Measure of state's importance and its estimation from migration data

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Abstract. From the sample observations relating to the behavior of transitions among the states, measures of importance of states have been developed. From common sense we see that number of visits to a place should be positively related to the importance of that state. From this measures we obtain a composite measure whether the states are all equal. Then estimation method of MEN which has been developed by us is used to find an estimator of the generator matrix involved in this and the estimator of this measures of equality.

1. Introduction

Internal migration model (Seal and Hossain, 2013b) tried estimation of the parameters involved in analyzing internal migration data and also testing (Seal and Hossain, 2013a) for the same. It is true that a person living in a place or state wishes to go to another place or state for different reasons like establishments, works, education etc. But after some time he goes to another place with the hope of fulfillment and there after spending some time, often he comes back to some of the previous places, as his memory consciously or unconsciously brings him for betterment. Our mind can decide this well either consciously or unconsciously. Therefore that person often and often will trace the pleasant or important states that he finds. Though our mind's adaptations are different, some of the common things like jobs, foods, financial positions have almost similar effects on all persons. So it is important to note the frequency with which a person traces a state. This helps in measuring the importance of that place or state. This idea is used in this work. Moreover a measure of equality of importance of these states is developed.

This is also important to note that we have chosen Markov model, which uses only the current states for further transitions. But when the states seem equal apparently, this is appropriate at least at this time of globalization. Otherwise where there are clear differences among the states, one can choose states easily. So when this is the situation mind acts at least unconsciously and therefore the factors depend grossly on current states.

Using mean recurrence function and its Laplace transformation we obtain a measure of importance of states. Also these are simplified by connecting this with the formulation of continuous Markov chain model with infinitesimal generator matrix for computation purpose. In Section 2 such measure is used to find the equality of importance of states or places. Then a sample drawing procedure is given and is used to estimate this measure. For estimating these matrix parameters we have developed Minimum Euclidean Norm (MEN) method. The idea here is to consider the Minimum Euclidean Norm between the parameter matrix and the corresponding sample matrix.

2. A measure of importance of a state via recurrence of that state

Suppose we have k states and let the states be numbered from 1, 2, ..., k. Let us suppose transitions occur at instants of time t_r, $r = 1, 2, \ldots, n$. Let the transitions to the states constitute a Markov process, but the intervals $(t_r - t_{r-1})$ have any distribution, this distribution may depend on the state from which the transition takes place as well as on the state to which the next transition takes place.

Let us take initial distribution of these k states to be q_1, q_2, ..., q_k, respectively, i.e.

$$q_j = P[X(0) = j], \; j = 1, 2, \ldots, k.$$

Also if it started from state j and entered in state i at time t for the first time with probability density $f_{ji}(t)$, then the unconditional distribution that a person is at state i in time t for the first time is

$$G_i(t) = \sum_{j=1}^{k} \int_0^t q_j f_{ji}(s)ds, \; i = 1, 2, \ldots, k. \tag{1}$$

We define the random variable $R_i(t)$, the number of repetitions into state i by time t which occur in $(0, t]$, so that,

$$R_i(t) = n_i, \text{ if } t_1 + \cdots + t_{n_i} \le t < t_1 + \cdots + t_{n_i} + t_{n_{i+1}}, \text{ for all } i = 1, 2, \cdots, k,$$

where t_r is the spacing between the $(r-1)$th and rth repetition to state i. The mean number of repetitions to be expected at state i during (0,t] is

$$m_i(t) = E[R_i(t)], \text{ for all } i = 1, 2, \cdots, k.$$

Then in order to decide the equal importance of the states we should know whether,

$$m_1(t) = m_2(t) = \cdots = m_k(t), \text{ for all } t \Longleftrightarrow \widetilde{m}_1(\theta) = \widetilde{m}_2(\theta) = \cdots = \widetilde{m}_k(\theta), \text{ for all } \theta,$$

where $\widetilde{m_i}(\theta)$ are the Laplace transformation of $m_i(t)$. We call $m_i(t)$ as mean recurrence function for the state i; $i = 1, 2, \cdots, k$.

