\, 0\,, \tag{3}$$

which is what we call as Čebyšev's inequality *today.*

It is interesting that Laplace had obtained the special case of the inequality (3)*, precisely when $\boldsymbol{p} = \boldsymbol{a} \geq 0$, long before Čebyšev* (see Mitrinović et al. (1993, p. 240)).

2. Bounds for discrete Čebyšev functional in probabilistic setting

In this section, we derive some new inequalities for the discrete Čebyšev functional, considering the probabilistic setting. For that purpose let us introduce simple random variables $\xi, \eta \colon \Omega \mapsto \mathbb{R}$, on the probability space $(\Omega, \mathcal{F}, \mathsf{P})$, that is their ranges are finite. Also, we assume that random vector (ξ, η) is distributed according to the law

$$\mathsf{P}(\xi = x_j, \eta = y_k) = p_{jk}, \quad j, k = \overline{1, n}\,; \quad \sum_{j=1}^n \sum_{k=1}^n p_{jk} = 1,$$

while

$$\mathsf{P}(\xi = x_j) = \mathsf{P}(\eta = y_j) = p_j = \sum_{k=1}^n p_{jk}; \quad j, k = \overline{1, n}\,.$$

Here $\boldsymbol{p} = (p_1, \cdots, p_n)$ can be taken by convention, as in Section 1.

Actually, we consider a subspace of the real Hilbert space of the finite second order random variables $L_2(\Omega, \mathcal{F}, \mathsf{P}) = \overline{\{\xi \colon \mathsf{E}|\xi|^2 < \infty\}}$. This H–space is equipped with a weighted inner product, defined by

$$\xi \circ \eta := \sum_{j=1}^n p_j x_j y_j\,,$$

and $\boldsymbol{p}$ is playing role of a weight–vector associated with n–tuples $(x_1, \cdots, x_n)$, $(y_1, \cdots, y_n)$ of values of r.v.s ξ, η respectively. By convention coordinates of $(x_1, \cdots, x_n)$, $(y_1, \cdots, y_n)$ are taken in increasing order throughout. Weighted inner product $\circ$ is symmetric, linear in both arguments and positive definite. Therefore the corresponding weighted 2–norm becomes

$$\|\xi\|_{2,\boldsymbol{p}} = \sqrt{\xi \circ \xi} = \sqrt{\mathsf{E}\xi^2}\,.$$

The Kolmogorov–type scalar product on the H–space $L_2(\Omega, \mathcal{F}, \mathsf{P})$ is

$$\langle \xi, \eta \rangle = \sum_{i=1}^n x_i y_i,$$

where the Euclidean norm is defined in the standard way as

$$\|\xi\|_2 \equiv \|\xi\|_{2,\boldsymbol{1}} = \sqrt{\langle \xi, \xi \rangle}.$$

Employing the weighted inner product definition we have

$$\mathfrak{T}_{\boldsymbol{p}}(\xi, \eta) = \xi \circ \eta - \xi \circ \boldsymbol{1} \cdot \eta \circ \boldsymbol{1},$$

where $\mathbf{1} = (1, \ldots, 1)_{1\times n}$. We point out, that $\mathfrak{T}_{\boldsymbol{p}}(\xi, \eta)$ significantly differs from the covariance functional $\mathrm{cov}(\xi, \eta)$ of r.v.s ξ, η, being

$$\mathrm{cov}(\xi, \eta) = \sum_{j,k=1}^{n} p_{jk} x_j y_k - \sum_{j=1}^{n} p_j x_j \sum_{k=1}^{n} p_k y_k = \mathsf{E}\xi\eta - \mathsf{E}\xi\, \mathsf{E}\eta = \mathsf{E}\xi\eta - \xi \circ \mathbf{1} \cdot \eta \circ \mathbf{1}\,;$$

also we have the relation

$$\mathfrak{T}_{\boldsymbol{p}}(\xi, \eta) + \mathsf{E}\xi\eta = \mathrm{cov}(\xi, \eta) + \xi \circ \eta\,.$$

