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Unbiased confidence intervals for distributions involving truncation parameter

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Abstract. Using the pivotal quantity method, a general solution is presented in this paper to obtain an unbiased confidence interval for families of distributions involving truncation parameter. Also, we show that for these families of distributions the unbiased confidence interval is equal to the shortest confidence interval. Several examples are given to demonstrate the performance of the proven theorems.

1. Introduction

The problem of unbiased and shortest confidence intervals have been studied by some authors. Guenther (1969, 1971) presented a method to obtain the shortest confidence intervals and the unbiased confidence intervals, respectively. Ferentions (1990) gave a general solution to find the shortest confidence intervals for families of distributions involving truncation parameters. Ferentions and Kourouklis (1990) gave a lemma which plays an important role in constructing confidence intervals with shortest length. Troendle and Frank (2001) presented unbiased confidence intervals for the odds ratio of two independent binomial samples with application to case-control data. Brent (2008) discussed about comparing equal-tail probability and unbiased confidence intervals for the intraclass correlation coefficient. Evans and Shakhatreh (2008) obtained some optimal properties of credible sets. In fact they presented a new credible region called *relative surprise regions* and showed that these regions are unbiased. Alizadeh et al. (2012) compared some intervals estimators for the Poisson distribution. A part of their study contains the comparison of Bayesian credible intervals as highest posterior density (HPD) and relative surprise (RS) credible intervals. They obviously conclude that the RS credible intervals have close competition with HPD credible intervals based on coverage probability and average of the length of credible intervals.

Let $X_1, X_2, \ldots, X_n$ be a random sample from a family of distributions with probability density function (p.d.f.)

$$f(x;\theta) = \frac{g(x)}{h(\theta)}\ ;\ a(\theta) \le x \le b(\theta), \tag{1}$$

where g is a function of x only, $h(\theta) = \int_{a(\theta)}^{b(\theta)} g(x)dx$ is a function of θ only and a, b are monotone functions of θ. In special cases, one of the extremities a or b may be a fixed. Let a^{-1}, b^{-1} be the inverse functions

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of a and b respectively, and $X_{(1)} = \min\{X_i\}$ and $X_{(n)} = \max\{X_i\}$ be the first and nth ordered statistics. Ferentions (1987) shown that $\hat{\theta}_1 = \min\{a^{-1}(X_{(1)}), b^{-1}(X_{(n)})\}$ is a sufficient statistic for θ, in which a is an increasing function and b is a decreasing one in (1). When the monotonicities of a and b are at the opposite directions, then the sufficient statistic is $\hat{\theta}_2 = \max\{a^{-1}(X_{(1)}), b^{-1}(X_{(n)})\}$, where $\hat{\theta}_1$ and $\hat{\theta}_2$ are also the maximum likelihood estimator for θ in the given family of distributions belonging to (1).

In this paper, by using the pivotal quantity method (Ferentions 1990), we find a general solution to obtain an unbiased confidence intervals with confidence coefficient $1-\alpha$ for families of distributions involving truncation parameter based on a random sample.

The organization of this paper is as follow. A general solution for constructing unbiased confidence intervals for parametric function, $h(\theta)$, is presented in Section 2, where we have a random sample from the given families of distributions in (1). Section 3 is devoted to several examples to show the performance of the proven theorems.

2. Unbiased Confidence Intervals

Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with probability density function $f(x;\theta)$. In using the standard method of obtaining a confidence interval for θ, one seek a random variable $Q(X_1, X_2, \ldots, X_n; \theta) = Q(\theta)$ whose distribution is independent of θ, where $Q(\theta)$ is named a pivotal quantity. Then in a number of standard situations, the probability statement $Pr(a < Q(\theta) < b) = 1 - \alpha$, is converted to $Pr(W_1 < \theta < W_2) = 1-\alpha$, where after observing $x_1, x_2, \ldots, x_n$, the specific numbers w_1, w_2 are calculated which form the end points of a $1 - \alpha$ confidence interval for θ. For (w_1, w_2) to be unbiased confidence interval, we must have

$$H(\theta, \theta') = Pr(W_1 < \theta' < W_2) \ \geq \ (\leq)\ 1 - \alpha \ \text{ if } \ \theta' = \theta \ (\theta' \neq \theta). \tag{2}$$

