ProbStat Forum, Volume 06, July 2013, Pages 35–41
ISSN 0974-3235

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A note on characterization related to distributional properties of random translation, contraction and dilation of generalized order statistics

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Abstract. The Erlang-truncated exponential distribution has been characterized through translation of two non-adjacent generalized order statistics (gos) and then the characterizing results are obtained for Pareto distribution through dilation of generalized order statistics (gos) and power function distribution through contraction of non-adjacent dual generalized order statistics (dgos). Further, the results are deduced for order statistics and adjacent dual generalized order statistics and generalized order statistics.

1. Introduction

Kamps (1995) introduced the concept of generalized order statistics (*gos*) as follows: Let X_1, X_2, $\cdots$, X_n be a sequence of independent and identically distributed (*iid*) random variables (*rv*) with the absolutely continuous distribution function (*df*) $F(x)$ and the probability density function (*pdf*) $f(x)$, $x \in (a, b)$. Let $n \in N$, $n \geq 2$, $k > 0$, $\tilde{m} = (m_1, m_2, \cdots, m_{n-1}) \in \Re^{n-1}$, $M_r = \sum_{j=r}^{n-1} m_j$, such that $\gamma_r = k + (n-r) + M_r > 0$ for all $r \in \{1, 2, \cdots, n-1\}$. If $m_1 = m_2 = \cdots = m_{n-1} = m$, then $X(r, n, m, k)$ is called the r^{th} $m - gos$ and its *pdf* is given as

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}^{(n)}}{(r-1)!} [\overline{F}(x)]^{\gamma_r^{(n)}-1} \left[\frac{1 - [\overline{F}(x)]^{m+1}}{m+1} \right]^{r-1} f(x), \quad a < x < b, \tag{1}$$

where

$$\gamma_r^{(n)} = k + (n-r)(m+1), \quad 1 \leq r \leq n,$$

$$C_{r-1}^{(n)} = \prod_{i=1}^{r} \gamma_i^{(n)}, \quad 1 \leq r \leq n.$$

Based on the generalized order statistics (*gos*), Burkschat *et al.* (2003) introduced the concept of the dual generalized order statistics (*dgos*) where the *pdf* of the r^{th} $m - dgos$ $X^*(r, n, m, k)$ is given as

$$f_{X^*(r,n,m,k)}(x) = \frac{C_{r-1}^{(n)}}{(r-1)!} [F(x)]^{\gamma_r^{(n)}-1} \left[\frac{1 - [F(x)]^{m+1}}{m+1} \right]^{r-1} f(x), \quad a < x < b,$$

Keywords. Order statistics, Generalized order statistics, Dual generalized order statistics, Random translation, Contraction, Dilation, Characterization of distributions, Erlang-truncated exponential, Power function, Pareto distributions
Received: 04 August 2012; Revised: 03 June 2013; Accepted: 11 July 2013
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which is obtained just by replacing $\overline{F}(x) = 1 - F(x)$ by $F(x)$. If support of the distribution $F(x)$ be over (a, b), then by convention, we will write $X(0, n, m, k) = a$ and $X^*(0, n, m, k) = b$.

Ahsanullah (2006) has characterized exponential distribution under random dilation for adjacent *gos*. In this paper, distributional properties of the *gos* have been used to characterize Erlang-truncated exponential distribution for non-adjacent *gos* under random translation, dilation and contraction, thus generalizing the results of Ahsanullah (2006). One may also refer to Arnold *et al*. (2008), Beutner and Kamps (2008), Nevzorov (2001), Navarro (2008), Oncel *et al*. (2005), Wesolowski and Ahsanullah (2004) and Castaño-Martínez *et al*. (2012) for the related results.

It may be seen that if Y is a measurable function of X with the relation $Y = h(X)$, then

$$Y(r, n, m, k) = h(X(r, n, m, k)) \tag{2}$$

if h is an increasing function, and

$$Y^*(r, n, m, k) = h(X^*(r, n, m, k)) \tag{3}$$

if h is a decreasing function, where $X(r, n, m, k)$ is the r^{th} $m - gos$ and $X^*(r, n, m, k)$ is the r^{th} $m - dgos$.