Theorem 2.1. *Let $\xi,\, \eta \in L_2(\Omega, \mathcal{F}, \mathsf{P})$ be nonnegative r.v.s, as defined above. Then, we have*

$$|\mathfrak{T}_{\boldsymbol{p}}(\xi, \eta)| \le \frac{1}{4} \min\left\{ \langle \xi, \eta \rangle,\, 8\sqrt{\mathsf{E}\xi^2 \cdot \mathsf{E}\eta^2} \right\}\,.$$

Proof. Using the Cauchy–Bunyakowsky–Schwarz–inequality we have that

$$\sum_{j=1}^{n} p_j x_j \sum_{j=1}^{n} p_j y_j \ge \Big(\sum_{j=1}^{n} p_j \sqrt{x_j y_j} \Big)^2 \ge \sum_{j=1}^{n} p_j^2 x_j y_j,$$

where we evaluate the middle expression by virtue of

$$\sum_{j=1}^{n} a_j^2 \le \Big(\sum_{j=1}^{n} a_j \Big)^2 \le 2 \sum_{j=1}^{n} a_j^2, \qquad \boldsymbol{a} \ge 0. \tag{4}$$

Now, it is easy to deduce that

$$\mathfrak{T}_{\boldsymbol{p}}(\xi, \eta) \le \sum_{j=1}^{n} p_j (1 - p_j) x_j y_j \le \max_{1\le k\le n} p_k(1 - p_k) \cdot \sum_{j=1}^{n} x_j y_j = \frac{1}{4} \sum_{j=1}^{n} x_j y_j\,.$$

Next, by triangle and the Cauchy–Bunyakowsky–Schwarz–inequality and by the elementary case of Young's inequality (Mitrinović et al., 1993, pp.379–389) we have

$$|\mathfrak{T}_{\boldsymbol{p}}(\xi, \eta)| \le |\xi \circ \eta| + |\xi \circ \mathbf{1}| \cdot |\eta \circ \mathbf{1}| \le 2\, \|\xi\|_{2,\boldsymbol{p}}\, \|\eta\|_{2,\boldsymbol{p}}\,,$$

which is the first asserted inequality. Using the conditions listed in the theorem, we can easily obtain the mentioned equality. This completes the proof of the Theorem. □

Theorem 2.2. *For nonnegative ξ, η, as defined above, we have*

$$|\mathrm{cov}(\xi, \eta)| \le \frac{n^2}{4} \min\left\{ \overline{X}_n \cdot \overline{Y}_n,\, \mathfrak{T}_{\boldsymbol{n}^{-1}}(\xi, \eta) \right\}\,, \tag{5}$$

where $\overline{X}_n, \overline{Y}_n$ stand for the sample means of ξ, η values.

Proof. Transforming (2) by the right–hand–side of well–known relation (Bognár et al., 2001, §2.1.5)

$$-\frac{1}{4} + p_{jk} \le p_j p_k \le p_{jk} + \frac{1}{4}\,, \tag{6}$$

we conclude

$$\begin{aligned}
\mathfrak{T}_{\boldsymbol{p}}(\xi, \eta) &= \frac{1}{2} \sum_{j,k=1}^{n} p_j p_k (x_j - x_k)(y_j - y_k) \le \frac{1}{2} \sum_{j,k=1}^{n} \Big(p_{jk} + \frac{1}{4} \Big)(x_j - x_k)(y_j - y_k) \\
&= \sum_{j=1}^{n} \Big\{ \sum_{k=1}^{n} p_{jk} \Big\} x_j y_j - \sum_{j,k=1}^{n} p_{jk} x_j y_k + \frac{1}{4} \Big\{ n \sum_{j=1}^{n} x_j y_j - \sum_{j=1}^{n} x_j \cdot \sum_{j=1}^{n} y_j \Big\} \\
&= \sum_{j=1}^{n} p_j x_j y_j - \sum_{j,k=1}^{n} p_{jk} x_j y_k + \frac{n^2}{4} \Big\{ \frac{1}{n} \sum_{j=1}^{n} x_j y_j - \frac{1}{n} \sum_{j=1}^{n} x_j \cdot \frac{1}{n} \sum_{j=1}^{n} y_j \Big\} \\
&= \mathfrak{T}_{\boldsymbol{p}}(\xi, \eta) - \mathrm{cov}(\xi, \eta) + \frac{n^2}{4}\, \mathfrak{T}_{\boldsymbol{n}^{-1}}(\xi, \eta)\,.
\end{aligned}$$

