ProbStat Forum, Volume 06, July 2013, Pages 18–28
ISSN 0974-3235

ProbStat Forum is an e-journal.
For details please visit www.probstat.org.in

# Inference for the parameters of generalized inverted family of distributions

**K. G. Potdar[a], D. T. Shirke[b]**

[a]*Department of Statistics, Ajara Mahavidyalaya, Ajara, Dist-Kolhapur, Maharashtra, India*-416505
[b]*Department of Statistics, Shivaji University, Kolhapur, Dist-Kolhapur, Maharashtra, India*-416004

**Abstract.** In this article, we introduce a generalized inverted scale family of distributions. Maximum likelihood estimators (MLEs) of scale and shape parameters are obtained. Asymptotic confidence intervals for both the parameters based on the MLE and log (MLE) are also constructed. Generalized inverted half-logistic distribution is considered as a member of the generalized inverted scale family. Simulation study is conducted to investigate performance of estimates and confidence intervals for this distribution. An illustration with real data is provided.

## 1. Introduction

A scale family of distributions plays an important role in a lifetime data analysis. Exponential distribution, Rayleigh distribution, half-logistic distribution etc. are some of the distributions more widely used to analyze lifetime data. Lin et al.v(1989) and Dey (2007) used inverted exponential distribution (IED) to analyze lifetime data. If $Y$ is exponential variate then $X = 1/Y$ has an inverted exponential distribution. Singh et al. (2012) discussed Bayes estimators of the parameters and reliability function of IED using Type-I as well as Type-II censored samples.

Generalized exponential distribution was introduced by Gupta and Kundu (1999, 2001a, 2001b). Abouammoh and Alshingiti (2009) generalized inverted exponential distribution (GIED) by introducing a shape parameter. They discussed statistical and reliability properties of GIED. They also studied estimation of both scale and shape parameters. Krishna and Kumar (2012) used Type-II censored data to estimate reliability characteristics of GIED. They proposed maximum likelihood estimation and least square estimation procedures. Potdar and Shirke (2012) discussed inference for the scale family of lifetime distributions based on progressively censored data.

In this article, a generalized inverted scale family of distributions is proposed by introducing a shape parameter to the scale family of distributions. Inferential procedures are considered for both the parameters of the family. In Section 2, we introduce the model and obtain maximum likelihood estimators (MLEs) for scale and shape parameters. Expression for elements of Fisher information matrix are derived in same section. Asymptotic confidence intervals (CIs) for scale and shape parameters are discussed. Generalized inverted half-logistic distribution (GIHD) is considered as a member of family in Section 3. The MLEs and CIs for the parameters of GIHD are studied. In Section 4, the performance of the MLEs and CIs are investigated using simulations. Real data example is given in Section 5. Conclusions are reported in Section 6.

*Keywords.* Generalized inverted half-logistic distribution, maximum likelihood estimation, confidence interval
Received: 22 September 2012; Revised: 14 May 2013; Accepted: 24 June 2013
*Email addresses:* potdarkiran.stat@gmail.com (K. G. Potdar), dts_stats@unishivaji.ac.in (D. T. Shirke)

## 2. Model and estimation of scale and shape parameter

Let $\mathbb{G}_\lambda$ be a scale family of lifetime distributions, where $\lambda$ is the scale parameter with cumulative distribution function (cdf) and probability density function (pdf) $G(\cdot)$ and $g(\cdot)$, respectively. We generalize this family by introducing a shape parameter $\alpha$ to obtain a generalized scale family of distributions. Let $Y$ be a random variable having distribution belonging to a generalized scale family of distributions. Suppose $F_Y(\cdot)$ is cdf of random variable $Y$. Let $X = (1/Y)$, then distribution of $X$ belongs to generalized inverted scale family of distributions. The cdf and pdf of the generalized inverted scale family of distributions are respectively given as

$$F_X(x;\lambda,\alpha) = 1 - \left[G\left(\frac{1}{\lambda x}\right)\right]^\alpha, \ x \geq 0, \ \alpha, \lambda > 0,$$

$$f_X(x;\lambda,\alpha) = \frac{\alpha}{\lambda x^2}\, g\left(\frac{1}{\lambda x}\right)\left[G\left(\frac{1}{\lambda x}\right)\right]^{\alpha-1}, \ x \geq 0, \ \alpha, \lambda > 0. \tag{1}$$

Some of the members of the above family are listed in the following Table 1.

Table 1 : *Some members of the generalized inverted scale family of distributions:*