Now we are interested in finding out $m_i(t)$, i.e. $E[R_i(t)]$ for all t and for all $i = 1, 2, \cdots, k$. Now we attempt to compute $p_{ji}(t)$ by decomposing the event in terms of the event, the first time one enters into state i. We can write the recursive equation in continuous time analog as

$$p_{ji}(t) = \int_0^t p_{ii}(t - s)dF_{ji}(s) = \int_0^t f_{ji}(s)p_{ii}(t - s)ds, \tag{2}$$

where $dF_{ji}(s)$ has the density function $f_{ji}(s)ds$.

Differentiating both sides of (2) we have,

$$p'_{ji}(t) = f_{ji}(t), \text{ for all } i, j = 1, 2, \cdots, k, \tag{3}$$

with initial condition $f_{ji}(0) = 0$, $p_{ii}(0) = 1$, for all i and j.

Again we know from the continuous time Markov chain

$$P'(t) = AP(t), \tag{4}$$

where $A = (a_{ij})$ is the infinitesimal matrix to be written as,

$$A = \begin{bmatrix} -a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & -a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & -a_{kk} \end{bmatrix}, \; 0 \le a_{ij} < \infty, \; \forall \, i, j; \; 0 \le a_{ii} \le \infty, \; \forall \, i; \; a_{ii} = \sum_{j=1, j \ne i}^{k} a_{ij}, \; \forall \, i. \tag{5}$$

Now we shall prove the following lemma which will be used in subsequent work.

Lemma 2.1. *For all $i, j = 1, 2, \cdots, k$, we have that $f_{ji}(t) = \sum_{l=1}^{k} a_{jl} p_{li}(t)$, where l is any intermediate state.*

Proof. From (4) we have that

$$p'_{ji}(t) = (a_{j1}, \cdots, a_{jk}) \begin{pmatrix} p_{1i}(t) \\ \vdots \\ p_{ki}(t) \end{pmatrix} = \sum_{l=1}^{k} a_{jl} p_{li}(t).$$

Thus from (3) we have that

$$f_{ji}(t) = p'_{ji}(t) = \sum_{l=1}^{k} a_{jl} p_{li}(t), \tag{6}$$

which completes the proof. $\square$

Now from (1) we have the unconditional distribution

$$G_i(t) = \sum_{j=1}^{k} \int_0^t q_j f_{ji}(s) ds.$$

Then from (6) we have that

$$G_i(t) = \sum_{j=1}^{k} q_j \int_0^t \sum_{l=1}^{k} a_{jl} p_{li}(s) ds = \sum_{j=1}^{k} \sum_{l=1}^{k} q_j a_{jl} \int_0^t p_{li}(s) ds. \tag{7}$$

Applying Laplace transformation we have that

$$\widetilde{G_i}(\theta) = L.T.\{G_i(t)\} = \sum_{j=1}^{k} \sum_{l=1}^{k} q_j a_{jl} \widetilde{p}_{li}(\theta), \tag{8}$$

where $\widetilde{p}_{li}(\theta) = L.T.\{p_{li}(t)\}$. Now $F_i(t)$, the distribution function of $f_{ii}(t)$, can be written as

$$F_i(t) = \int_0^t f_{ii}(s) ds = \int_0^t \sum_{l=1}^{k} a_{il} p_{li}(s) ds$$

Applying Laplace transformation we get,

$$label2.9\widetilde{F_i}(\theta) = L.T.\{F_i(t)\} = \int_0^\infty e^{-\theta t} dF_i(t) = \sum_{l=1}^{k} a_{il} \widetilde{p}_{li}(\theta). \tag{9}$$

Thus we get (Ross, 1996)

$$\widetilde{m}_i(\theta) = \frac{\widetilde{G_i}(\theta)}{1 - \widetilde{F_i}(\theta)} = \frac{\sum_{j=1}^{k} \sum_{l=1}^{k} q_j a_{jl} \widetilde{p}_{li}(\theta)}{1 - \sum_{l=1}^{k} a_{il} \widetilde{p}_{li}(\theta)} \text{ , for all } i. \tag{10}$$

Again we have that

$$P(t) = I + \sum_{n=1}^{\infty} \frac{A^n t^n}{n!}.$$

Now we have Laplace transformation of this $P(t)$ in the following lemma.