This is in fact

$$\operatorname{cov}(\xi,\eta) \le \frac{n^2}{4}\,\mathfrak{T}_{\boldsymbol{n}^{-1}}(\xi,\eta)\,.$$

Now, repeating this estimation procedure applying $p_j p_k \ge p_{jk} - \frac{1}{4}$, we clearly get

$$\big|\operatorname{cov}(\xi,\eta)\big| \le \frac{n^2}{4}\,\mathfrak{T}_{\boldsymbol{n}^{-1}}(\xi,\eta)\,.$$

On the other hand, also by (6) we have

$$\big|\operatorname{cov}(\xi,\eta)\big| = \Big|\sum_{j,k=1}^{n}(p_{jk} - p_j p_k)x_j y_k\Big| \le \frac{1}{4}\sum_{j,k=1}^{n}|x_j||y_k|,$$

which finishes the proof of (5). $\square$

Open Problem 1. *It is well known that* $\operatorname{cov}(\xi,\eta)$ *measures the strength of the correlation between r.v.s* ξ,η *and*

$$|\operatorname{cov}(\xi,\eta)| \le \sqrt{\mathsf{D}\xi\cdot\mathsf{D}\eta}\,.$$

Also there are only few examples when there exists absolute constant $\lambda \in (0,1)$ *such, that*

$$|\operatorname{cov}(\xi,\eta)| \le \lambda\cdot\sqrt{\mathsf{D}\xi\cdot\mathsf{D}\eta}\,,$$

see e.g. Móri and Székely (1986) *in which the sharp constant* $\lambda = \sqrt{15}/4$ *has been obtained for* ξ,η *being the sample mean and the sample variance of an independent sample of size* $n \ge 2$*, respectively. Regarding bound* (5) *we pose an* Open Problem *about the existence of such* $\lambda \in (0,1)$*, that*

$$\max\big\{\overline{X}_n\cdot\overline{Y}_n,\,\mathfrak{T}_{\boldsymbol{n}^{-1}}(\xi,\eta)\big\} \le \frac{4\lambda}{n^2}\sqrt{\mathsf{D}\xi\cdot\mathsf{D}\eta}\,.$$

Theorem 2.3. *Let* $r, s,\ r^{-1}+s^{-1} = 1,\ r > 1$*, be conjugated Hölder exponents and* $\xi,\ \eta$ *be finite second order r.v.s, as defined above. Then*

$$\big|\mathfrak{T}_{\boldsymbol{p}}(\xi,\eta)\big| \le \frac{1}{2}\,\min\big\{|x_n - x_1|\,|y_n - y_1|, M\big\}\,,$$

where

$$M = \Big\{\sum_{j,k=1}^{n} p_j^{\max\{1,r-1\}}\,|x_j - x_k|^r\Big\}^{1/r}\Big\{\sum_{j,k=1}^{n} p_k^{\max\{1,s-1\}}\,|y_j - y_k|^s\Big\}^{1/s}.$$