| Sr. No. | Distribution | Generalized distribution (Y) and Generalized inverted distribution (X=1/Y) |
|---|---|---|
| 1 | **Scale family**<br>cdf= $G\left(\frac{y}{\lambda}\right)$<br>pdf= $\frac{1}{\lambda}g\left(\frac{y}{\lambda}\right)$ | **Generalized scale family**<br>$F_Y(y;\lambda,\alpha) = \left[G\left(\frac{y}{\lambda}\right)\right]^\alpha$<br>$f_Y(y;\lambda,\alpha) = \frac{\alpha}{\lambda}\, g\left(\frac{y}{\lambda}\right)\left[G\left(\frac{y}{\lambda}\right)\right]^{\alpha-1}$<br>**Generalized inverted scale family**<br>$F_X(x;\lambda,\alpha) = 1 - \left[G\left(\frac{1}{\lambda x}\right)\right]^\alpha$<br>$f_X(x;\lambda,\alpha) = \frac{\alpha}{\lambda x^2}\, g\left(\frac{1}{\lambda x}\right)\left[G\left(\frac{1}{\lambda x}\right)\right]^{\alpha-1}$ |
| 2 | **Exponential distribution**<br>cdf=$1 - e^{-y/\lambda}$<br>pdf=$\frac{1}{\lambda}e^{-y/\lambda}$ | **Generalized exponential distribution**<br>$F_Y(y;\lambda,\alpha) = \left[1 - e^{-y/\lambda}\right]^\alpha$<br>$f_Y(y;\lambda,\alpha) = \frac{\alpha}{\lambda}\, e^{-y/\lambda}\left[1 - e^{-y/\lambda}\right]^{\alpha-1}$<br>(Gupta and Kundu, 1999)<br>**Generalized inverted exponential distribution**<br>$F_X(x;\lambda,\alpha) = 1 - \left[1 - e^{-1/(\lambda x)}\right]^\alpha$<br>$f_X(x;\lambda,\alpha) = \frac{\alpha}{\lambda x^2}\, e^{-1/(\lambda x)}\left[1 - e^{-1/(\lambda x)}\right]^{\alpha-1}$<br>(Abouammoh and Alshingiti, 2009) |
| 3 | **Half-logistic distribution**<br>cdf=$\frac{1-e^{-y/\lambda}}{1+e^{-y/\lambda}}$<br>pdf=$\frac{2e^{-y/\lambda}}{\lambda\left(1+e^{-y/\lambda}\right)^2}$ | **Generalized half-logistic distribution**<br>$F_Y(y;\lambda,\alpha) = \left[\frac{1-e^{-y/\lambda}}{1+e^{-y/\lambda}}\right]^\alpha$<br>$f_Y(y;\lambda,\alpha) = \frac{2\alpha}{\lambda}\, e^{-y/\lambda}\frac{\left(1-e^{-y/\lambda}\right)^{(\alpha-1)}}{\left(1+e^{-y/\lambda}\right)^{(\alpha+1)}}$<br>**Generalized inverted half-logistic distribution**<br>$F_X(x;\lambda,\alpha) = 1 - \left[\frac{1-e^{-1/(\lambda x)}}{1+e^{-1/(\lambda x)}}\right]^\alpha$<br>$f_X(x;\lambda,\alpha) = \frac{2\alpha}{\lambda x^2}\, e^{-1/(\lambda x)}\frac{\left[1-e^{-1/(\lambda x)}\right]^{\alpha-1}}{\left[1+e^{-1/(\lambda x)}\right]^{\alpha+1}}$ |
| 4 | **Rayleigh distribution**<br>cdf=$1 - e^{-(y/\lambda)^2}$<br>pdf=$\frac{2y}{\lambda^2}e^{-(y/\lambda)^2}$ | **Generalized Rayleigh distribution**<br>$F_Y(y;\lambda,\alpha) = \left[1 - e^{-(y/\lambda)^2}\right]^\alpha$<br>$f_Y(y;\lambda,\alpha) = \frac{2\alpha y}{\lambda^2}\, e^{-(y/\lambda)^2}\left[1 - e^{-(y/\lambda)^2}\right]^{(\alpha-1)}$<br>**Generalized inverted Rayleigh distribution**<br>$F_X(x;\lambda,\alpha) = 1 - \left[1 - e^{-(1/(\lambda x))^2}\right]^\alpha$<br>$f_X(x;\lambda,\alpha) = \frac{2\alpha}{\lambda^2 x^3}\, e^{-(1/(\lambda x))^2}\left[1 - e^{-(1/(\lambda x))^2}\right]^{\alpha-1}$ |

In the following, we discuss method of finding MLE of $\alpha$ and $\lambda$.

*2.1. Maximum likelihood estimation*

Suppose we observe lifetimes of $n$ units having lifetime distribution given in equation (1). The likelihood function for the observed data is given by,

$$l(\underline{x}|\lambda,\alpha)=\prod_{i=1}^{n} f_X\left(x_i;\lambda,\alpha\right)=\prod_{i=1}^{n}\frac{\alpha}{\lambda x_i^2}\, g\left(\frac{1}{\lambda x_i}\right)\left[G\left(\frac{1}{\lambda x_i}\right)\right]^{\alpha-1}.$$

Then log-likelihood function is given by

$$L=n\log(\alpha)-n\log(\lambda)-2\sum_{i=1}^{n}\log(x_i)+\sum_{i=1}^{n}\log\left[g\left(\frac{1}{\lambda x_i}\right)\right]+(\alpha-1)\sum_{i=1}^{n}\log\left[G\left(\frac{1}{\lambda x_i}\right)\right]. \tag{2}$$

When $\lambda$ is known, the MLE of $\alpha$ is the solution of $\frac{dL}{d\alpha}=0$. Thus the MLE of $\alpha$ is the solution of the equation

$$\frac{n}{\alpha}+\sum_{i=1}^{n}\log\left[G\left(\frac{1}{\lambda x_i}\right)\right]=0. \tag{3}$$

Therefore, when $\lambda$ is known the MLE of $\alpha$ is

$$\hat{\alpha}=-\frac{n}{\sum_{i=1}^{n}\log\left[G\left(\frac{1}{\lambda x_i}\right)\right]}. \tag{4}$$

Similarly, when $\alpha$ is known, the MLE of $\lambda$ is the solution of $\frac{dL}{d\lambda}=0$. Thus the MLE of $\lambda$ is solution of the equation

$$-\frac{n}{\lambda}-\frac{1}{\lambda^2}\sum_{i=1}^{n}\frac{g'\left(\frac{1}{\lambda x_i}\right)}{x_i g\left(\frac{1}{\lambda x_i}\right)}-\frac{\alpha-1}{\lambda^2}\sum_{i=1}^{n}\frac{G'\left(\frac{1}{\lambda x_i}\right)}{x_i G\left(\frac{1}{\lambda x_i}\right)}=0. \tag{5}$$

When both parameters $\alpha$ and $\lambda$ are unknown, the MLEs of $\alpha$ and $\lambda$ are the solutions of the two simultaneous equations (3) and (5). We substitute $\hat{\alpha}$ in equation (4) into equation (5) so as to get a nonlinear equation in $\lambda$ only. Such nonlinear equation does not have closed form solution. Therefore, we use Newton-Raphson method to compute $\hat{\lambda}$. In Newton-Raphson method, we have to choose initial value of $\lambda$. We use least square estimate as initial value of $\lambda$. Ng (2005) discussed estimation of model parameters of modified Weibull distribution where the empirical distribution function is computed as (see Meeker and Escober, 1998)

$$\hat{F}\left(x_{(i)}\right)=1-\prod_{j=1}^{i}\left(1-\hat{p}_j\right)$$

with

$$\hat{p}_j=\frac{1}{n-j+1},\ \text{for } j=1,2,\ldots,m.$$

The estimate of the parameters can be obtained by least square fit of simple linear regression:

$$y_i=\beta x_{(i)},\ \text{with } \beta=\lambda,$$

$$y_i=\frac{1}{G^{-1}\left[\frac{\left(1-\hat{F}\left(x_{(i-1)}\right)\right)^{1/\alpha}+\left(1-\hat{F}\left(x_{(i)}\right)\right)^{1/\alpha}}{2}\right]},\ \text{for } i=1,2,\ldots,m,$$

$\hat{F}(x_{(0)}) = 0$.