We will denote

(i) $X \sim Erlang - truncated\ exp(\beta(\alpha_\lambda))$ if X has an Erlang-truncated exponential distribution with the *df*

$$F(x) = [1 - e^{-\beta(\alpha_\lambda)x}],\ 0 \le x < \infty,\ \beta > 0, \lambda > 0,$$

where $\alpha_\lambda = 1 - e^{-\lambda}$.

(ii) $X \sim Par(\beta(\alpha_\lambda))$ if X has a Pareto distribution with the *df*

$$F(x) = 1 - x^{-\beta(\alpha_\lambda)},\ 1 < x < \infty,\ \beta > 0, \lambda > 0.$$

(iii) $X \sim pow(\beta(\alpha_\lambda))$ if X has a power function distribution with the *df*

$$F(x) = x^{\beta(\alpha_\lambda)},\ 0 < x < 1,\ \beta > 0, \lambda > 0.$$

It may further be noted that if $\log X \sim Erlang - truncated\ exp(\beta(\alpha_\lambda))$, then

$$X \sim Par(\beta(\alpha_\lambda)), \tag{4}$$

and if $-\log X \sim Erlang - truncated\ exp(\beta(\alpha_\lambda))$, then

$$X \sim pow(\beta(\alpha_\lambda)) \tag{5}$$

It has been assumed here throughout that the *df* is differentiable *w.r.t.* its argument.

2. Characterization results

Theorem 2.1. *Let $X(r, n, m, k)$ be the r^{th} $m - gos$ from a sample with absolutely continuous df $F(x)$ and pdf $f(x)$. Then for $1 \le r < n_2 \le n_1$, we have that*

$$X(n_1 - n_2 + r - j, n_1 - j, m, k) \overset{d}{=} X(r, n_2, m, k) + X(n_1 - n_2 - j, n_1, m, k),\ j = 0, 1,$$

where $X(n_1 - n_2 - j, n_1, m, k)$ is independent of $X(r, n_2, m, k)$ if and only if the random variable (rv) X_1 has Erlang-truncated exp($\beta(\alpha_\lambda)$) distribution and $X \overset{d}{=} Y$ denotes that X and Y have the same df.

Proof. To prove the necessary part, let the moment generating function (mgf) of $X(n_1-n_2+r,n_1,m,k)$ be $M_{X_{(n_1-n_2+r,n_1,m,k)}}(t)$, then

$$X(n_1-n_2+r,n_1,m,k) \ \stackrel{d}{=}\ X(r,n_2,m,k)+Y$$

implies

$$M_{X_{(n_1-n_2+r,n_1,m,k)}}(t) \ \stackrel{d}{=}\ M_{X_{(r,n_2,m,k)}}(t)M_Y(t).$$

Since for the Erlang-truncated $\exp(\beta(\alpha_\lambda))$ distribution we have that

$$M_{X_{(r,n_2,m,k)}}(t)=\frac{C_{r-1}^{(n_2)}}{(m-1)^r}\frac{\Gamma\Big(\frac{\gamma_r^{(n_2)}}{(m+1)}-\frac{t}{\beta(\alpha_\lambda)(m+1)}\Big)}{\Gamma\Big(r-\frac{t}{\beta(\alpha_\lambda)(m+1)}+\frac{\gamma_r^{(n_2)}}{m+1}\Big)}=\prod_{i=1}^{r}\Big(1-\frac{t}{\beta(\alpha_\lambda)\gamma_i^{(n_2)}}\Big)^{-1},$$

therefore

$$M_Y(t)=\frac{M_{X_{(n_1-n_2+r,n_1,m,k)}}(t)}{M_{X_{(r,n_2,m,k)}}(t)}=\frac{C_{n_1-n_2-1}^{(n_1)}}{(m+1)^{n_1-n_2}}\frac{\Gamma\Big(\frac{\gamma_r^{(n_2)}}{(m+1)}-\frac{t}{\beta(\alpha_\lambda)(m+1)}+r\Big)}{\Gamma\Big(\frac{\gamma_r^{(n_2)}}{(m+1)}-\frac{t}{\beta(\alpha_\lambda)(m+1)}+(n_1-n_2+r)\Big)}$$

as $C_{n_1-n_2+r-1}^{(n_1)}=C_{r-1}^{(n_1)}C_{n_1-n_2-1}^{(n_1)}$; $\gamma_{n_1-n_2+r}^{(n_1)}=\gamma_r^{(n_2)}$ and $\gamma_{n_1-n_2}^{(n_1)}=\gamma_r^{(n_2)}+r(m+1)$.