Proof. From (2) it follows that

$$\begin{aligned}\big|\mathfrak{T}_{\boldsymbol{p}}(\xi,\eta)\big| &\le \frac{1}{2}\sum_{j,k=1}^{n} p_j p_k\,|x_j - x_k||y_j - y_k| \\ &\le \frac{1}{2}\max_{1\le j,k\le n}|x_j - x_k|\cdot\max_{1\le j,k\le n}|y_j - y_k|\sum_{j,k=1}^{n} p_j p_k = \frac{1}{2}|x_n - x_1|\,|y_n - y_1|\,,\end{aligned}$$

since $(x_j), (y_j)$ increase. Now, because we have conjugated exponents r, s, $r > 1$, by using weighted Hölder inequality, again by the Korkin's identity, we have

$$\begin{aligned}
\big|\mathfrak{T}_{\boldsymbol{p}}(\xi,\eta)\big| &\le \frac{1}{2}\sum_{j,k=1}^{n} p_j\,|x_j-x_k|\,p_k\,|y_j-y_k| \\
&\le \frac{1}{2}\Bigg(\sum_{j,k=1}^{n}\{p_j\,|x_j-x_k|\}^r\Bigg)^{1/r}\Bigg(\sum_{j,k=1}^{n}\{p_k\,|y_j-y_k|\}^s\Bigg)^{1/s} \\
&\le \frac{1}{2}\Bigg(\sum_{j,k=1}^{n} p_j^{r-1}\,|x_j-x_k|^r\Bigg)^{1/r}\Bigg(\sum_{j,k=1}^{n} p_k^{s-1}\,|y_j-y_k|^s\Bigg)^{1/s}
\end{aligned} \tag{7}$$

and also we have

$$\begin{aligned}
\big|\mathfrak{T}_{\boldsymbol{p}}(\xi,\eta)\big| &\le \frac{1}{2}\sum_{j,k=1}^{n} p_jp_k\,|x_j-x_k|\,|y_j-y_k| \\
&\le \frac{1}{2}\Bigg(\sum_{j,k=1}^{n} p_jp_k\,|x_j-x_k|^r\Bigg)^{1/r}\Bigg(\sum_{j,k=1}^{n} p_jp_k\,|y_j-y_k|^s\Bigg)^{1/s} \\
&\le \frac{1}{2}\Bigg(\sum_{j,k=1}^{n} p_j\,|x_j-x_k|^r\Bigg)^{1/r}\Bigg(\sum_{j,k=1}^{n} p_k\,|y_j-y_k|^s\Bigg)^{1/s}.
\end{aligned} \tag{8}$$

Now, from (7) and (8), we get immediately the asserted constant M. $\square$

Now, let us recall the following result, which will be used in the sequel.

Lemma 2.4 (Weighted Pólya–Szegő type inequality for sums). (Pogány, 2005, p. 118) *Assume that*

$$0 < m_1 \le x_j \le M_1, \quad j = \overline{1,n}, \quad 0 < m_2 \le y_k \le M_2, \quad k = \overline{1,m}\,.$$

Also, let $p_{ij} \ge 0$ *and* $\sum_{i=1}^n \sum_{j=1}^m p_{ij} = 1$. *Then*

$$\sum_{j=1}^{n} q_j x_j^2 \sum_{k=1}^{m} r_k y_k^2 \le \frac{1}{4}\left(\sqrt{\frac{m_1m_2}{M_1M_2}}+\sqrt{\frac{M_1M_2}{m_1m_2}}\right)^2\left(\sum_{j=1}^{n}\sum_{k=1}^{m} p_{jk}x_jy_k\right)^2 \tag{9}$$

where

$$\sum_{k=1}^{m} p_{jk} = q_j, \quad j = \overline{1,n}, \quad \sum_{j=1}^{n} p_{jk} = r_k, \quad k = \overline{1,m}\,.$$

Inequality (9) *becomes an equality if and only if* $m = n$ *and*

$$\begin{aligned}
&\kappa = \frac{M_1m_2n}{m_1M_2+M_1m_2} \in \mathbb{N}_0, \\
&x_{l_1} = \cdots x_{l_\kappa} = m_1,\ x_{l_{\kappa+1}} = \cdots = x_{l_n} = M_1, \\
&y_{l_1} = \cdots y_{l_\kappa} = M_2,\ y_{l_{\kappa+1}} = \cdots = y_{l_n} = m_2,\ l_j \in \{1,\dots,n\}, \\
&p_{jk} = 0,\ (j,k) \in \{l_1,\dots,l_\kappa\} \times \{l_{\kappa+1},\dots,l_n\} \cup \{l_{\kappa+1},\dots,l_n\} \times \{l_1,\dots,l_\kappa\}.
\end{aligned}$$

Now, we can state the next result.