The least square estimates of $\lambda$ is given by

$$\hat{\lambda}_0 = \frac{\sum_{i=1}^{m} x_{(i)} y_i}{\sum_{i=1}^{m} x_{(i)}^2}.$$

We use $\hat{\lambda}_0$ as an initial value of $\lambda$ to obtain the MLE $\hat{\lambda}$ using Newton-Raphson method. Then we obtain $\hat{\alpha}$ using equation (4). We use these MLEs $\hat{\alpha}$ and $\hat{\lambda}$ to obtain Confidence intervals for $\alpha$ and $\lambda$.

### 2.2. Fisher information matrix

Log-likelihood function $L$ is described by equation (2). Now, Fisher information matrix of $\theta = (\alpha, \lambda)'$ is

$$I(\theta) = -E \begin{bmatrix} \frac{d^2L}{d\alpha^2} & \frac{d^2L}{d\alpha d\lambda} \\ \frac{d^2L}{d\lambda d\alpha} & \frac{d^2L}{d\lambda^2} \end{bmatrix},$$

where

$$\frac{d^2L}{d\alpha^2} = -\frac{n}{\alpha^2},$$

$$\frac{d^2L}{d\alpha d\lambda} = \frac{d^2L}{d\lambda d\alpha} = -\frac{1}{\lambda^2} \sum_{i=1}^{n} \frac{G'\left(\frac{1}{\lambda x_i}\right)}{x_i G\left(\frac{1}{\lambda x_i}\right)},$$

$$\begin{aligned} \frac{d^2L}{d\lambda^2} &= \frac{n}{\lambda^2} + \frac{1}{\lambda^4} \sum_{i=1}^{n} \frac{g\left(\frac{1}{\lambda x_i}\right) g''\left(\frac{1}{\lambda x_i}\right) - \left[g'\left(\frac{1}{\lambda x_i}\right)\right]^2 + 2\lambda x_i g\left(\frac{1}{\lambda x_i}\right) g'\left(\frac{1}{\lambda x_i}\right)}{\left[x_i g\left(\frac{1}{\lambda x_i}\right)\right]^2} \\ &\quad + \frac{\alpha - 1}{\lambda^4} \sum_{i=1}^{n} \frac{G\left(\frac{1}{\lambda x_i}\right) G''\left(\frac{1}{\lambda x_i}\right) - \left[G'\left(\frac{1}{\lambda x_i}\right)\right]^2 + 2\lambda x_i G\left(\frac{1}{\lambda x_i}\right) G'\left(\frac{1}{\lambda x_i}\right)}{\left[x_i G\left(\frac{1}{\lambda x_i}\right)\right]^2}, \end{aligned}$$

and

$$g'(\cdot) = \frac{d}{d\lambda} g(\cdot),\ G'(\cdot) = \frac{d}{d\lambda} G(\cdot),\ g''(\cdot) = \frac{d^2}{d\lambda^2} g(\cdot) \text{ and } G''(\cdot) = \frac{d^2}{d\lambda^2} G(\cdot).$$

To obtain expectation of the above expression is tedious job. Therefore, we use the observed Fisher information matrix which is given by

$$I(\underline{\hat{\theta}}) = \begin{bmatrix} -\frac{d^2L}{d\alpha^2} & -\frac{d^2L}{d\alpha d\lambda} \\ -\frac{d^2L}{d\lambda d\alpha} & -\frac{d^2L}{d\lambda^2} \end{bmatrix}_{\alpha=\hat{\alpha}, \lambda=\hat{\lambda}.}$$

The asymptotic variance-covariance matrix of the MLEs is the inverse of $I(\underline{\hat{\theta}})$. After obtaining inverse matrix, we get variance of $\hat{\alpha}$ and variance of $\hat{\lambda}$. We use these terms to obtain confidence intervals for $\alpha$ and $\lambda$ respectively.

*2.3. Confidence interval*

Assuming asymptotic normal distribution for the MLEs, CIs for $\alpha$ and $\lambda$ are constructed. Let $\hat{\alpha}$ and $\hat{\lambda}$ are the MLE of $\alpha$ and $\lambda$ respectively. Let $\hat{\sigma}^2(\hat{\alpha})$ and $\hat{\sigma}^2(\hat{\lambda})$ is the estimated variances of $\hat{\alpha}$ and $\hat{\lambda}$ respectively. Therefore, $100(1-\xi)\%$ asymptotic CIs for $\alpha$ and $\lambda$ are respectively given by,

$$\left(\hat{\alpha} - \tau_{\xi/2}\sqrt{\hat{\sigma}^2(\hat{\alpha})}, \hat{\alpha} + \tau_{\xi/2}\sqrt{\hat{\sigma}^2(\hat{\alpha})}\right) \text{ and } \left(\hat{\lambda} - \tau_{\xi/2}\sqrt{\hat{\sigma}^2(\hat{\lambda})}, \hat{\lambda} + \tau_{\xi/2}\sqrt{\hat{\sigma}^2(\hat{\lambda})}\right), \tag{6}$$

where $\tau_{\xi/2}$ is the upper $100(1-\xi)^{th}$ percentile of standard normal distribution. Meeker and Escober (1998) reported that the asymptotic CI based on log(MLE) has better coverage probability. An approximate $100(1-\xi)\%$ CI for $\log(\alpha)$ and $\log(\lambda)$ are

$$\left(\log(\hat{\alpha}) - \tau_{\xi/2}\sqrt{\hat{\sigma}^2(\log(\hat{\alpha}))}, \log(\hat{\alpha}) + \tau_{\xi/2}\sqrt{\hat{\sigma}^2(\log(\hat{\alpha}))}\right)$$

and

$$\left(\log(\hat{\lambda}) - \tau_{\xi/2}\sqrt{\hat{\sigma}^2(\log(\hat{\lambda}))}, \log(\hat{\lambda}) + \tau_{\xi/2}\sqrt{\hat{\sigma}^2(\log(\hat{\lambda}))}\right),$$

where $\hat{\sigma}^2(\log(\hat{\alpha}))$ is the estimated variance of $\log(\hat{\alpha})$ which is approximately obtained by $\hat{\sigma}^2(\log(\hat{\alpha})) \approx \frac{\hat{\sigma}^2(\hat{\alpha})}{\hat{\alpha}^2}$. $\hat{\sigma}^2(\log(\hat{\lambda}))$ is the estimated variance of $\log(\hat{\lambda})$ which is approximately obtained by $\hat{\sigma}^2(\log(\hat{\lambda})) \approx \frac{\hat{\sigma}^2(\hat{\lambda})}{\hat{\lambda}^2}$. Hence, an approximate $100(1-\xi)\%$ CIs for $\alpha$ and $\lambda$ are respectively given by,

$$\left(\hat{\alpha}e^{\left(-\frac{\tau_{\xi/2}\sqrt{\hat{\sigma}^2(\hat{\alpha})}}{\hat{\alpha}}\right)}, \hat{\alpha}e^{\left(\frac{\tau_{\xi/2}\sqrt{\hat{\sigma}^2(\hat{\alpha})}}{\hat{\alpha}}\right)}\right) \text{ and } \left(\hat{\lambda}e^{\left(-\frac{\tau_{\xi/2}\sqrt{\hat{\sigma}^2(\hat{\lambda})}}{\hat{\lambda}}\right)}, \hat{\lambda}e^{\left(\frac{\tau_{\xi/2}\sqrt{\hat{\sigma}^2(\hat{\lambda})}}{\hat{\lambda}}\right)}\right).$$

We use above discussed inferential procedures of generalized inverted scale family to generalized inverted half-logistic distribution (GIHD).