But this is the mgf of $X(n_1-n_2,n_1,m,k)$, the $(n_1-n_2)^{th}$ $m-gos$ from a sample of size n_1 drawn from Erlang-truncated $\exp(\beta(\alpha_\lambda))$ and hence the result.

For the proof of sufficiency part, we have by the convolution method

$$\begin{aligned}
f_{X(n_1-n_2+r,n_1,m,k)}(y) &= \int_0^y f_{X(r,n_2,m,k)}(x)f_Y(y-x)\,dx\\
&= \frac{\beta(\alpha_\lambda)\ C_{n_1-n_2-1}^{(n_1)}}{(n_1-n_2-1)!(m+1)^{n_1-n_2-1}}\int_0^y [e^{-\beta(\alpha_\lambda)(y-x)}]^{\gamma_{n_1-n_2}^{(n_1)}}\\
&\quad \times[1-(e^{-\beta(\alpha_\lambda)(y-x)})^{m+1}]^{n_1-n_2-1}f_{X(r,n_2,m,k)}(x)dx, \qquad (6)
\end{aligned}$$

as $\gamma_{n_1-n_2}^{(n_1)}=\gamma_{n_1-n_2-1}^{(n_1-1)}$.

Differentiating both the sides of (6) $w.r.t.$ y, we get

$$\begin{aligned}
\frac{d}{dy}f_{X(n_1-n_2+r,n_1,m,k)}(y) &= \frac{\beta(\alpha_\lambda)(n_1-n_2-1)(m+1)\ C_{n_1-n_2-1}^{(n_1)}}{(n_1-n_2-1)!(m+1)^{n_1-n_2-1}}\int_0^y \beta(\alpha_\lambda)[e^{-\beta(\alpha_\lambda)(y-x)}]^{\gamma_{n_1-n_2}^{(n_1)}+(m+1)}\\
&\quad \times[1-(e^{-\beta(\alpha_\lambda)(y-x)})^{m+1}]^{n_1-n_2-2}f_{X(r,n_2,m,k)}(x)\ dx\\
&\quad -\frac{\beta(\alpha_\lambda)\ \gamma_{n_1-n_2}^{(n_1)}\ C_{n_1-n_2-1}^{(n_1)}}{(n_1-n_2-1)!(m+1)^{n_1-n_2-1}}\int_0^y \beta(\alpha_\lambda)[e^{-\beta(\alpha_\lambda)(y-x)}]^{\gamma_{n_1-n_2}^{(n_1)}}\\
&\quad \times[1-(e^{-\beta(\alpha_\lambda)(y-x)})^{m+1}]^{n_1-n_2-1}f_{X(r,n_2,m,k)}(x)\ dx
\end{aligned}$$

Now since,

$$\begin{aligned}
f_{X(n_1-n_2,n_1,m,k)}(x) &= \frac{\beta(\alpha_\lambda)\ C_{n_1-n_2-1}^{(n_1)}}{(n_1-n_2-1)!(m+1)^{n_1-n_2-1}}[e^{-\beta(\alpha_\lambda)x}]^{\gamma_{n_1-n_2}^{(n_1)}}\\
&\quad \times[1-(e^{-\beta(\alpha_\lambda)x})^{m+1}]^{n_1-n_2-2}[1-(e^{-\beta(\alpha_\lambda)x})^{m+1}]
\end{aligned}$$

$$= \frac{\beta(\alpha_\lambda)\ C^{(n_1)}_{n_1-n_2-1}}{(n_1-n_2-1)!(m+1)^{n_1-n_2-1}}[e^{-\beta(\alpha_\lambda)x}]^{\gamma^{(n_1)}_{n_1-n_2}}[1-(e^{-\beta(\alpha_\lambda)x})^{m+1}]^{n_1-n_2-2}$$

$$-\frac{\beta(\alpha_\lambda)\ C^{(n_1)}_{n_1-n_2-1}}{(n_1-n_2-1)!(m+1)^{n_1-n_2-1}}[e^{-\beta(\alpha_\lambda)x}]^{\gamma^{(n_1)}_{n_1-n_2}+(m+1)}[1-(e^{-\beta(\alpha_\lambda)x})^{m+1}]^{n_1-n2-2}$$