Theorem 2.5. *Let r.v.s ξ, η be as defined above. Then, we have*

$$\big|\mathfrak{T}_{\boldsymbol{p}}(\xi,\eta)\big| < \left\{1+\frac{1}{2}\left(\sqrt{\frac{x_1y_1}{x_ny_n}}+\sqrt{\frac{x_ny_n}{x_1y_1}}\right)^2\right\}\cdot \xi\circ\eta\,. \tag{10}$$

Proof. Without loss of any generality we can assume that all elements x_j from the range of ξ differ, and so do all y_k values, therefore the sufficient equality conditions are obviously violated. This means that the asserted inequality is strict.

Employing the weighted discrete Pólya–Szegő type inequality (9) to the ranges of ξ,η, bearing in mind that $m=n$; $m_1=x_1, M_1=x_n; m_2=y_1, M_2=y_n$; $r_j=q_j=p_j$ we can write

$$\begin{aligned}
\big|\mathfrak{T}_{\boldsymbol{p}}(\xi,\eta)\big| &\le \sum_{j=1}^n p_jx_jy_j+\sum_{j=1}^n p_jx_j\sum_{j=1}^n p_jy_j\\
&< \sum_{j=1}^n p_jx_jy_j+\frac{1}{4}\left(\sqrt{\frac{x_1y_1}{x_ny_n}}+\sqrt{\frac{x_ny_n}{x_1y_1}}\right)^2\left(\sum_{j,k=1}^n p_{jk}\sqrt{x_iy_j}\right)^2\\
&\le \sum_{j=1}^n p_jx_jy_j+\frac{1}{2}\left(\sqrt{\frac{x_1y_1}{x_ny_n}}+\sqrt{\frac{x_ny_n}{x_1y_1}}\right)^2\sum_{j,k=1}^n p_{jk}^2x_jy_k\\
&< \left\{1+\frac{1}{2}\left(\sqrt{\frac{x_1y_1}{x_ny_n}}+\sqrt{\frac{x_ny_n}{x_1y_1}}\right)^2\right\}\cdot\sum_{j=1}^n p_jx_jy_j,
\end{aligned}$$

so the assertion (10). □

In the following we would need the reverse Cauchy–Bunyakowsky–Schwarz–inequality, also known as Cassels' inequality.

Lemma 2.6 (Weighted discrete Cassels' inequality for sums). (Dragomir, 2003, p. 2). *If the positive n–tuples $(x_1,\cdots,x_n)$ and $(y_1,\cdots,y_n)$ satisfy the condition*

$$0<m\le\frac{x_j}{y_j}\le M<\infty,\qquad j=\overline{1,n},$$

then

$$\sum_{k=1}^n p_kx_k^2\sum_{k=1}^n p_ky_k^2\le\frac{(M+m)^2}{4mM}\left(\sum_{k=1}^n p_kx_ky_k\right)^2, \tag{11}$$

where $\boldsymbol{p}=(p_1,\cdots,p_n)>0$.