## 3. Application to generalized inverted half-logistic distribution

Consider a member of the scale family of distributions, namely half-logistic distribution with scale parameter $\lambda$. Let $X$ be generalized inverted half-logistic random variable. The cdf and pdf of $X$ are respectively

$$F_X(x) = 1 - \left[\frac{1-e^{-1/(\lambda x)}}{1+e^{-1/(\lambda x)}}\right]^{\alpha}, \ x \geq 0, \ \alpha, \lambda > 0,$$

and

$$f_X(x) = \frac{2\alpha}{\lambda x^2} e^{-1/(\lambda x)} \frac{\left(1-e^{-1/(\lambda x)}\right)^{\alpha-1}}{\left(1+e^{-1/(\lambda x)}\right)^{\alpha+1}}, \ x \geq 0, \ \alpha, \lambda > 0.$$

*3.1. Maximum likelihood estimation*

The log-likelihood function for generalized inverted half-logistic distribution from equation (2) is

$$\begin{aligned} L &= n\log(2) + n\log(\alpha) - n\log(\lambda) - 2\sum_{i=1}^{n}\log(x_i) - \frac{1}{\lambda}\sum_{i=1}^{n}\frac{1}{x_i} + (\alpha-1)\sum_{i=1}^{n}\log\left[1-e^{-1/(\lambda x_i)}\right] \\ &\quad -(\alpha+1)\sum_{i=1}^{n}\log\left[1+e^{-1/(\lambda x_i)}\right], \end{aligned}$$

and

$$\frac{dL}{d\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \log\left[1 - e^{-1/(\lambda x_i)}\right] - \sum_{i=1}^{n} \log\left[1 + e^{-1/(\lambda x_i)}\right] = 0. \tag{7}$$

When $\lambda$ is known, the MLE of $\alpha$ is given by,

$$\hat{\alpha} = \frac{n}{\sum_{i=1}^{n} \log\left(1 + e^{-1/(\lambda x_i)}\right) - \sum_{i=1}^{n} \log\left(1 - e^{-1/(\lambda x_i)}\right)}. \tag{8}$$

Consider

$$\frac{dL}{d\lambda} = -\frac{n}{\lambda} + \frac{1}{\lambda^2}\sum_{i=1}^{n}\frac{1}{x_i} - \frac{(\alpha-1)}{\lambda^2}\sum_{i=1}^{n}\frac{e^{-1/(\lambda x_i)}}{x_i\left(1 - e^{-1/(\lambda x_i)}\right)} - \frac{(\alpha+1)}{\lambda^2}\sum_{i=1}^{n}\frac{e^{-1/(\lambda x_i)}}{x_i\left(1 + e^{-1/(\lambda x_i)}\right)} = 0. \tag{9}$$

When $\alpha$ is known, the MLE of $\lambda$ is solution of the equation $\frac{dL}{d\lambda} = 0$. When both parameters $\alpha$ and $\lambda$ are unknown, the MLEs of $\alpha$ and $\lambda$ are the solutions of the two simultaneous equations (7) and (9). We substitute $\hat{\alpha}$ in equation (8) into equation (9), we get

$$\begin{aligned} 0 \quad = \quad & -\frac{n}{\lambda} + \frac{1}{\lambda^2}\sum_{i=1}^{n}\frac{1}{x_i} - \frac{n\left[\sum_{i=1}^{n}\frac{e^{-1/(\lambda x_i)}}{x_i\left(1-e^{-1/(\lambda x_i)}\right)} + \sum_{i=1}^{n}\frac{e^{-1/(\lambda x_i)}}{x_i\left(1+e^{-1/(\lambda x_i)}\right)}\right]}{\lambda^2\left[\sum_{i=1}^{n}\log\left(1 + e^{-1/(\lambda x_i)}\right) - \sum_{i=1}^{n}\log\left(1 - e^{-1/(\lambda x_i)}\right)\right]} \\ & + \frac{1}{\lambda^2}\left[\sum_{i=1}^{n}\frac{e^{-1/(\lambda x_i)}}{x_i\left(1 - e^{-1/(\lambda x_i)}\right)} - \sum_{i=1}^{n}\frac{e^{-1/(\lambda x_i)}}{x_i\left(1 + e^{-1/(\lambda x_i)}\right)}\right]. \end{aligned}$$

This is nonlinear equation in $\lambda$ only, which does not have closed form solution. Therefore, we use Newton-Raphson method to compute $\hat{\lambda}$. We replace $\lambda$ by MLE $\hat{\lambda}$ in equation (8), we get MLE of $\alpha$.

*3.2. Fisher information matrix*

Taking second derivatives of log-likelihood function L, we have

$$\frac{d^2L}{d\alpha^2} = -\frac{n}{\alpha^2},$$

$$\frac{d^2L}{d\alpha d\lambda} = \frac{d^2L}{d\lambda d\alpha} = -\frac{1}{\lambda^2}\sum_{i=1}^{n}\frac{e^{-1/(\lambda x_i)}}{x_i\left(1 - e^{-1/(\lambda x_i)}\right)} - \frac{1}{\lambda^2}\sum_{i=1}^{n}\frac{e^{-1/(\lambda x_i)}}{x_i\left(1 + e^{-1/(\lambda x_i)}\right)},$$

$$\begin{aligned} \frac{d^2L}{d\lambda^2} \quad = \quad & \frac{n}{\lambda^2} - \frac{2}{\lambda^3}\sum_{i=1}^{n}\frac{1}{x_i} - \frac{(\alpha-1)}{\lambda^4}\sum_{i=1}^{n}\frac{e^{-1/(\lambda x_i)}\left[1 - 2\lambda x_i\left(1 - e^{-1/(\lambda x_i)}\right)\right]}{x_i^2\left(1 - e^{-1/(\lambda x_i)}\right)^2} \\ & - \frac{(\alpha+1)}{\lambda^4}\sum_{i=1}^{n}\frac{e^{-1/(\lambda x_i)}\left[1 - 2\lambda x_i\left(1 + e^{-1/(\lambda x_i)}\right)\right]}{x_i^2\left(1 + e^{-1/(\lambda x_i)}\right)^2}. \end{aligned}$$

Using these expressions in (6), we obtain observed Fisher information matrix.

