

A note on characterization related to distributional properties of random translation, contraction and dilation of generalized order statistics

Imtiyaz A. Shah

Department of Statistics, University of Kashmir, Srinagar Jammu and Kashmir-190 006, India

Abstract. The Erlang-truncated exponential distribution has been characterized through translation of two non-adjacent generalized order statistics (gos) and then the characterizing results are obtained for Pareto distribution through dilation of generalized order statistics (gos) and power function distribution through contraction of non-adjacent dual generalized order statistics (dgos). Further, the results are deduced for order statistics and adjacent dual generalized order statistics and generalized order statistics.

1. Introduction

Kamps (1995) introduced the concept of generalized order statistics (*gos*) as follows: Let X_1, X_2, \dots, X_n be a sequence of independent and identically distributed (*iid*) random variables (*rv*) with the absolutely continuous distribution function (*df*) $F(x)$ and the probability density function (*pdf*) $f(x)$, $x \in (a, b)$. Let $n \in \mathbb{N}$, $n \geq 2$, $k > 0$, $\tilde{m} = (m_1, m_2, \dots, m_{n-1}) \in \mathfrak{R}^{n-1}$, $M_r = \sum_{j=r}^{n-1} m_j$, such that $\gamma_r = k + (n-r) + M_r > 0$ for all $r \in \{1, 2, \dots, n-1\}$. If $m_1 = m_2 = \dots = m_{n-1} = m$, then $X(r, n, m, k)$ is called the r^{th} m -*gos* and its *pdf* is given as

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}^{(n)}}{(r-1)!} [\bar{F}(x)]^{\gamma_r^{(n)}-1} \left[\frac{1 - [\bar{F}(x)]^{m+1}}{m+1} \right]^{r-1} f(x), \quad a < x < b, \quad (1)$$

where

$$\gamma_r^{(n)} = k + (n-r)(m+1), \quad 1 \leq r \leq n,$$

$$C_{r-1}^{(n)} = \prod_{i=1}^r \gamma_i^{(n)}, \quad 1 \leq r \leq n.$$

Based on the generalized order statistics (*gos*), Burkschat *et al.* (2003) introduced the concept of the dual generalized order statistics (*dgos*) where the *pdf* of the r^{th} m -*dgos* $X^*(r, n, m, k)$ is given as

$$f_{X^*(r,n,m,k)}(x) = \frac{C_{r-1}^{(n)}}{(r-1)!} [F(x)]^{\gamma_r^{(n)}-1} \left[\frac{1 - [F(x)]^{m+1}}{m+1} \right]^{r-1} f(x), \quad a < x < b,$$

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Email address: masoom.immy03@gmail.com (Imtiyaz A. Shah)

which is obtained just by replacing $\bar{F}(x) = 1 - F(x)$ by $F(x)$. If support of the distribution $F(x)$ be over (a, b) , then by convention, we will write $X(0, n, m, k) = a$ and $X^*(0, n, m, k) = b$.

Ahsanullah (2006) has characterized exponential distribution under random dilation for adjacent *gos*. In this paper, distributional properties of the *gos* have been used to characterize Erlang-truncated exponential distribution for non-adjacent *gos* under random translation, dilation and contraction, thus generalizing the results of Ahsanullah (2006). One may also refer to Arnold *et al.* (2008), Beutner and Kamps (2008), Nevzorov (2001), Navarro (2008), Oncel *et al.* (2005), Wesolowski and Ahsanullah (2004) and Castaño-Martínez *et al.* (2012) for the related results.

It may be seen that if Y is a measurable function of X with the relation $Y = h(X)$, then

$$Y(r, n, m, k) = h(X(r, n, m, k)) \tag{2}$$

if h is an increasing function, and

$$Y^*(r, n, m, k) = h(X^*(r, n, m, k)) \tag{3}$$

if h is a decreasing function, where $X(r, n, m, k)$ is the r^{th} $m - gos$ and $X^*(r, n, m, k)$ is the r^{th} $m - dgos$.

We will denote

(i) $X \sim Erlang - truncated\ exp(\beta(\alpha_\lambda))$ if X has an Erlang-truncated exponential distribution with the *df*

$$F(x) = [1 - e^{-\beta(\alpha_\lambda)x}], \quad 0 \leq x < \infty, \quad \beta > 0, \lambda > 0,$$

where $\alpha_\lambda = 1 - e^{-\lambda}$.