Theorem 2.7. *Let discrete random variables ξ, η be as defined above. Then*

$$\big|\mathfrak{T}_{\boldsymbol{p}}(\xi,\eta)\big|\le\left\{1+n\frac{(x_1y_1+x_ny_n)^2}{4x_1x_ny_1y_n}\right\}\cdot\xi\circ\eta\,.$$

Proof. It is enough to observe that $m=x_1/y_n, M=x_n/y_1$. Than, using the triangle inequality, weighted Cassels' inequality (11) and by right–hand–side of (4), we obtain the asserted result. □

3. Discrete Čebyšev functional built in Lipschitz function class

In the following, we would introduce so–called *Lipschitz function* $f\colon \Xi \mapsto \Upsilon$, where (Ξ, d) and (Υ, d) are two given metric spaces and $d(x, y) = |x - y|$, $x, y \in \Xi, \Upsilon \subseteq \mathbb{R}$. So, a function f is said to be Lipschitz function, or precisely, it is said to be *uniform Lipschitz of order* $\alpha > 0$ on Ξ, if there exists an absolute constant $L > 0$ such that

$$|f(x) - f(y)| \le L|x - y|^\alpha \qquad x, y \in \Xi\,.$$

Here L is the Lipschitz constant and the class consisting of such functions we write $\mathrm{Lip}_L(\alpha)$.

Theorem 3.1. *Let* r, s, $r^{-1} + s^{-1} = 1$, $r > 1$, *be conjugated Hölder exponents and let* $\boldsymbol{a} = (a_1, \dots, a_n)$, $\boldsymbol{b} = (b_1, \dots, b_n) \in \mathbb{R}^n$ *and* $\boldsymbol{p} \ge 0$ *such that* $\sum_{j=1}^n p_j = 1$. *Also, assume that* $f \in \mathrm{Lip}_{L_f}(\alpha_f)$ *and* $g \in \mathrm{Lip}_{L_g}(\alpha_g)$. *Then*

$$|\mathfrak{T}_{\boldsymbol{p}}(f(\boldsymbol{a}), g(\boldsymbol{b}))| \le L_f L_g\, |a_n - a_1|^{\alpha_f} |b_n - b_1|^{\alpha_g}\, \min\{1, M_1\}\;,$$

where $f(\boldsymbol{x}) := (f(x_1), \cdots, f(x_n))$, $\boldsymbol{x} \in \{\boldsymbol{a}, \boldsymbol{b}\}$ *and*

$$M_1 = n \left(\sum_{j=1}^n p_j^r \right)^{1/r} \left(\sum_{j=1}^n p_j^s \right)^{1/s}. \tag{12}$$

Proof. As r, s, $r > 1$ are conjugated exponents, by virtue of the discrete weighted Hölder inequality from (2) it holds:

$$\begin{aligned} |\mathfrak{T}_{\boldsymbol{p}}(f(\boldsymbol{a}), g(\boldsymbol{b}))| &\le \frac{1}{2} \sum_{j,k=1}^n \{p_j p_k\}^{1/r+1/s}\, |f(a_j) - f(a_k)|\; |g(b_j) - g(b_k)| \\ &\le \frac{1}{2} \left(\sum_{j,k=1}^n p_j p_k |f(a_j) - f(a_k)|^r \right)^{1/r} \left(\sum_{j,k=1}^n p_j p_k |g(b_j) - g(b_i)|^s \right)^{1/s}. \end{aligned} \tag{13}$$

Because of $f \in \mathrm{Lip}_{L_f}(\alpha_f), g \in \mathrm{Lip}_{L_g}(\alpha_g)$, having in mind that coordinates of $\boldsymbol{a}, \boldsymbol{b}$ can be taken increasing without loss of generality, we estimate the right–hand–side of (13) in the following way:

$$\begin{aligned} |\mathfrak{T}_{\boldsymbol{p}}(f(\boldsymbol{a}), g(\boldsymbol{b}))| &\le \frac{L_f L_g}{2} \left(\sum_{j,k=1}^n p_j p_k |a_j - a_k|^{\alpha_f r} \right)^{1/r} \left(\sum_{j,k=1}^n p_j p_k |b_j - b_k|^{\alpha_g s} \right)^{1/s} \\ &= L_f L_g\, |a_n - a_1|^{\alpha_f} |b_n - b_1|^{\alpha_g} \left(\sum_{j,k=1}^n p_j p_k \right)^{1/r+1/s} \\ &= L_f L_g\, |a_n - a_1|^{\alpha_f} |b_n - b_1|^{\alpha_g}\,. \end{aligned}$$