## 4. Performance Study

A simulation study is carried out to study the performance of MLEs of $\alpha$ and $\lambda$ when the GIHD as lifetime distribution. We consider bias and mean square error (MSE) to compare MLEs. Asymptotic CIs based on MLEs and log-transformed MLEs are compared with their confidence levels.

Simulation is carried out for ($\alpha$, $\lambda$)=$(0.5, 0.5)$, $(0.5, 1)$, $(1, 0.5)$, $(1, 1)$ with sample size $n = 20, \ldots, 100$. Newton-Raphson method is used to compute MLE of $\lambda$. For each sample size, 10000 sets of observations were generated. The MLE, bias and MSE of $\hat{\alpha}$ and $\hat{\lambda}$ are displayed in the Tables 2 to 5 for various values of $\alpha$ and $\lambda$. Further, the confidence levels and lengths for the CIs based on MLE and log(MLE) of $\alpha$ are given in Tables 6 to 9 and the confidence levels and lengths for the CIs based on MLE and log(MLE) of $\lambda$ are given in Tables 10 to 13.

Table-2: *MLE, Bias, MSE of* $\hat{\alpha}$ *and* $\hat{\lambda}$ *when* $\alpha = 0.5$ *and* $\lambda = 0.5$.

| | MLE of $\alpha$ | | | MLE of $\lambda$ | | |
|---|---|---|---|---|---|---|
| $n$ | $\hat{\alpha}$ | Bias | MSE | $\hat{\lambda}$ | Bias | MSE |
| 20 | 0.5558 | 0.0558 | 0.0294 | 0.4850 | −0.0150 | 0.0214 |
| 30 | 0.5378 | 0.0378 | 0.0165 | 0.4875 | −0.0125 | 0.0143 |
| 40 | 0.5264 | 0.0264 | 0.0113 | 0.4932 | −0.0068 | 0.0110 |
| 50 | 0.5225 | 0.0225 | 0.0083 | 0.4924 | −0.0076 | 0.0085 |
| 60 | 0.5164 | 0.0164 | 0.0066 | 0.4953 | −0.0047 | 0.0071 |
| 80 | 0.5131 | 0.0131 | 0.0047 | 0.4952 | −0.0048 | 0.0054 |
| 100 | 0.5090 | 0.0090 | 0.0036 | 0.4974 | −0.0026 | 0.0043 |

Table-3: *MLE, Bias, MSE of* $\hat{\alpha}$ *and* $\hat{\lambda}$ *when* $\alpha = 0.5$ *and* $\lambda = 1$.

| | MLE of $\alpha$ | | | MLE of $\lambda$ | | |
|---|---|---|---|---|---|---|
| $n$ | $\hat{\alpha}$ | Bias | MSE | $\hat{\lambda}$ | Bias | MSE |
| 20 | 0.5553 | 0.0553 | 0.0301 | 0.9682 | −0.0318 | 0.0846 |
| 30 | 0.5352 | 0.0352 | 0.0161 | 0.9808 | −0.0192 | 0.0564 |
| 40 | 0.5263 | 0.0263 | 0.0110 | 0.9884 | −0.0116 | 0.0428 |
| 50 | 0.5203 | 0.0203 | 0.0083 | 0.9873 | −0.0127 | 0.0349 |
| 60 | 0.5175 | 0.0175 | 0.0065 | 0.9874 | −0.0126 | 0.0279 |
| 80 | 0.5126 | 0.0126 | 0.0047 | 0.9915 | −0.0085 | 0.0215 |
| 100 | 0.5110 | 0.0110 | 0.0037 | 0.9918 | −0.0082 | 0.0173 |

Table-4: *MLE, Bias, MSE of* $\hat{\alpha}$ *and* $\hat{\lambda}$ *when* $\alpha = 1$ *and* $\lambda = 0.5$.

| | MLE of $\alpha$ | | | MLE of $\lambda$ | | |
|---|---|---|---|---|---|---|
| $n$ | $\hat{\alpha}$ | Bias | MSE | $\hat{\lambda}$ | Bias | MSE |
| 20 | 1.1481 | 0.1481 | 0.1864 | 0.4852 | −0.0148 | 0.0141 |
| 30 | 1.0925 | 0.0925 | 0.0890 | 0.4893 | −0.0107 | 0.0093 |
| 40 | 1.0680 | 0.0680 | 0.0595 | 0.4921 | −0.0079 | 0.0071 |
| 50 | 1.0526 | 0.0526 | 0.0445 | 0.4948 | −0.0052 | 0.0057 |
| 60 | 1.0436 | 0.0436 | 0.0348 | 0.4955 | −0.0045 | 0.0047 |
| 80 | 1.0330 | 0.0330 | 0.0240 | 0.4961 | −0.0039 | 0.0035 |
| 100 | 1.0237 | 0.0237 | 0.0185 | 0.4975 | −0.0025 | 0.0027 |

Table-5: *MLE, Bias, MSE of* $\hat{\alpha}$ *and* $\hat{\lambda}$ *when* $\alpha = 1$ *and* $\lambda = 1$.

| | MLE of $\alpha$ | | | MLE of $\lambda$ | | |
|---|---|---|---|---|---|---|
| $n$ | $\hat{\alpha}$ | Bias | MSE | $\hat{\lambda}$ | Bias | MSE |
| 20 | 1.1439 | 0.1439 | 0.1968 | 0.9753 | −0.0247 | 0.0554 |
| 30 | 1.0895 | 0.0895 | 0.0915 | 0.9806 | −0.0194 | 0.0370 |
| 40 | 1.0645 | 0.0645 | 0.0575 | 0.9891 | −0.0109 | 0.0282 |
| 50 | 1.0522 | 0.0522 | 0.0440 | 0.9880 | −0.0120 | 0.0223 |
| 60 | 1.0428 | 0.0428 | 0.0342 | 0.9884 | −0.0116 | 0.0190 |
| 80 | 1.0322 | 0.0322 | 0.0251 | 0.9926 | −0.0074 | 0.0143 |
| 100 | 1.0252 | 0.0252 | 0.0186 | 0.9940 | −0.0060 | 0.0113 |