Lemma 2.2. *If* $P(t) = I + \sum_{n=1}^{\infty} \frac{A^n t^n}{n!}$ *then*

$$\widetilde{P}(\theta) = \frac{1}{\theta} \left[I + \sum_{n=1}^{\infty} \frac{A^n}{\theta^n} \right] = \frac{1}{\theta} \left[I + \frac{A}{\theta} \left(I - \frac{A}{\theta} \right)^{-1} \right].$$

Proof. We have that $p_{ij}(t)$, the (i, j)th element of $P(t)$ can be written as

$$p_{ij}(t) = \sum_{n=1}^{\infty} \frac{(A^n)_{ij}}{n!} t^n, \text{ for all } i \neq j.$$

Now

$$
\begin{aligned}
\widetilde{p}_{ij}(\theta) = L.T.\{p_{ij}(t)\} &= \int_0^{\infty} e^{-\theta t} p_{ij}(t) dt, \text{ for all } i \neq j \\
&= \int_0^{\infty} e^{-\theta t} \sum_{n=1}^{\infty} \frac{(A^n)_{ij}}{n!} t^n dt \\
&= \sum_{n=1}^{\infty} \frac{(A^n)_{ij}}{n!} \int_0^{\infty} e^{-\theta t} t^n dt \\
&= \sum_{n=1}^{\infty} \frac{(A^n)_{ij}}{n!} \cdot \frac{n!}{\theta^{n+1}} \\
&= \sum_{n=1}^{\infty} \frac{(A^n)_{ij}}{\theta^{n+1}}.
\end{aligned}
\tag{11}
$$

Again

$$p_{ii}(t) = 1 + \sum_{n=1}^{\infty} \frac{(A^n)_{ii}}{n!} t^n,$$

so we get

$$
\begin{aligned}
\widetilde{p}_{ii}(\theta) &= \int_0^{\infty} e^{-\theta t} p_{ii}(t) dt \\
&= \int_0^{\infty} 1 \cdot e^{-\theta t} dt + \sum_{n=1}^{\infty} \frac{(A^n)_{ii}}{n!} \int_0^{\infty} e^{-\theta t} t^n dt \\
&= \frac{1}{\theta} + \sum_{n=1}^{\infty} \frac{(A^n)_{ii}}{\theta^{n+1}}.
\end{aligned}
\tag{12}
$$

Therefore from (11) and (12) we get

$$\widetilde{P}(\theta) = \frac{1}{\theta} \left[I + \sum_{n=1}^{\infty} \frac{A^n}{\theta^n} \right] = \frac{1}{\theta} \left[I + \frac{A}{\theta} \left(I - \frac{A}{\theta} \right)^{-1} \right]. \tag{13}$$

that completes the proof. $\square$

Now we are giving in the following some measures of equality of states. The measures of equality of states are being given in terms of the Laplace transformation of the mean recurrence time

$$\sup_{\theta} \sum_{i=1}^{k} (\widetilde{m}_i(\theta) - \widetilde{m}(\theta))^2, \tag{14}$$

or

$$\int_0^\infty \sum_{i=1}^k (\widetilde{m}_i(\theta) - \widetilde{m}(\theta))^2 \, d\theta, \tag{15}$$

where

$$\widetilde{m}(\theta) = \frac{1}{k} \sum_{i=1}^k \widetilde{m}_i(\theta), \text{ for all } \theta.$$

If the value of the above two measure is small then we say that the states are equally important.