Now, it is left to derive M_1. Again, by using discrete Hölder inequality with the same pair of conjugated parameters r, s and the definition of Lipschitz functions f and g we get:

$$\begin{aligned}
|\mathfrak{T}_{\boldsymbol{p}}(f(\boldsymbol{a}), g(\boldsymbol{b}))| &\le \frac{1}{2}\sum_{j,k=1}^{n} \{p_j\,|f(a_j)-f(a_k)|\}\cdot\{p_k\,|g(b_j)-g(b_k)|\} \\
&\le \frac{1}{2}\left(\sum_{j,k=1}^{n} p_j^r\,|f(a_j)-f(a_k)|^r\right)^{1/r}\left(\sum_{j,k=1}^{n} p_k^s\,|g(b_j)-g(b_k)|^s\right)^{1/s} \\
&\le L_f L_g\,|a_n-a_1|^{\alpha_f}|b_n-b_1|^{\alpha_g}\left(\sum_{j,k=1}^{n} p_j^r\right)^{1/r}\left(\sum_{j,k=1}^{n} p_k^s\right)^{1/s} \\
&= L_f L_g\,|a_n-a_1|^{\alpha_f}|b_n-b_1|^{\alpha_g}\, n\left(\sum_{j=1}^{n} p_j^r\right)^{1/r}\left(\sum_{j=1}^{n} p_j^s\right)^{1/s}
\end{aligned}$$

which completes the proof of the Theorem. $\square$

Now, we discuss certain results which follow by the previous theorem. First, we would need the well–known fact that every Lipschitz function $f\colon \Xi \mapsto \Upsilon$ is uniformly continuous, i.e. for every $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all points $x, y \in \Xi$ such that $|x - y| < \delta$.

Corollary 3.2. *Let $\boldsymbol{a} = (a_1, \dots, a_n)$, $\boldsymbol{b} = (b_1, \dots, b_n)$ be two n–tuples coming from the metric space Ξ and let $f, g\colon \Xi \mapsto \Upsilon$ be Lipschitz functions, $f \in \mathrm{Lip}_{L_f}(\alpha_f)$ and $g \in \mathrm{Lip}_{L_g}(\alpha_g)$. Then $\mathfrak{T}_{\boldsymbol{p}}(f(\boldsymbol{a}), f(\boldsymbol{b}))$ is uniformly continuous.*

Proof. Desired result follows directly by Theorem 3.1, and uniform continuity of the functions f, g. It is enough to observe that for all $\max\{|a_n - a_1|, |b_n - b_1|\} < \delta$ it follows that

$$|\mathfrak{T}_{\boldsymbol{p}}(f(\boldsymbol{a}), g(\boldsymbol{b}))| < L_f L_g \delta^{\alpha_f+\alpha_g} \min\{1, M_1\} =: \epsilon\,,$$

where M_1 is given with (12). $\square$

Corollary 3.3. *If f, $g\colon \Xi \mapsto \Upsilon$ are injective Lipschitz functions between metric spaces, $f \in \mathrm{Lip}_{L_f}(\alpha_f)$ and $g \in \mathrm{Lip}_{L_g}(\alpha_g)$, corresponding metric d is discrete and $\boldsymbol{a} = (a_1, \dots, a_n)$, $\boldsymbol{b} = (b_1, \dots, b_n)$ are two n–tuples from Ξ, then*

$$|\mathfrak{T}_{\boldsymbol{p}}(f(\boldsymbol{a}), g(\boldsymbol{b}))| \le L_f L_g \min\{1, M_1\}\,, \tag{14}$$

where M_1 is given with (12).

Proof. Assertion (14) follows from Theorem 3.1 and definition of a discrete metric:

$$d(x, y) = \begin{cases} 1, & \text{if } x \ne y \\ 0, & \text{if } x = y \end{cases}.$$

So the asserted result. $\square$

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