Table-6: *Confidence levels and lengths of CIs for $\alpha$ when $\alpha = 0.5$ and $\lambda = 0.5$.*

| $n$ | MLE(90%) | | MLE(95%) | | Log-MLE(90%) | | Log-MLE(95%) | |
|---|---|---|---|---|---|---|---|---|
| | Level | Length | Level | Length | Level | Length | Level | Length |
| 20 | 0.9167 | 0.4773 | 0.9625 | 0.5688 | 0.8878 | 0.4922 | 0.9417 | 0.5941 |
| 30 | 0.9113 | 0.3744 | 0.9589 | 0.4461 | 0.8884 | 0.3820 | 0.9401 | 0.4590 |
| 40 | 0.9044 | 0.3160 | 0.9568 | 0.3766 | 0.8910 | 0.3208 | 0.9390 | 0.3847 |
| 50 | 0.9086 | 0.2802 | 0.9577 | 0.3338 | 0.8925 | 0.2835 | 0.9439 | 0.3395 |
| 60 | 0.9054 | 0.2523 | 0.9542 | 0.3006 | 0.8952 | 0.2548 | 0.9461 | 0.3048 |
| 80 | 0.9045 | 0.2168 | 0.9524 | 0.2583 | 0.8960 | 0.2184 | 0.9467 | 0.2610 |
| 100 | 0.9049 | 0.1921 | 0.9553 | 0.2289 | 0.9018 | 0.1932 | 0.9507 | 0.2308 |

Table-7: *Confidence levels and lengths of CIs for $\alpha$ when $\alpha = 0.5$ and $\lambda = 1$.*

| $n$ | MLE(90%) | | MLE(95%) | | Log-MLE(90%) | | Log-MLE(95%) | |
|---|---|---|---|---|---|---|---|---|
| | Level | Length | Level | Length | Level | Length | Level | Length |
| 20 | 0.9153 | 0.4771 | 0.9628 | 0.5685 | 0.8897 | 0.4920 | 0.9421 | 0.5938 |
| 30 | 0.9123 | 0.3724 | 0.9598 | 0.4437 | 0.8914 | 0.3800 | 0.9446 | 0.4566 |
| 40 | 0.9057 | 0.3160 | 0.9563 | 0.3765 | 0.8901 | 0.3207 | 0.9443 | 0.3846 |
| 50 | 0.9032 | 0.2789 | 0.9548 | 0.3323 | 0.8903 | 0.2822 | 0.9428 | 0.3380 |
| 60 | 0.9081 | 0.2529 | 0.9556 | 0.3013 | 0.8989 | 0.2554 | 0.9480 | 0.3056 |
| 80 | 0.9053 | 0.2165 | 0.9515 | 0.2580 | 0.8963 | 0.2182 | 0.9450 | 0.2608 |
| 100 | 0.9028 | 0.1929 | 0.9547 | 0.2299 | 0.8943 | 0.1941 | 0.9475 | 0.2318 |

Table-8: *Confidence levels and lengths of CIs for $\alpha$ when $\alpha = 1$ and $\lambda = 0.5$.*

| $n$ | MLE(90%) | | MLE(95%) | | Log-MLE(90%) | | Log-MLE(95%) | |
|---|---|---|---|---|---|---|---|---|
| | Level | Length | Level | Length | Level | Length | Level | Length |
| 20 | 0.9272 | 1.1189 | 0.9656 | 1.3333 | 0.8848 | 1.1647 | 0.9414 | 1.4111 |
| 30 | 0.9179 | 0.8536 | 0.9627 | 1.0172 | 0.8908 | 0.8757 | 0.9433 | 1.0547 |
| 40 | 0.9141 | 0.7173 | 0.9589 | 0.8547 | 0.8909 | 0.7310 | 0.9454 | 0.8779 |
| 50 | 0.9090 | 0.6297 | 0.9554 | 0.7503 | 0.8950 | 0.6391 | 0.9444 | 0.7663 |
| 60 | 0.9048 | 0.5683 | 0.9534 | 0.6772 | 0.8916 | 0.5754 | 0.9463 | 0.6892 |
| 80 | 0.9109 | 0.4856 | 0.9554 | 0.5786 | 0.9023 | 0.4901 | 0.9486 | 0.5863 |
| 100 | 0.9045 | 0.4294 | 0.9546 | 0.5116 | 0.8986 | 0.4325 | 0.9496 | 0.5170 |

Table-9: *Confidence levels and lengths of CIs for $\alpha$ when $\alpha = 1$ and $\lambda = 1$.*

| $n$ | MLE(90%) | | MLE(95%) | | Log-MLE(90%) | | Log-MLE(95%) | |
|---|---|---|---|---|---|---|---|---|
| | Level | Length | Level | Length | Level | Length | Level | Length |
| 20 | 0.9260 | 1.1148 | 0.9670 | 1.3283 | 0.8838 | 1.1605 | 0.9423 | 1.4061 |
| 30 | 0.9121 | 0.8511 | 0.9595 | 1.0142 | 0.8891 | 0.8731 | 0.9398 | 1.0515 |
| 40 | 0.9126 | 0.7147 | 0.9594 | 0.8516 | 0.8933 | 0.7283 | 0.9453 | 0.8747 |
| 50 | 0.9102 | 0.6294 | 0.9557 | 0.7500 | 0.8916 | 0.6389 | 0.9458 | 0.7660 |
| 60 | 0.9099 | 0.5677 | 0.9550 | 0.6765 | 0.8927 | 0.5748 | 0.9473 | 0.6885 |
| 80 | 0.9035 | 0.4853 | 0.9541 | 0.5782 | 0.8947 | 0.4898 | 0.9469 | 0.5858 |
| 100 | 0.9067 | 0.4302 | 0.9553 | 0.5126 | 0.8973 | 0.4333 | 0.9497 | 0.5179 |