In spite of the above the following standardized measure has desirable properties

$$M_{sd} = \frac{\displaystyle\sum_{\substack{i,j=1 \\ i \neq j}}^k \left(\int_0^\infty (\widetilde{m}_i(\theta) - \widetilde{m}_j(\theta))^2 \, d\theta \right)^{1/2}}{\displaystyle\sum_{i=1}^k \left(\int_0^\infty \widetilde{m}_i^2(\theta) d\theta \right)^{1/2}}. \tag{16}$$

We have the following properties of M_{sd} :

(i) It lies between 0 and $2(k-1)$. By Minkowski's inequality in function space of L_2 norm we have

$$\left(\int_0^\infty (\widetilde{m}_i(\theta) - \widetilde{m}_j(\theta))^2 \, d\theta \right)^{1/2} \leq \left(\int_0^\infty \widetilde{m}_i^2(\theta) d\theta \right)^{1/2} + \left(\int_0^\infty \widetilde{m}_j^2(\theta) d\theta \right)^{1/2}.$$

So

$$\begin{aligned}
\sum_{\substack{i,j=1 \\ i \neq j}}^k \left(\int_0^\infty (\widetilde{m}_i(\theta) - \widetilde{m}_j(\theta))^2 \, d\theta \right)^{1/2} &\leq \sum_{\substack{i,j=1 \\ i \neq j}}^k \left(\int_0^\infty \widetilde{m}_i^2(\theta) d\theta \right)^{1/2} + \sum_{\substack{i,j=1 \\ i \neq j}}^k \left(\int_0^\infty \widetilde{m}_j^2(\theta) d\theta \right)^{1/2} \\
&= 2(k-1) \sum_{i=1}^k \left(\int_0^\infty \widetilde{m}_i^2(\theta) d\theta \right)^{1/2}.
\end{aligned}$$

Therefore

$$0 \leq \frac{\displaystyle\sum_{\substack{i,j=1 \\ i \neq j}}^k \left(\int_0^\infty (\widetilde{m}_i(\theta) - \widetilde{m}_j(\theta))^2 \, d\theta \right)^{1/2}}{\displaystyle\sum_{i=1}^k \left(\int_0^\infty \widetilde{m}_i^2(\theta) d\theta \right)^{1/2}} \leq 2(k-1).$$

(ii) Perfect equality of importance implies that it has value zero. This follows easily from above.

The algorithm to evaluate these measure for a given generator matrix is: For a generator matrix we find $\widetilde{P}$ from (13). Then using this and (10) as obtained from the idea of recurrence of states we obtain $\widetilde{m}_i(\theta)$. Finally, we get (14), (15) and (16) as in the above Section 2.

3. Estimation of the measure from matrix observations by method of minimum Euclidean norm

Let us take observations at time points $T = t_1, t_2, \cdots, t_n$ as

$$N^{(i)} = \begin{bmatrix} n_{11}^{(i)} & n_{12}^{(i)} & \cdots & n_{1k}^{(i)} \\ n_{21}^{(i)} & n_{22}^{(i)} & \cdots & n_{2k}^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ n_{k1}^{(i)} & n_{k2}^{(i)} & \cdots & n_{kk}^{(i)} \end{bmatrix}, \text{ for all } i = 1, 2, \ldots, n,$$

where $n_{rs}^{(i)}$ is the number of persons transited from state r to state s during $(t_{i-1}, t_i]$.

Then we have the estimated transition probability matrix by

$$\widehat{P^{(i)}} = \begin{bmatrix} \frac{n_{11}^{(i)}}{n_{1\cdot}^{(i)}} & \frac{n_{12}^{(i)}}{n_{1\cdot}^{(i)}} & \cdots & \frac{n_{1k}^{(i)}}{n_{1\cdot}^{(i)}} \\ \frac{n_{21}^{(i)}}{n_{2\cdot}^{(i)}} & \frac{n_{22}^{(i)}}{n_{2\cdot}^{(i)}} & \cdots & \frac{n_{2k}^{(i)}}{n_{2\cdot}^{(i)}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{n_{k1}^{(i)}}{n_{k\cdot}^{(i)}} & \frac{n_{k2}^{(i)}}{n_{k\cdot}^{(i)}} & \cdots & \frac{n_{kk}^{(i)}}{n_{k\cdot}^{(i)}} \end{bmatrix}, \text{ for all } i = 1, 2, \cdots, n.$$