(ii) $X \sim Par(\beta(\alpha_\lambda))$ if X has a Pareto distribution with the *df*

$$F(x) = 1 - x^{-\beta(\alpha_\lambda)}, \quad 1 < x < \infty, \quad \beta > 0, \lambda > 0.$$

(iii) $X \sim pow(\beta(\alpha_\lambda))$ if X has a power function distribution with the *df*

$$F(x) = x^{\beta(\alpha_\lambda)}, \quad 0 < x < 1, \quad \beta > 0, \lambda > 0.$$

It may further be noted that if $\log X \sim Erlang - truncated\ exp(\beta(\alpha_\lambda))$, then

$$X \sim Par(\beta(\alpha_\lambda)), \tag{4}$$

and if $-\log X \sim Erlang - truncated\ exp(\beta(\alpha_\lambda))$, then

$$X \sim pow(\beta(\alpha_\lambda)) \tag{5}$$

It has been assumed here throughout that the *df* is differentiable *w.r.t.* its argument.

2. Characterization results

Theorem 2.1. Let $X(r, n, m, k)$ be the r^{th} $m - gos$ from a sample with absolutely continuous *df* $F(x)$ and *pdf* $f(x)$. Then for $1 \leq r < n_2 \leq n_1$, we have that

$$X(n_1 - n_2 + r - j, n_1 - j, m, k) \stackrel{d}{=} X(r, n_2, m, k) + X(n_1 - n_2 - j, n_1, m, k), \quad j = 0, 1,$$

where $X(n_1 - n_2 - j, n_1, m, k)$ is independent of $X(r, n_2, m, k)$ if and only if the random variable (*rv*) X_1 has Erlang-truncated $\exp(\beta(\alpha_\lambda))$ distribution and $X \stackrel{d}{=} Y$ denotes that X and Y have the same *df*.

Proof. To prove the necessary part, let the moment generating function (*mgf*) of $X(n_1 - n_2 + r, n_1, m, k)$ be $M_{X(n_1 - n_2 + r, n_1, m, k)}(t)$, then

$$X(n_1 - n_2 + r, n_1, m, k) \stackrel{d}{=} X(r, n_2, m, k) + Y$$

implies

$$M_{X(n_1 - n_2 + r, n_1, m, k)}(t) \stackrel{d}{=} M_{X(r, n_2, m, k)}(t)M_Y(t).$$

Since for the Erlang-truncated $\exp(\beta(\alpha_\lambda))$ distribution we have that

$$M_{X(r, n_2, m, k)}(t) = \frac{C_{r-1}^{(n_2)}}{(m-1)^r} \frac{\Gamma\left(\frac{\gamma_r^{(n_2)}}{(m+1)} - \frac{t}{\beta(\alpha_\lambda)(m+1)}\right)}{\Gamma\left(r - \frac{t}{\beta(\alpha_\lambda)(m+1)} + \frac{\gamma_r^{(n_2)}}{m+1}\right)} = \prod_{i=1}^r \left(1 - \frac{t}{\beta(\alpha_\lambda)\gamma_i^{(n_2)}}\right)^{-1},$$

therefore

$$M_Y(t) = \frac{M_{X(n_1 - n_2 + r, n_1, m, k)}(t)}{M_{X(r, n_2, m, k)}(t)} = \frac{C_{n_1 - n_2 - 1}^{(n_1)}}{(m+1)^{n_1 - n_2}} \frac{\Gamma\left(\frac{\gamma_r^{(n_2)}}{(m+1)} - \frac{t}{\beta(\alpha_\lambda)(m+1)} + r\right)}{\Gamma\left(\frac{\gamma_r^{(n_2)}}{(m+1)} - \frac{t}{\beta(\alpha_\lambda)(m+1)} + (n_1 - n_2 + r)\right)}$$

as $C_{n_1 - n_2 + r - 1}^{(n_1)} = C_{r-1}^{(n_1)}C_{n_1 - n_2 - 1}^{(n_1)}$; $\gamma_{n_1 - n_2 + r}^{(n_1)} = \gamma_r^{(n_2)}$ and $\gamma_{n_1 - n_2}^{(n_1)} = \gamma_r^{(n_2)} + r(m+1)$.