Throughout this paper we assume that $H(\theta, \theta')$ depends only on a function of θ and θ', say γ. Consequently, instead of $H(\theta, \theta')$ we will use the simpler H(γ). When Q(θ) is a continuous random variable, (2) imply

$$H(\gamma) = \ (<)\ 1 - \alpha \quad if \quad \gamma = (\neq)\ 1. \tag{3}$$

In other words, $H(\gamma)$ has a unique maximum at $\gamma = 1$ (Guenther, 1971).

The following lemma state that confidence intervals are invariant under reparameterization.

Lemma 2.1. *Suppose that (T_1, T_2) be an unbiased confidence interval for θ with confidence coefficient $1-\alpha$ and let τ be a monotone function of θ.*

i) If τ is a strictly increasing function of θ, then $\Big(\tau(T_1), \tau(T_2)\Big)$ is an unbiased confidence interval for $\tau(\theta)$ with confidence coefficient $1 - \alpha$.

ii) If τ is a strictly decreasing function of θ, then $\Big(\tau(T_2), \tau(T_1)\Big)$ is an unbiased confidence interval for $\tau(\theta)$ with confidence coefficient $1 - \alpha$.

Proof. (i) We have

$$\begin{aligned} 1 - \alpha &= Pr(T_1 < \theta < T_2) \\ &= Pr(\tau(T_1) < \tau(\theta) < \tau(T_2)). \end{aligned}$$

This means that $(\tau(T_1), \tau(T_2))$ is a confidence interval for $\tau(\theta)$ with confidence coefficient $1 - \alpha$.

(ii) is similar to proof (i). □

Usually the construction of confidence intervals is based on a suitable point estimator. In the following theorems we are going to get an unbiased confidence interval for $h(\theta)$ based on maximum likelihood estimator and sufficient statistic introduced by Ferentins (1987), when we have a random sample from families of distributions involving truncation parameter.

Theorem 2.2. *Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with p.d.f. $f(x;\theta) = g(x)/h(\theta)$; $a(\theta) \leq x \leq b(\theta)$ in which h and b are two increasing functions of θ, and a is a decreasing function of θ. Then the random interval $\left(h(\hat{\theta}_2), h(\hat{\theta}_2)\alpha^{-1/n}\right)$ is an unbiased confidence interval for $h(\theta)$ with confidence coefficient $1-\alpha$, where $\hat{\theta}_2 = \max\{a^{-1}(X_{(1)}), b^{-1}(X_{(n)})\}$.*

Proof. Following Ferentions (1990), the p.d.f. of random variable $Q = h(\hat{\theta}_2)/h(\theta)$ is obtained as $f_Q(q) = nq^{n-1}$; $0 \leq q \leq 1$, which is independent of θ. Therefore Q can be considered as a pivotal quantity to find an unbiased confidence interval for $h(\theta)$ as in the following

$$Pr(a < Q < b) = 1-\alpha,$$

or equivalently

$$Pr\left(\frac{h(\hat{\theta}_2)}{b} < h(\theta) < \frac{h(\hat{\theta}_2)}{a}\right) = 1-\alpha. \tag{4}$$

Therefore for every $\theta = \theta'$, we have

$$Pr\left(\frac{h(\hat{\theta}_2)}{b} < h(\theta') < \frac{h(\hat{\theta}_2)}{a}\right) = 1-\alpha,$$

and we can write

$$Pr\left(\frac{h(\hat{\theta}_2)}{bh(\theta)} < \frac{h(\theta')}{h(\theta)} < \frac{h(\hat{\theta}_2)}{ah(\theta)}\right) = 1-\alpha, \ for\ h(\theta) > 0.$$