implying that

$$\frac{\beta(\alpha_\lambda)\ (n_1-n_2-1)(m+1)C^{(n_1)}_{n_1-n_2-1}}{(n_1-n_2-1)!(m+1)^{n_1-n_2-1}}[e^{-\beta(\alpha_\lambda)x}]^{\gamma^{(n_1)}_{n_1-n_2}+(m+1)}[1-(e^{-\beta(\alpha_\lambda)x})^{m+1}]^{n_1-n_2-2}$$

$$= \frac{C^{(n_1)}_{n_1-n_2-1}}{C^{(n_1-1)}_{n_1-n_2-2}} f_{X(n_1-n_2-1,n_1-1,m,k)}(x) - (m+1)(n_1-n_2-1) f_{X(n_1-n_2,n_1,m,k)}(x)$$

and after noting that $C^{(n_1)}_{n_1-n_2-1} = \gamma^{(n_1)}_1\ C^{(n_1-1)}_{n_1-n_2-2}$ and $\frac{\gamma^{(n_1)}_r}{(m+1)} + (n_1-n_2-j+r) = \frac{\gamma^{(n_1)}_j}{(m+1)}$; $\gamma^{(n_2)}_{r+j} = \gamma^{(n_1)}_{n_1-n_2+r+j}$. This leads to

$$\frac{d}{dy} f_{X(n_1-n_2+r,n_1,m,k)}(y) = \beta(\alpha_\lambda)\gamma^{(n_1)}_1[f_{X(n_1-n_2+r-1,n_1-1,m,k)}(y) - f_{X(n_1-n_2+r,n_1,m,k)}(y)],$$

or

$$f_{X(n_1-n_2+r,n_1,m,k)}(y) = \beta(\alpha_\lambda)\gamma^{(n_1)}_1[F_{X(n_1-n_2+r-1,n_1-1,m,k)}(y) - F_{X(n_1-n_2+r,n_1,m,k)}(y)]. \tag{7}$$

Now (Kamps, 1995)

$$F_{X(n_1-n_2+r-1,n_1-1,m,k)}(y) - F_{X(n_1-n_2+r,n_1,m,k)}(y) = \frac{C^{(n_1-1)}_{n_1-n_2+r-2}}{(n_1-n_2+r-1)!(m+1)^{n_1-n_2+r-1}} \times[\bar{F}(y)]^{\gamma^{(n_1)}_{n_1-n_2+r}}[1-(\bar{F}(y))^{m+1}]^{n_1-n_2+r-1}. \tag{8}$$

Therefore, in view of (1), (7) and (8), we have that

$$\frac{f(y)}{\overline{F}(y)} = \beta(\alpha_\lambda)$$

implying that

$$\overline{F}(y) = e^{-\beta(\alpha_\lambda)y}$$

and hence the proof. □

Corollary 2.2. *Let $X(r,n,m,k)$ be the r^{th} $m-gos$ from a sample with absolutely continuous df $F(x)$ and pdf $f(x)$. Then for $1 \le r < n_2 \le n_1$, we have that*

$$X(n_1-n_2+r-j, n_1-j, m, k) \stackrel{d}{=} X(r,n_2,m,k)X(n_1-n_2-j,n_1,m,k),\ j=0,1, \tag{9}$$

where $X(n_1-n_2-j,n_1,m,k)$ is independent of $X(r,n_2,m,k)$ if and only if X_1 has $Par(\beta(\alpha_\lambda))$ distribution.

Proof. Here the product $X(r,n_2,m,k)X(n_1-n_2-j,n_1,m,k)$ in (9) is called random dilation of $X(r,n_2,m,k)$ (Beutner and Kamps, 2008). Note that

$$\log X(n_1-n_2+r,n_1,m,k) \stackrel{d}{=} \log X(r,n_2,m,k) + \log X(n_1-n_2,n_1m,k)$$

implies

$$X(n_1-n_2+r,n_1,m,k) \stackrel{d}{=} X(r,n_2,m,k)X(n_1-n_2,n_1,m,k)$$

and the proof follows in view of (2) and (4). □

Remark 2.3. At $j = 0$ and $\beta(\alpha_\lambda) = \alpha$, we have

$$X(r+1, n_1, m, k) \stackrel{d}{=} X(r, n_2, m, k)X(1, n_1, m, k)$$

as obtained by Beutner and Kamps (2008).