But this is the *mgf* of $X(n_1 - n_2, n_1, m, k)$, the $(n_1 - n_2)^{th}$ *m-gos* from a sample of size n_1 drawn from Erlang-truncated $\exp(\beta(\alpha_\lambda))$ and hence the result.

For the proof of sufficiency part, we have by the convolution method

$$\begin{aligned} f_{X(n_1 - n_2 + r, n_1, m, k)}(y) &= \int_0^y f_{X(r, n_2, m, k)}(x)f_Y(y-x) dx \\ &= \frac{\beta(\alpha_\lambda) C_{n_1 - n_2 - 1}^{(n_1)}}{(n_1 - n_2 - 1)!(m+1)^{n_1 - n_2 - 1}} \int_0^y [e^{-\beta(\alpha_\lambda)(y-x)}] \gamma_{n_1 - n_2}^{(n_1)} \\ &\quad \times [1 - (e^{-\beta(\alpha_\lambda)(y-x)})^{m+1}]^{n_1 - n_2 - 1} f_{X(r, n_2, m, k)}(x) dx, \end{aligned} \tag{6}$$

as $\gamma_{n_1 - n_2}^{(n_1)} = \gamma_{n_1 - n_2 - 1}^{(n_1)}$.

Differentiating both the sides of (6) *w.r.t.* y , we get

$$\begin{aligned} \frac{d}{dy} f_{X(n_1 - n_2 + r, n_1, m, k)}(y) &= \frac{\beta(\alpha_\lambda)(n_1 - n_2 - 1)(m+1) C_{n_1 - n_2 - 1}^{(n_1)}}{(n_1 - n_2 - 1)!(m+1)^{n_1 - n_2 - 1}} \int_0^y \beta(\alpha_\lambda) [e^{-\beta(\alpha_\lambda)(y-x)}] \gamma_{n_1 - n_2}^{(n_1) + (m+1)} \\ &\quad \times [1 - (e^{-\beta(\alpha_\lambda)(y-x)})^{m+1}]^{n_1 - n_2 - 2} f_{X(r, n_2, m, k)}(x) dx \\ &\quad - \frac{\beta(\alpha_\lambda) \gamma_{n_1 - n_2}^{(n_1)} C_{n_1 - n_2 - 1}^{(n_1)}}{(n_1 - n_2 - 1)!(m+1)^{n_1 - n_2 - 1}} \int_0^y \beta(\alpha_\lambda) [e^{-\beta(\alpha_\lambda)(y-x)}] \gamma_{n_1 - n_2}^{(n_1)} \\ &\quad \times [1 - (e^{-\beta(\alpha_\lambda)(y-x)})^{m+1}]^{n_1 - n_2 - 1} f_{X(r, n_2, m, k)}(x) dx \end{aligned}$$

Now since,

$$\begin{aligned} f_{X(n_1 - n_2, n_1, m, k)}(x) &= \frac{\beta(\alpha_\lambda) C_{n_1 - n_2 - 1}^{(n_1)}}{(n_1 - n_2 - 1)!(m+1)^{n_1 - n_2 - 1}} [e^{-\beta(\alpha_\lambda)x}] \gamma_{n_1 - n_2}^{(n_1)} \\ &\quad \times [1 - (e^{-\beta(\alpha_\lambda)x})^{m+1}]^{n_1 - n_2 - 2} [1 - (e^{-\beta(\alpha_\lambda)x})^{m+1}] \end{aligned}$$

$$= \frac{\beta(\alpha_\lambda) C_{n_1-n_2-1}^{(n_1)}}{(n_1 - n_2 - 1)!(m + 1)^{n_1-n_2-1}} [e^{-\beta(\alpha_\lambda)x}]^{\gamma_{n_1-n_2}^{(n_1)}} [1 - (e^{-\beta(\alpha_\lambda)x})^{m+1}]^{n_1-n_2-2}$$