The result is similar for $h(\theta) < 0$. Let $\gamma = \frac{h(\theta')}{h(\theta)}$, then we have

$$\begin{aligned} H(\gamma) &= Pr\left(\frac{Q}{b} < \gamma < \frac{Q}{a}\right) \\ &= Pr(a\gamma < Q < b\gamma) \\ &= \int_{a\gamma}^{b\gamma} f_Q(q)dq \\ &= \int_{a\gamma}^{b\gamma} nq^{n-1}dq \\ &= (b^n - a^n)\gamma^n. \end{aligned} \tag{5}$$

Also, we can write

$$Pr(a < Q < b) = b^n - a^n = 1-\alpha. \tag{6}$$

Therefore, using (5) and (6), we can write

$$H(\gamma) = (1-\alpha)\gamma^n.$$

We have $H(1) = 1-\alpha$, this means that the maximum of the function H occurs in $\gamma = 1$ and (3) is holds. Furthermore, considering the distribution of Q, we must be careful to have $b\gamma < 1$. On the other hand, we can choose γ greater than 1 such that $H(\gamma) > 1-\alpha$, which is not matched to (3). Thus we must choose $b = 1$ and by replacing it in (6), we get $a = \alpha^{\frac{1}{n}}$. Consequently, by replacing $a = \alpha^{\frac{1}{n}}$ and $b = 1$ in (4), the random interval $\left(h(\hat{\theta}_2), h(\hat{\theta}_2)\alpha^{-1/n}\right)$ is an unbiased confidence interval for $h(\theta)$ with confidence coefficient $1-\alpha$. □

Theorem 2.3. *Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with p.d.f. $f(x;\theta) = g(x)/h(\theta)$; $a(\theta) \le x \le b(\theta)$ in which h and b are two decreasing functions of θ, and a is a increasing function of θ. Then the random interval $\left(h(\hat{\theta}_1), h(\hat{\theta}_1)\alpha^{-\frac{1}{n}}\right)$ is an unbiased confidence interval for $h(\theta)$ with confidence coefficient $1-\alpha$, in which $\hat{\theta}_1 = \min\{a^{-1}(X_{(1)}), b^{-1}(X_{(n)})\}$.*

Proof. The proof is similar to the proof of Theorem 2.2. □

3. Examples

Example 3.1. Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with p.d.f. $f(x;\theta) = \frac{1}{\theta} I_{[0,\theta)}(x)$, where $I_A(x)$ is the indicator function of A. We are going to obtain an unbiased confidence interval for θ with confidence coefficient $1-\alpha$. By Theorem 2.2, we can write $a(\theta) = 0$, $b(\theta) = \theta$, $h(\theta) = \theta$, $g(x) = 1$ and therefore $\hat{\theta}_2 = X_{(n)}$. Then, an unbiased confidence interval for $h(\theta) = \theta$ with confidence coefficient $1-\alpha$ is $\left(X_{(n)}, X_{(n)}\alpha^{-\frac{1}{n}}\right)$. For instance, if $n = 5$ and $\alpha = 0.025$, we get $\alpha^{-\frac{1}{n}} = 2.09$, and hence an unbiased confidence interval for θ with confidence coefficient 0.975 is $\left(X_{(5)}, 2.09X_{(5)}\right)$.