Remark 2.4. Let $X_{r:n}$ be the r^{th} order statistic from a sample with absolutely continuous *df* $F(x)$ and *pdf* $f(x)$. Then for $1 \leq r < n_2 \leq n_1$, we have that

$$X_{n_1-n_2+r-j:n_1-j} \stackrel{d}{=} X_{r:n_2} \, . \, X_{n_1-n_2-j:n_1}, \; j = 0, 1,$$

where $X_{n_1-n_2-j:n_1}$ is independent of $X_{r:n_2}$ if and only if the random variable (rv) X_1 has Par($\beta(\alpha_\lambda)$) distribution.

At $j = 0$ and $\beta(\alpha_\lambda) = \alpha$, we have $X_{r+1:n_1} \stackrel{d}{=} X_{r:n_1-1}X_{1:n_1}$, as given in Castaño-Martínez *et al.* (2012). This is of the form $X_{s:n_1} \stackrel{d}{=} X_{r:n_2}V$, again an unsolved problem (Arnold *et al.*, 2008).

Corollary 2.5. *Let $X^*(r, n, m, k)$ be the r^{th} $m-dgos$ from a sample with absolutely continuous df $F(x)$, and pdf $f(x)$. Then for $1 \leq r < n_2 \leq n_1$, we have that*

$$X^*(n_1 - n_2 + r - j, n_1 - j, m, k) \stackrel{d}{=} X^*(r, n_2, m, k)X^*(n_1 - n_2 - j, n_1, m, k), \; j = 0, 1, \tag{10}$$

where $X^*(n_1 - n_2 - j, n_1, m, k)$ and is independent of $X^*(r, n_2, m, k)$ if and only if the random variable (rv) X_1 has pow($\beta(\alpha_\lambda)$) distribution.

Proof. Here the product $X^*(r, n_2, m, k)X^*(n_1 - n_2 - j, n_1, m, k)$ in (10) is called random contraction of $X^*(r, n_2, m, k)$ (Beutner and Kamps, 2008). It may be noted that

$$-\log X(n_1 - n_2 + r, n_1, m, k) \stackrel{d}{=} -\log X(r, n_2, m, k) - \log X(n_1 - n_2, n_1, m, k)$$

implies

$$X^*(n_1 - n_2 + r, n_1, m, k) \stackrel{d}{=} X^*(r, n_2, m, k)X^*(n_1 - n_2, n_1, m, k)$$

and the result follows with an appeal to (3) and (5). □

Remark 2.6. Let $X_{r:n}$ be the r^{th} order statistic from a sample with absolutely continuous *df* $F(x)$ and *pdf* $f(x)$. Then for $1 \leq r < n_2 \leq n_1$, we have that

$$X_{r:n_1-j} \stackrel{d}{=} X_{r:n_2} \, . \, X_{n_2+1:n_1-j}, \; j = 0, 1,$$

where $X_{n_2+1:n_1-j}$ is independent of $X_{r:n_2}$, if and only if the random variable (rv) X_1 has pow($\beta(\alpha_\lambda)$) distribution. This is of the form $X_{r:n_1} \stackrel{d}{=} X_{r:n_2}W$, which at $r = 1$ and $\beta(\alpha_\lambda) = \alpha$, reduces to $X_{1:n_1} \stackrel{d}{=} X_{1:n_2}W$. This is discussed by Arnold *et al*. (2008).

At $j = 0$ and $\beta(\alpha_\lambda) = \alpha$, we have

$$X_{r:n_1} \stackrel{d}{=} X_{r:n_1-1}X_{n_1:n_1},$$

where $X_{n1:n1} \sim pow(\alpha n_1)$ as given by Wesolowski and Ahsanullah (2004) and Castaño-Martínez *et al.* (2012).