$$- \frac{\beta(\alpha_\lambda) C_{n_1-n_2-1}^{(n_1)}}{(n_1 - n_2 - 1)!(m + 1)^{n_1-n_2-1}} [e^{-\beta(\alpha_\lambda)x}]^{\gamma_{n_1-n_2}^{(n_1)} + (m+1)} [1 - (e^{-\beta(\alpha_\lambda)x})^{m+1}]^{n_1-n_2-2}$$

implying that

$$\frac{\beta(\alpha_\lambda) (n_1 - n_2 - 1)(m + 1)C_{n_1-n_2-1}^{(n_1)}}{(n_1 - n_2 - 1)!(m + 1)^{n_1-n_2-1}} [e^{-\beta(\alpha_\lambda)x}]^{\gamma_{n_1-n_2}^{(n_1)} + (m+1)} [1 - (e^{-\beta(\alpha_\lambda)x})^{m+1}]^{n_1-n_2-2}$$

$$= \frac{C_{n_1-n_2-1}^{(n_1)}}{C_{n_1-n_2-2}^{(n_1-1)}} f_{X(n_1-n_2-1, n_1-1, m, k)}(x) - (m + 1)(n_1 - n_2 - 1) f_{X(n_1-n_2, n_1, m, k)}(x)$$

and after noting that $C_{n_1-n_2-1}^{(n_1)} = \gamma_1^{(n_1)} C_{n_1-n_2-2}^{(n_1-1)}$ and $\frac{\gamma_r^{(n_1)}}{(m+1)} + (n_1-n_2-j+r) = \frac{\gamma_j^{(n_1)}}{(m+1)}$; $\gamma_{r+j}^{(n_2)} = \gamma_{n_1-n_2+r+j}^{(n_1)}$. This leads to

$$\frac{d}{dy} f_{X(n_1-n_2+r, n_1, m, k)}(y) = \beta(\alpha_\lambda) \gamma_1^{(n_1)} [f_{X(n_1-n_2+r-1, n_1-1, m, k)}(y) - f_{X(n_1-n_2+r, n_1, m, k)}(y)],$$

or

$$f_{X(n_1-n_2+r, n_1, m, k)}(y) = \beta(\alpha_\lambda) \gamma_1^{(n_1)} [F_{X(n_1-n_2+r-1, n_1-1, m, k)}(y) - F_{X(n_1-n_2+r, n_1, m, k)}(y)]. \tag{7}$$

Now (Kamps, 1995)

$$F_{X(n_1-n_2+r-1, n_1-1, m, k)}(y) - F_{X(n_1-n_2+r, n_1, m, k)}(y) = \frac{C_{n_1-n_2+r-2}^{(n_1-1)}}{(n_1 - n_2 + r - 1)!(m + 1)^{n_1-n_2+r-1}}$$

$$\times [\bar{F}(y)]^{\gamma_{n_1-n_2+r}^{(n_1)}} [1 - (\bar{F}(y))^{m+1}]^{n_1-n_2+r-1}. \tag{8}$$

Therefore, in view of (1), (7) and (8), we have that

$$\frac{f(y)}{\bar{F}(y)} = \beta(\alpha_\lambda)$$

implying that

$$\bar{F}(y) = e^{-\beta(\alpha_\lambda)y}$$

and hence the proof. \square

Corollary 2.2. Let $X(r, n, m, k)$ be the r^{th} m -gos from a sample with absolutely continuous df $F(x)$ and pdf $f(x)$. Then for $1 \leq r < n_2 \leq n_1$, we have that

$$X(n_1 - n_2 + r - j, n_1 - j, m, k) \stackrel{d}{=} X(r, n_2, m, k)X(n_1 - n_2 - j, n_1, m, k), \quad j = 0, 1, \tag{9}$$

where $X(n_1 - n_2 - j, n_1, m, k)$ is independent of $X(r, n_2, m, k)$ if and only if X_1 has $Par(\beta(\alpha_\lambda))$ distribution.