Table-10: *Confidence levels and lengths of CIs for $\lambda$ when $\alpha = 0.5$ and $\lambda = 0.5$.*

| $n$ | MLE(90%) | | MLE(95%) | | Log-MLE(90%) | | Log-MLE(95%) | |
|---|---|---|---|---|---|---|---|---|
| | Level | Length | Level | Length | Level | Length | Level | Length |
| 20 | 0.8376 | 0.4667 | 0.8785 | 0.5561 | 0.8709 | 0.4854 | 0.9194 | 0.5880 |
| 30 | 0.8512 | 0.3832 | 0.8967 | 0.4566 | 0.8783 | 0.3933 | 0.9264 | 0.4738 |
| 40 | 0.8689 | 0.3364 | 0.9095 | 0.4008 | 0.8793 | 0.3430 | 0.9360 | 0.4121 |
| 50 | 0.8729 | 0.3003 | 0.9162 | 0.3579 | 0.8864 | 0.3051 | 0.9394 | 0.3659 |
| 60 | 0.8798 | 0.2764 | 0.9231 | 0.3294 | 0.8902 | 0.2801 | 0.9403 | 0.3356 |
| 80 | 0.8850 | 0.2392 | 0.9252 | 0.2850 | 0.8909 | 0.2415 | 0.9412 | 0.2889 |
| 100 | 0.8867 | 0.2152 | 0.9359 | 0.2564 | 0.8925 | 0.2169 | 0.9467 | 0.2593 |

Table-11: *Confidence levels and lengths of CIs for $\lambda$ when $\alpha = 0.5$ and $\lambda = 1$.*

| $n$ | MLE(90%) | | MLE(95%) | | Log-MLE(90%) | | Log-MLE(95%) | |
|---|---|---|---|---|---|---|---|---|
| | Level | Length | Level | Length | Level | Length | Level | Length |
| 20 | 0.8390 | 0.9313 | 0.8749 | 1.1098 | 0.8691 | 0.9687 | 0.9220 | 1.1733 |
| 30 | 0.8650 | 0.7723 | 0.9008 | 0.9202 | 0.8810 | 0.7928 | 0.9320 | 0.9550 |
| 40 | 0.8738 | 0.6743 | 0.9132 | 0.8035 | 0.8862 | 0.6876 | 0.9375 | 0.8261 |
| 50 | 0.8751 | 0.6032 | 0.9175 | 0.7188 | 0.8851 | 0.6128 | 0.9380 | 0.7350 |
| 60 | 0.8827 | 0.5504 | 0.9265 | 0.6558 | 0.8927 | 0.5576 | 0.9423 | 0.6681 |
| 80 | 0.8824 | 0.4792 | 0.9298 | 0.5710 | 0.8925 | 0.4839 | 0.9400 | 0.5790 |
| 100 | 0.8845 | 0.4285 | 0.9305 | 0.5106 | 0.8919 | 0.4319 | 0.9402 | 0.5163 |

Table-12: *Confidence levels and lengths of CIs for $\lambda$ when $\alpha = 1$ and $\lambda = 0.5$.*

| $n$ | MLE(90%) | | MLE(95%) | | Log-MLE(90%) | | Log-MLE(95%) | |
|---|---|---|---|---|---|---|---|---|
| | Level | Length | Level | Length | Level | Length | Level | Length |
| 20 | 0.8475 | 0.3773 | 0.8881 | 0.4496 | 0.8691 | 0.3871 | 0.9243 | 0.4662 |
| 30 | 0.8661 | 0.3106 | 0.9079 | 0.3701 | 0.8810 | 0.3159 | 0.9306 | 0.3791 |
| 40 | 0.8717 | 0.2706 | 0.9174 | 0.3224 | 0.8824 | 0.2740 | 0.9357 | 0.3283 |
| 50 | 0.8789 | 0.2436 | 0.9257 | 0.2903 | 0.8874 | 0.2461 | 0.9397 | 0.2945 |
| 60 | 0.8813 | 0.2227 | 0.9273 | 0.2653 | 0.8881 | 0.2246 | 0.9402 | 0.2686 |
| 80 | 0.8889 | 0.1930 | 0.9342 | 0.2300 | 0.8943 | 0.1943 | 0.9448 | 0.2321 |
| 100 | 0.8935 | 0.1733 | 0.9387 | 0.2065 | 0.8961 | 0.1741 | 0.9469 | 0.2079 |

Table-13: *Confidence levels and lengths of CIs for $\lambda$ when $\alpha = 1$ and $\lambda = 1$.*

| $n$ | MLE(90%) | | MLE(95%) | | Log-MLE(90%) | | Log-MLE(95%) | |
|---|---|---|---|---|---|---|---|---|
| | Level | Length | Level | Length | Level | Length | Level | Length |
| 20 | 0.8578 | 0.7590 | 0.8980 | 0.9044 | 0.8793 | 0.7787 | 0.9295 | 0.9378 |
| 30 | 0.8659 | 0.6230 | 0.9099 | 0.7424 | 0.8835 | 0.6337 | 0.9333 | 0.7605 |
| 40 | 0.8803 | 0.5446 | 0.9212 | 0.6489 | 0.8876 | 0.5516 | 0.9391 | 0.6608 |
| 50 | 0.8792 | 0.4863 | 0.9233 | 0.5795 | 0.8866 | 0.4913 | 0.9388 | 0.5879 |
| 60 | 0.8787 | 0.4442 | 0.9254 | 0.5293 | 0.8885 | 0.4479 | 0.9403 | 0.5357 |
| 80 | 0.8821 | 0.3864 | 0.9271 | 0.4605 | 0.8860 | 0.3889 | 0.9395 | 0.4646 |
| 100 | 0.8870 | 0.3461 | 0.9349 | 0.4124 | 0.8915 | 0.3479 | 0.9459 | 0.4154 |

Bias, MSE of MLEs of various values of $\alpha$ and $\lambda$ are reported in Tables 2 to 5. For small values of $\alpha$ and $\lambda$, MLEs show better performance. The bias and MSE of the estimates are relatively smaller for small value of parameter. The Bias and MSE of estimates of $\alpha$ are not affected due to different values of $\lambda$. Similarly, bias and MSE of estimates of $\lambda$ are not affected for different values of $\alpha$. The bias and MSE of the MLEs decrease with increase in sample size $n$.

Confidence levels and lengths of CIs of $\alpha$ for various values of $\alpha$ and $\lambda$ are reported in Tables 6 to 9. When sample size is small, lengths and levels of CIs based on MLE are moderately large. Increase in sample size reduces the lengths of CIs and confidence levels approach to nominal levels. Confidence levels of CIs based on log-transformed MLE increase and lengths of CIs decrease as sample size increases.

Confidence levels and lengths of CIs of $\lambda$ for various values of $\alpha$ and $\lambda$ are reported in Tables 10 to 13. When sample size is small, CIs based on MLE show poor confidence level as compared to CIs based on log-transformed MLE. When sample size is large confidence levels of CIs based on log- transformed MLE are close to nominal levels. In both MLE and log-transformed MLE case, increase in sample size considerably reduces the length of CIs and increases in confidence levels.