Theorem 2.7. *Let $X(r, n, m, k)$ be the r^{th} $m-gos$ from a sample with absolutely continuous df $F(x)$ and pdf $f(x)$. Then for $1 \leq r < n_2 \leq n_1$,*

$$X(n_1 - n_2 + r - j, n_1 - j, m, k) \stackrel{d}{=} X(n_1 - n_2, n_1, m, k) + X(r, n_2 - j, m, k), \; j = 0, 1,$$

where $X(r, n_2 - j, m, k)$ is independent of $X(n_1 - n_2, n_1, m, k)$ if and only if the random variable (rv) X_1 has Erlang-truncated exp($\beta(\alpha_\lambda)$) distribution.

Proof. To prove of the necessary part follows from Theorem 2.1. To prove the sufficiency part, we have

$$f_{X(n_1-n_2+r,n_1,m,k)}(y) = \int_0^y f_{X(n_1-n_2,n_1,m,k)}(x) f_Y(y-x)\ dx$$
$$= \frac{\beta(\alpha_\lambda)\ C_{r-1}^{(n_2)}}{(r-1)!(m+1)^{r-1}} \int_0^y [e^{-\beta(\alpha_\lambda)(y-x)}]^{\gamma_r^{(n_2)}} [1-(e^{-\alpha(y-x)})^{m+1}]^{r-1} f_{X(n_1-n_2,n_1,m,k)}(x) dx. \quad (11)$$

Differentiating both the sides of (11) $w.r.t.\ y$, we get

$$\begin{aligned}\frac{d}{dy} f_{X(n_1-n_2+r,n_1,m,k)}(y) &= \frac{\beta(\alpha_\lambda)(r-1)(m+1)\ C_{r-1}^{(n_2)}}{(r-1)!(m+1)^{r-1}} \int_0^y \beta(\alpha_\lambda)[e^{-\beta(\alpha_\lambda)(y-x)}]^{\gamma_r^{(n_2)}+(m+1)} \\ &\quad \times [1-(e^{-\beta(\alpha_\lambda)(y-x)})^{m+1}]^{r-2} f_{X(n_1-n_2,n_1,m,k)}(x)\ dx \\ &\quad - \frac{\beta(\alpha_\lambda)\ \gamma_r^{(n_2)}\ C_{r-1}^{(n_2)}}{(r-1)!(m+1)^{r-1}} \int_0^y \beta(\alpha_\lambda)[e^{-\beta(\alpha_\lambda)(y-x)}]^{\gamma_r^{(n_2)}} \\ &\quad \times [1-(e^{-\beta(\alpha_\lambda)(y-x)})^{m+1}]^{r-1} f_{X(n_1-n_2,n_1,m,k)}(x)\ dx \\ &= \beta(\alpha_\lambda)\gamma_r^{(n_2)}[f_{X(n_1-n_2+r-1,n_1,m,k)}(y) - f_{X(n_1-n_2+r,n_1,m,k)}(y)]\end{aligned}$$

or,

$$f_{X(n_1-n_2+r,n_1,m,k)}(y) = \beta(\alpha_\lambda)\gamma_r^{(n_2)}[F_{X(n_1-n_2+r-1,n_1,m,k)}(y) - F_{X(n_1-n_2+r,n_1,m,k)}(y)]. \quad (12)$$

Now (Kamps, 1995)

$$\begin{aligned}F_{X(n_1-n_2+r-1,n_1,m,k)}(y) \ - \ F_{X(n_1-n_2+r,n_1,m,k)}(y) &= \frac{C_{n_1-n_2+r-2}^{(n_1)}}{(n_1-n_2+r-1)!(m+1)^{n_1-n_2+r-1}} \\ &\quad \times [\bar{F}(y)]^{\gamma_{n_1-n_2+r}^{(n_1)}} [1-(\bar{F}(y))^{m+1}]^{n_1-n_2+r-1}. \quad (13)\end{aligned}$$

Therefore, in view of (1), (12) and (14), we have

$$\frac{f(y)}{\bar{F}(y)} = \beta(\alpha_\lambda)$$

implying that

$$\bar{F}(y) = e^{-\beta(\alpha_\lambda)y}$$

and the Theorem is proved. □

Corollary 2.8. *Let $X(r,n,m,k)$ be the r^{th} $m-gos$ from a sample with absolutely continuous df $F(x)$ and pdf $f(x)$. Then for $1 \le r < n_2 \le n_1$,*

$$X(n_1-n_2+r-j,n_1,m,k) \stackrel{d}{=} X(n_1-n_2,n_1,m,k)X(r,n_2-j,m,k),\ j=0,1,$$

where $X(r,n_2-j,m,k)$ is independent of $X(n_1-n_2,n_1,m,k)$ if and only if X_1 has $Par(\beta(\alpha_\lambda))$ distribution.