Proof. Here the product $X(r, n_2, m, k)X(n_1 - n_2 - j, n_1, m, k)$ in (9) is called random dilation of $X(r, n_2, m, k)$ (Beutner and Kamps, 2008). Note that

$$\log X(n_1 - n_2 + r, n_1, m, k) \stackrel{d}{=} \log X(r, n_2, m, k) + \log X(n_1 - n_2, n_1, m, k)$$

implies

$$X(n_1 - n_2 + r, n_1, m, k) \stackrel{d}{=} X(r, n_2, m, k)X(n_1 - n_2, n_1, m, k)$$

and the proof follows in view of (2) and (4). \square

Remark 2.3. At $j = 0$ and $\beta(\alpha_\lambda) = \alpha$, we have

$$X(r + 1, n_1, m, k) \stackrel{d}{=} X(r, n_2, m, k)X(1, n_1, m, k)$$

as obtained by Beutner and Kamps (2008).

Remark 2.4. Let $X_{r:n}$ be the r^{th} order statistic from a sample with absolutely continuous *df* $F(x)$ and *pdf* $f(x)$. Then for $1 \leq r < n_2 \leq n_1$, we have that

$$X_{n_1-n_2+r-j:n_1-j} \stackrel{d}{=} X_{r:n_2} \cdot X_{n_1-n_2-j:n_1}, \quad j = 0, 1,$$

where $X_{n_1-n_2-j:n_1}$ is independent of $X_{r:n_2}$ if and only if the random variable (rv) X_1 has $\text{Par}(\beta(\alpha_\lambda))$ distribution.

At $j = 0$ and $\beta(\alpha_\lambda) = \alpha$, we have $X_{r+1:n_1} \stackrel{d}{=} X_{r:n_1-1}X_{1:n_1}$, as given in Castaño-Martínez *et al.* (2012). This is of the form $X_{s:n_1} \stackrel{d}{=} X_{r:n_2}V$, again an unsolved problem (Arnold *et al.*, 2008).

Corollary 2.5. Let $X^*(r, n, m, k)$ be the r^{th} $m - \text{dgos}$ from a sample with absolutely continuous *df* $F(x)$, and *pdf* $f(x)$. Then for $1 \leq r < n_2 \leq n_1$, we have that

$$X^*(n_1 - n_2 + r - j, n_1 - j, m, k) \stackrel{d}{=} X^*(r, n_2, m, k)X^*(n_1 - n_2 - j, n_1, m, k), \quad j = 0, 1, \tag{10}$$

where $X^*(n_1 - n_2 - j, n_1, m, k)$ and is independent of $X^*(r, n_2, m, k)$ if and only if the random variable (rv) X_1 has $\text{pow}(\beta(\alpha_\lambda))$ distribution.

Proof. Here the product $X^*(r, n_2, m, k)X^*(n_1 - n_2 - j, n_1, m, k)$ in (10) is called random contraction of $X^*(r, n_2, m, k)$ (Beutner and Kamps, 2008). It may be noted that

$$-\log X(n_1 - n_2 + r, n_1, m, k) \stackrel{d}{=} -\log X(r, n_2, m, k) - \log X(n_1 - n_2, n_1, m, k)$$

implies

$$X^*(n_1 - n_2 + r, n_1, m, k) \stackrel{d}{=} X^*(r, n_2, m, k)X^*(n_1 - n_2, n_1, m, k)$$

and the result follows with an appeal to (3) and (5). \square

Remark 2.6. Let $X_{r:n}$ be the r^{th} order statistic from a sample with absolutely continuous *df* $F(x)$ and *pdf* $f(x)$. Then for $1 \leq r < n_2 \leq n_1$, we have that

$$X_{r:n_1-j} \stackrel{d}{=} X_{r:n_2} \cdot X_{n_2+1:n_1-j}, \quad j = 0, 1,$$

where $X_{n_2+1:n_1-j}$ is independent of $X_{r:n_2}$, if and only if the random variable (rv) X_1 has $\text{pow}(\beta(\alpha_\lambda))$ distribution. This is of the form $X_{r:n_1} \stackrel{d}{=} X_{r:n_2}W$, which at $r = 1$ and $\beta(\alpha_\lambda) = \alpha$, reduces to $X_{1:n_1} \stackrel{d}{=} X_{1:n_2}W$. This is discussed by Arnold *et al.* (2008).

At $j = 0$ and $\beta(\alpha_\lambda) = \alpha$, we have

$$X_{r:n_1} \stackrel{d}{=} X_{r:n_1-1}X_{n_1:n_1},$$

where $X_{n_1:n_1} \sim \text{pow}(\alpha n_1)$ as given by Wesolowski and Ahsanullah (2004) and Castaño-Martínez *et al.* (2012).