Again we know

$$P(t_i - t_{i-1}) = e^{A(t_i - t_{i-1})}, \text{ for all } i = 1, 2, \cdots, n,$$

where A is the infinitesimal generator matrix given by (5). Thus we can write

$$\prod_{i=1}^{n} e^{A(t_i - t_{i-1})} \cong \prod_{i=1}^{n} \widehat{P^{(i)}} \implies e^{A(t_n - t_1)} \cong \widehat{P^*},$$

where

$$\widehat{P^*} = \prod_{i=1}^{n} \widehat{P^{(i)}}.$$

So our problem is to find out A such that

$$\left\| e^{A(t_n - t_1)} - \widehat{P^*} \right\|$$

becomes minimum. Thus we try to get the estimated value of A by our MEN method. To do the above we can write $\widehat{P^*}$ as

$$\widehat{P^*} = [\alpha_1 \ \alpha_2 \ \cdots \ \alpha_k]$$

where each α_j for $j = 1, 2, \cdots, k$ is a column vector.

It is to be noted that usually rank $\widehat{P^{(i)}} = k - 1$, for $i = 1, 2, \cdots, n$ as sum of each row is 1. Therefore rank $\widehat{P^*}$ may be assumed to be $k - 1$. So in order to extract maximum information from $\widehat{P^*}$ we shall find $B = e^{A(t_n - t_1)}$ of rank $k - 1$ and the procedure is as follows.

Let $Q_1, Q_2, \cdots, Q_{k-1}$ be the first $(k - 1)$ eigenvector of $\widehat{P^*}\widehat{P^*}'$. Let B^* be the value of B for which $\left\| B - \widehat{P^*} \right\|$ is minimum. Then the column vectors of B^* are the following (Rao, 1976)

$$\begin{aligned} B_1^* &= (Q_1'\alpha_1)Q_1 + (Q_2'\alpha_1)Q_2 + \cdots\cdots + (Q_{k-1}'\alpha_1)Q_{k-1} \\ B_2^* &= (Q_1'\alpha_2)Q_1 + (Q_2'\alpha_2)Q_2 + \cdots\cdots + (Q_{k-1}'\alpha_2)Q_{k-1} \\ &\vdots \qquad \vdots \\ B_k^* &= (Q_1'\alpha_k)Q_1 + (Q_2'\alpha_k)Q_2 + \cdots\cdots + (Q_{k-1}'\alpha_k)Q_{k-1}. \end{aligned}$$

Then we get the matrix $B^* = [B_1^* \; B_2^* \; \cdots \; B_k^*]$. Now we find $\xi_1, \xi_2, \cdots, \xi_{k-1}$ right eigenvectors of B^*, $\eta_1, \eta_2 \cdots, \eta_{k-1}$ left eigenvectors of B^*, and $\lambda_1, \lambda_2, \cdots, \lambda_{k-1}$ eigenvalues of B^*. Then we have

$$B^* = \sum_{i=1}^{k-1} \lambda_i \xi_i \eta_i'.$$

Now we need the following definition and useful result for the exponential matrix.

Definition 3.1 (Karlin and Taylor (1975)). *For a square matrix A we have that*

$$e^A = I + \sum_{l=1}^{\infty} \frac{A^l}{l!} = \sum_{l=0}^{\infty} \frac{A^l}{l!}.$$

Now let $X_1, X_2, \cdots, X_k$ be the right eigenvectors and $Y_1, Y_2, \cdots, Y_k$ be the left eigenvectors respectively corresponding to the eigenvalues $q_1, q_2, \cdots, q_k$ of A. Then we can write

$$A = \sum_{i=1}^{k} \frac{q_i}{a_i} X_i Y_i',$$

where $a_i = X_i' \cdot Y_i$ for $i = 1, 2, \cdots, k$. Then

$$\sum_{l=0}^{\infty} \frac{A^l}{l!} = \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{i=1}^{k} \frac{q_i^l}{a_i} X_i Y_i' = \sum_{i=1}^{k} \sum_{l=0}^{\infty} \frac{q_i^l}{a_i l!} X_i Y_i' = \sum_{i=1}^{k} \frac{1}{a_i} e^{q_i} X_i Y_i'.$$