Proof. Consider

$$\log X(n_1-n_2+r,n_1,m,k) \stackrel{d}{=} \log X(n_1-n_2,n_1,m,k) + \log X(r,n_2,m,k)$$

implies

$$X(n_1-n_2+r,n_1,m,k) \stackrel{d}{=} X(n_1-n_2,n_1,m,k)X(r,n_2,m,k)$$

and the proof follows in view of (2) and (4). □

Remark 2.9. Let $X_{r:n}$ be the r^{th} order statistic from a sample with absolutely continuous the *df* $F(x)$ and *pdf* $f(x)$. Then for $1 \le r < n_2 \le n_1$, we have that

$$X_{n_1-n_2+r-j:n_1} \stackrel{d}{=} X_{n_1-n_2:n_1} X_{r:n_2-j},\ j = 0, 1,$$

where $X_{r:n_2-j}$ is independent of $X_{n_1-n_2:n_1}$ if and only if the random variable (rv) X_1 has $\mathrm{Par}(\beta(\alpha_\lambda))$ distribution.

At $j = n_1 - n_2$ and $\beta(\alpha_\lambda) = \alpha$, we have that $X_{n_1-n_2+r:n_1} \stackrel{d}{=} X_{n_1-n_2:n_1} X_{r:n_2}$, that is $X_{s:n_1} \stackrel{d}{=} X_{r:n_1} X_{s-r:n_1-r}$ as obtained by Castaño-Martínez *et al.* (2012).

Corollary 2.10. *Let $X^*(r, n, m, k)$ be the r^{th} $m - dgos$ from a sample with absolutely continuous df $F(x)$ and pdf $f(x)$. Then for $1 \le r < n_2 \le n_1$,*

$$X^*(n_1 - n_2 + r - j, n_1, m, k) \stackrel{d}{=} X^*(n_1 - n_2, n_1, m, k) X^*(r, n_2 - j, m, k),\ j = 0, 1, \tag{14}$$

where $X^*(r, n_2 - j, m, k)$ and is independent of $X^*(n_1 - n_2, n_1, m, k)$ if and only if $X_1 \sim pow(\beta(\alpha_\lambda)$.

Proof. This can be shown by considering

$$-\log X(n_1 - n_2 + r, n_1, m, k) \stackrel{d}{=} -\log X(n_1 - n_2, n_1, m, k) - \log X(r, n_2, m, k)$$

implies

$$X^*(n_1 - n_2 + r, n_2, m, k) \stackrel{d}{=} X^*(n_1 - n_2, n_1, m, k) X^*(r, n_2, m, k)$$

and the result follows with an appeal to (3) and (5). □

Remark 2.11. Let $X_{r:n}$ be the r^{th} order statistic from a sample with absolutely continuous the *df* $F(x)$ and *pdf* $f(x)$. Then for $1 \le r < n_2 \le n_1$, we have that

$$X_{n_2-r-j+1:n_1} \stackrel{d}{=} X_{n_2+1:n_1} \,.\, X_{n_2-r-j+1:n_2-j},\ j = 0, 1,$$

where $X_{n_2-r-j+1:n_2-j}$ is independent of $X_{n_2+1:n_1}$ if and only if the random variable (rv) X_1 has $\mathrm{pow}(\beta(\alpha_\lambda))$ distribution.

At $r = 1$ and $\beta(\alpha_\lambda) = \alpha$, this reduces $X_{n_1-j:n_1} \stackrel{d}{=} X_{n_2+1:n_1} X_{n_1-j:n_1-j}$ or $X_{r:n_1} \stackrel{d}{=} X_{r+1:n_1} X_{r:r}$, where $X_{r:r} \sim pow(r\alpha)$ as obtained by Navarro (2008) and Castaño-Martínez *et al.* (2012).

Acknowledgements

The author is thankful to the Referee and the Editor for their fruitful suggestions and comments.

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