Example 3.2. Let X_1, X_2, $\ldots$, X_n be a random sample from a distribution with p.d.f. $f(x;\theta) = \frac{1}{2\theta} I_{[-\theta,\theta]}(x)$. We are going to obtain an unbiased confidence interval for θ with confidence coefficient $1-\alpha$. By Theorem 2.2, we can write $a(\theta) = -\theta$, $b(\theta) = \theta$, $h(\theta) = 2\theta$, $g(x) = 1$ and therefore $\hat{\theta}_2 = \max(-X_{(1)}, X_{(n)})$. Then, an unbiased confidence interval for $h(\theta) = 2\theta$ with confidence coefficient $1-\alpha$ is $\left(2\hat{\theta}_2, 2\hat{\theta}_2\alpha^{-\frac{1}{n}}\right)$. For instance, if $n = 20$ and $\alpha = 0.025$, we get $\alpha^{-\frac{1}{n}} = 1.2$, and hence an unbiased confidence interval for 2θ with confidence coefficient 0.975 is $\left(2\hat{\theta}_2, 2.4\hat{\theta}_2\right)$. Also using Lemma 2.1, an unbiased confidence interval for θ with confidence coefficient 0.975 is $\left(\hat{\theta}_2, 1.2\hat{\theta}_2\right)$.

Example 3.3. Let X_1, X_2, $\ldots$, X_n be a random sample from a distribution with p.d.f. $f(x;\theta) = \frac{2x}{\theta^2} I_{(0,\theta]}(x)$. We are going to obtain an unbiased confidence interval for θ^2 with confidence coefficient $1-\alpha$. By Theorem 2.2, we can write $a(\theta) = 0$, $b(\theta) = \theta$, $h(\theta) = \theta^2$, $g(x) = 2x$ and therefore $\hat{\theta}_2 = X_{(n)}$. Then, an unbiased confidence interval for θ^2 with confidence coefficient $1-\alpha$ is $\left(X^2_{(n)}, X^2_{(n)}\alpha^{-\frac{1}{n}}\right)$. For instance, if $n = 5$ and $\alpha = 0.05$, we get $\alpha^{-\frac{1}{n}} = 1.82$, and hence an unbiased confidence interval for θ^2 with confidence coefficient 0.95 is $\left(X_{(5)}, 1.82X_{(5)}\right)$.

Example 3.4. Let X_1, X_2, $\ldots$, X_n be a random sample from a distribution with p.d.f. $f(x;\theta) = \frac{\theta}{(\theta-1)x^2} I_{(1,\theta]}(x)$. We are going to obtain an unbiased confidence interval for θ with confidence coefficient $1-\alpha$. By Theorem 2.2, we can write $a(\theta) = 1$, $b(\theta) = \theta$, $h(\theta) = \frac{\theta-1}{\theta}$, $g(x) = \frac{1}{x^2}$ and therefore $\hat{\theta}_2 = X_{(n)}$. Then, an unbiased confidence interval for $h(\theta) = \frac{\theta-1}{\theta}$ with confidence coefficient $1-\alpha$ is $\left(\frac{X_{(n)}-1}{X_{(n)}}, \frac{X_{(n)}-1}{X_{(n)}}\alpha^{-\frac{1}{n}}\right)$. For instance, if $n = 20$ and $\alpha = 0.05$, we get $\alpha^{-\frac{1}{n}} = 1.16$, and hence an unbiased confidence interval for $\frac{\theta-1}{\theta}$ with confidence coefficient 0.95 is $\left(\frac{X_{(20)}-1}{X_{(20)}}, 1.16\frac{X_{(20)}-1}{X_{(20)}}\right)$. Also using Lemma 2.1, an unbiased confidence interval for θ with confidence coefficient $1-\alpha$ is $\left(X_{(n)}, \frac{X_{(n)}}{\alpha^{-\frac{1}{n}}+(1-\alpha^{-\frac{1}{n}})X_{(n)}}\right)$. For instance, several 0.95 and 0.99 unbiased and shortest confidence intervals for θ are given in Table 1 based on different sample sizes.

n	$1-\alpha$	unbiased and shortest confidence intervals for θ
10	0.95	$\left(X_{(10)}, \frac{X_{(10)}}{1.35-0.35X_{(10)}}\right)$
10	0.99	$\left(X_{(10)}, \frac{X_{(10)}}{1.58-0.58X_{(10)}}\right)$
20	0.95	$\left(X_{(20)}, \frac{X_{(20)}}{1.16-0.16X_{(20)}}\right)$
20	0.99	$\left(X_{(20)}, \frac{X_{(20)}}{1.26-0.26X_{(20)}}\right)$
30	0.95	$\left(X_{(30)}, \frac{X_{(30)}}{1.1-0.1X_{(30)}}\right)$
30	0.99	$\left(X_{(30)}, \frac{X_{(30)}}{1.17-0.17X_{(30)}}\right)$

Table 1: Several unbiased and shortest confidence intervals for θ.