Thus, we have that

$$e^A = \sum_{i=1}^{k} \frac{1}{a_i} e^{q_i} X_i Y_i'.$$

We show that X_j are right eigenvectors of e^A:

$$\left(\sum_{i=1}^{k} \frac{1}{a_i} e^{q_i} X_i Y_i' \right) \cdot X_j = \sum_{i=1}^{k} \frac{1}{a_i} e^{q_i} X_i (Y_i' \cdot X_j) = \frac{1}{a_j} e^{q_j} X_j a_j = e^{q_j} X_j,$$

which implies that e^{q_j} is the corresponding eigenvalue.

So we have the following lemma.

Lemma 3.2. *$X_1, X_2, \cdots, X_k$ are the right eigenvectors and $Y_1, Y_2, \cdots, Y_k$ are the left eigenvectors, respectively, of e^A corresponding to the eigenvalues $e^{q_1}, e^{q_2}, \cdots, e^{q_k}$.*

Lemma 3.3. *If*

$$A = \sum_{i=1}^{k} \frac{q_i}{a_i} X_i Y_i'$$

then

$$e^{At} = \sum_{i=1}^{k} \frac{e^{tq_i}}{a_i} X_i Y_i',$$

i.e. e^{At} has eigenvalues $(e^{q_i})^t$ with corresponding eigenvectors X_i, Y_i.

Lemma 3.4. *If e^{At} has eigenvalues $\lambda_1(t), \lambda_2(t), \cdots, \lambda_k(t)$, then*

$$A = \frac{1}{t} \sum_{i=1}^{k} \frac{\log \lambda_i(t)}{a_i} X_i Y_i'.$$

Proof. e^{At} has eigenvalues $\lambda_1(t), \lambda_2(t), \cdots, \lambda_k(t)$, which implies that $\lambda_i(t) = \lambda_i^t$ as in previous lemma, where λ_i's are eigenvalues of e^A. Also $\lambda_i = e^{q_i}$ for some i, when q_i's are eigenvalues of A. Thus,

$$\lambda_i^t = e^{tq_i} \quad \Rightarrow \quad \log \lambda_i^t = tq_i \Rightarrow q_i = \frac{1}{t} \log \lambda_i^t = \frac{1}{t} \log \lambda_i(t).$$

□

Thus we have the result for exponential matrix.

Theorem 3.5. *If $e^A = \sum_{i=1}^{k} \dfrac{\lambda_i}{d_i} \xi_i \eta_i'$, where $d_i = \xi_i \cdot \eta_i$ and ξ_i's are right eigenvectors, η_i's are left eigenvectors of e^A corresponding to eigenvalues λ_i, then we have $A = \sum_{i=1}^{k} \dfrac{\log \lambda_i}{d_i} \xi_i \eta_i'$.*

Proof. Follows as in above lemma. □

With the above lemma and theorem we suggest the following. We take the estimated value of A as

$$\widehat{A} = \frac{1}{t} \sum_{i=1}^{k-1} \frac{\log \lambda_i(t)}{d_i} \xi_i \eta_i',$$

where $t = t_n - t_1$. Using this $\widehat{A}$, we get $\widehat{P(\theta)}$ from (13) and finally we get $\widehat{m_i(\theta)}$ from (10) for all $i = 1, 2, \cdots, k$.

4. Concluding remarks

We could not generalize this measure when repetitions to state i occur at random times having a distribution, partly because the mathematical calculations will be tedious, though from practical point of view, that is important. Even distribution of spacing should be modified from transition to transition, e.g., shift in location, because with the passage of time it is to be expected.

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