## 5. Real Data Example

Lawless (1982) provided real data, which represents the number of million revolutions before failure for each of 23 ball bearings in a life test: 17.88, 28.92, 33, 41.52, 42.12, 45.6, 48.4, 51.84, 51.96, 54.12, 55.56, 67.8, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.4. Krishna and Kumar (2012) considered this data and fitted different five models namely exponential, inverted exponential, weibull, gamma and generalized inverted exponential distribution. To test the goodness of fit, they considered four criteria such as (i) negative log-likelihood (ii) Kolmogorov- Smirnov (K-S) statistic (iii) Akaike's information criterion (AIC) (iv) Bayesian information criterion (BIC). In short AIC and BIC are respectively given by,

$$AIC = 2k - 2\log(L) \text{ and } BIC = k\log(n) - 2\log(L),$$

where $k$ is the number of parameters in the reliability model, $L$ is the maximized value of the likelihood function for the estimated model and $n$ is the number of the observations in the given data set.

The maximum likelihood estimators of parameters, values of negative log-likelihood, AIC, BIC and K-S values are presented in Table 14.

Table-14: *The MLEs, negative log-likelihood, AIC, BIC and K-S value.*

| Sr.No. | Distribution | MLEs | $-LogL$ | AIC | BIC | K-S |
|---|---|---|---|---|---|---|
| 1 | GIHD($\lambda$, $\alpha$) | $\hat\lambda$=0.007166<br>$\hat\alpha$=3.3735 | 113.8679 | 231.7358 | 234.0068 | 0.06086 |
| 2 | GIED($\lambda$, $\alpha$) | $\hat\lambda$=129.9959<br>$\hat\alpha$=5.3076 | 113.5490 | 231.0980 | 233.3690 | 0.0703 |
| 3 | IED($\lambda$) | $\hat\lambda$=55.0551 | 121.7259 | 245.4519 | 246.5874 | 0.3060 |
| 4 | Exp ($\theta$) | $\hat\theta$=0.0138 | 121.4338 | 244.8675 | 246.0030 | 0.2622 |
| 5 | Gamma($\alpha$, $\beta$) | $\hat\alpha$=0.0557<br>$\hat\beta$=4.0244 | 113.0298 | 230.0596 | 232.3306 | 0.1233 |
| 6 | Weibull($\alpha$, $\beta$) | $\hat\alpha$=2.1018<br>$\hat\beta$=81.8745 | 113.6920 | 231.3839 | 233.6549 | 0.1510 |

According to K-S test for goodness of fit, the order of best fit among above six models is given by

**Best**: GIHD→GIED→Gamma→Weibull→Exponential→IED :**Worst**

According to negative log-likelihood criterion, AIC and BIC for goodness of fit, the order of best fit among above six models is given by

**Best**: Gamma→GIED→Weibull→GIHD→Exponential→IED :**Worst**

For this real data set, we construct confidence intervals based on MLE and log-transformed MLE of $\alpha$ and $\lambda$. The confidence intervals of $\alpha$ and $\lambda$ and its lengths are displayed in Table 15 and Table 16 respectively.

Table-15: *MLE, Confidence intervals for $\alpha$ and their lengths.*

| MLE | Based on MLE | | Based on log-MLE | |
|---|---|---|---|---|
| | 90% CI | 95% CI | 90% CI | 95% CI |
| 3.3735 | (1.3328, 5.4141)<br>Length=4.0813 | (0.9418, 5.8051)<br>Length=4.8633 | (1.8423, 6.1772)<br>Length=4.3349 | (1.6407, 6.9362)<br>Length=5.2955 |

Table-16: *MLE, Confidence intervals for $\lambda$ and their lengths.*

| MLE | Based on MLE | | Based on log-MLE | |
|---|---|---|---|---|
| | 90% CI | 95% CI | 90% CI | 95% CI |
| 0.0072 | (0.0050, 0.0093) | (0.0046, 0.0098) | (0.0053, 0.0097) | (0.0050, 0.0103) |
| | Length=0.0043 | Length=0.0052 | Length=0.0044 | Length=0.0053 |

## 6. Conclusion

This article introduces a generalized inverted scale family of distributions having scale and shape parameters. Point and interval estimation procedures for the parameters of the family are discussed. As a member of family, GIHD is considered and through simulation study, performance of estimators and confidence intervals are studied. In this study, both MLE and CI of parameters give better performance. Expressions given in this article can also be used for generalized inverted exponential distribution, generalized inverted Rayleigh distribution etc.

## Acknowledgement

The first author wishes to thank University Grants Commission, New Delhi, India for providing Fellowship under Faculty Improvement Programme to carry out this research.

## References

Abouammoh, A.M., Alshingiti, A.M. (2009) Reliability estimation of Generalized inverted exponential distribution, J. Statist. Comput. Simul. 79(11), 1301–1315.

Dey, S. (2007) Inverted exponential distribution as a life distribution model from a Bayesian viewpoint, Data Sci. J. 6, 107–113.

Gupta, R.D., Kundu, D. (1999) Generalized exponential distribution, Aust. N. Z. J. Statist. 41(2), 173–183.

Gupta, R.D., Kundu, D. (2001a) Exponentiated exponential distribution: An alternative to gamma and Weibull distributions, Biomet. J. 43(1), 117–130.

Gupta, R.D., Kundu, D. (2001b) Generalized exponential distribution: Different methods of estimations, J. Statist. Comput. Simul. 69(4), 315–338.

Krishna, H., Kumar, K. (2013) Reliability estimation in generalized inverted exponential distribution with progressively type II censored sample, J. Statist. Comput. Simul. 83(6), 1007–1019.

Lawless, J.F. (1982) Statistical models and methods for lifetime data, John Wiley and Sons, New York.

Lin, C.T., Duran, B.S., Lewis, T.O. (1989) Inverted gamma as life distribution, Microelectron Reliab. 29(4), 619–626.

Meeker, W.Q., Escober, L.A. (1998) Statistical Methods for Reliability Data, John Wiley and Sons, New York.

Ng, H.K.T. (2005) Parameter estimation for a modified Weibull distribution for progressively Type-II censored samples, IEEE Trans. Reliab. 54, 374–380.

Potdar, K.G., Shirke, D.T. (2012) Inference for the scale parameter of lifetime distribution of $k$-unit parallel system based on progressively censored data, J. Statist. Comput. Simul., DOI: 10.1080/00949655.2012.700314.

Singh, S.K., Singh., U., Kumar, D. (2012) Bayes estimators of the reliability function and parameter of inverted exponential distribution using informative and non-informative priors, J. Statist. Comput. Simul., DOI: 10.1080/00949655.2012.690156.