Example 3.5. Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with p.d.f. $f(x;\theta) = \frac{\theta}{x^2} I_{(\theta,\infty)}(x)$. We are going to find an unbiased confidence interval for θ with confidence coefficient $1-\alpha$. By Theorem 2.3, we can write $a(\theta) = \theta$, $b(\theta) = \infty$, $h(\theta) = \frac{1}{\theta}$, $g(x) = \frac{1}{x^2}$ and $\hat{\theta}_1 = X_{(1)}$, and then an unbiased confidence interval for $h(\theta) = \frac{1}{\theta}$ with confidence coefficient $1-\alpha$ is $\left(\frac{1}{X_{(1)}}, \frac{1}{X_{(1)}}\alpha^{-\frac{1}{n}}\right)$. For instance, if $n = 20$ and $\alpha = 0.1$, we get $\alpha^{-\frac{1}{n}} = 1.12$, and hence an unbiased confidence interval for $\frac{1}{\theta}$ with confidence coefficient 0.90 is $\left(\frac{1}{X_{(1)}}, 1.12\frac{1}{X_{(1)}}\right)$. Also, using Lemma 2, an unbiased confidence interval for θ with confidence coefficient 0.90 is $\left(0.89X_{(1)}, X_{(1)}\right)$.

Example 3.6. Let $X_1, X_2, \ldots, X_n$ be a random sample from a Pareto distribution $f(x;\theta) = \frac{k\theta^k}{x^{k+1}} I_{(\theta,\infty)}(x)$, $k > 0$. We are going to find an unbiased confidence interval for θ with confidence coefficient $1-\alpha$. It must be note that this example is an extension of Example 3.5. By Theorem 2.3, we can write $a(\theta) = \theta$, $b(\theta) = \infty$, $h(\theta) = \frac{1}{\theta^k}$, $g(x) = \frac{k}{x^{k+1}}$ and therefore $\hat{\theta}_1 = X_{(1)}$, and then an unbiased confidence interval for $h(\theta) = \frac{1}{\theta^k}$ with confidence coefficient $1-\alpha$ is $\left(\frac{1}{X^k_{(1)}}, \frac{1}{X^k_{(1)}}\alpha^{-\frac{1}{n}}\right)$. For instance, if $n = 35$, $k = 3$ and $\alpha = 0.05$, we get $\alpha^{-\frac{1}{n}} = 1.08$, and hence an unbiased confidence interval for $\frac{1}{\theta^3}$ with confidence coefficient 0.95 is $\left(\frac{1}{X^3_{(1)}}, 1.08\frac{1}{X^3_{(1)}}\right)$. Also, using Lemma 2, an unbiased confidence interval for θ with confidence coefficient $1-\alpha$ is $\left(\alpha^{\frac{k}{n}}X_{(1)}, X_{(1)}\right)$. For instance, several 0.95 and 0.99 unbiased and shortest confidence intervals for θ are given in Table 2 based on different sample sizes.

n	k	$1-\alpha$	unbiased and shortest confidence intervals for θ
10	3	0.95	$(0.41X_{(1)}, X_{(1)})$
10	3	0.99	$(0.25X_{(1)}, X_{(1)})$
20	5	0.95	$(0.47X_{(1)}, X_{(1)})$
20	5	0.99	$(0.31X_{(1)}, X_{(1)})$
30	7	0.95	$(0.49X_{(1)}, X_{(1)})$
30	7	0.99	$(0.34X_{(1)}, X_{(1)})$

Table 2: Several unbiased and shortest confidence intervals for θ.

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