Theorem 2.7. Let $X(r, n, m, k)$ be the r^{th} $m - \text{gos}$ from a sample with absolutely continuous *df* $F(x)$ and *pdf* $f(x)$. Then for $1 \leq r < n_2 \leq n_1$,

$$X(n_1 - n_2 + r - j, n_1 - j, m, k) \stackrel{d}{=} X(n_1 - n_2, n_1, m, k) + X(r, n_2 - j, m, k), \quad j = 0, 1,$$

where $X(r, n_2 - j, m, k)$ is independent of $X(n_1 - n_2, n_1, m, k)$ if and only if the random variable (rv) X_1 has $\text{Erlang-truncated exp}(\beta(\alpha_\lambda))$ distribution.

Proof. To prove of the necessary part follows from Theorem 2.1. To prove the sufficiency part, we have

$$\begin{aligned} f_{X(n_1-n_2+r, n_1, m, k)}(y) &= \int_0^y f_{X(n_1-n_2, n_1, m, k)}(x) f_Y(y-x) dx \\ &= \frac{\beta(\alpha_\lambda) C_{r-1}^{(n_2)}}{(r-1)!(m+1)^{r-1}} \int_0^y [e^{-\beta(\alpha_\lambda)(y-x)}] \gamma_r^{(n_2)} [1 - (e^{-\alpha(y-x)})^{m+1}]^{r-1} f_{X(n_1-n_2, n_1, m, k)}(x) dx. \end{aligned} \quad (11)$$

Differentiating both the sides of (11) w.r.t. y , we get

$$\begin{aligned} \frac{d}{dy} f_{X(n_1-n_2+r, n_1, m, k)}(y) &= \frac{\beta(\alpha_\lambda)(r-1)(m+1) C_{r-1}^{(n_2)}}{(r-1)!(m+1)^{r-1}} \int_0^y \beta(\alpha_\lambda) [e^{-\beta(\alpha_\lambda)(y-x)}] \gamma_r^{(n_2)+(m+1)} \\ &\quad \times [1 - (e^{-\beta(\alpha_\lambda)(y-x)})^{m+1}]^{r-2} f_{X(n_1-n_2, n_1, m, k)}(x) dx \\ &\quad - \frac{\beta(\alpha_\lambda) \gamma_r^{(n_2)} C_{r-1}^{(n_2)}}{(r-1)!(m+1)^{r-1}} \int_0^y \beta(\alpha_\lambda) [e^{-\beta(\alpha_\lambda)(y-x)}] \gamma_r^{(n_2)} \\ &\quad \times [1 - (e^{-\beta(\alpha_\lambda)(y-x)})^{m+1}]^{r-1} f_{X(n_1-n_2, n_1, m, k)}(x) dx \\ &= \beta(\alpha_\lambda) \gamma_r^{(n_2)} [f_{X(n_1-n_2+r-1, n_1, m, k)}(y) - f_{X(n_1-n_2+r, n_1, m, k)}(y)] \end{aligned}$$

or,

$$f_{X(n_1-n_2+r, n_1, m, k)}(y) = \beta(\alpha_\lambda) \gamma_r^{(n_2)} [F_{X(n_1-n_2+r-1, n_1, m, k)}(y) - F_{X(n_1-n_2+r, n_1, m, k)}(y)]. \quad (12)$$

Now (Kamps, 1995)

$$\begin{aligned} F_{X(n_1-n_2+r-1, n_1, m, k)}(y) - F_{X(n_1-n_2+r, n_1, m, k)}(y) &= \frac{C_{n_1-n_2+r-2}^{(n_1)}}{(n_1-n_2+r-1)!(m+1)^{n_1-n_2+r-1}} \\ &\quad \times [\bar{F}(y)]^{\gamma_{n_1-n_2+r}^{(n_1)}} [1 - (\bar{F}(y))^{m+1}]^{n_1-n_2+r-1}. \end{aligned} \quad (13)$$

Therefore, in view of (1), (12) and (14), we have

$$\frac{f(y)}{\bar{F}(y)} = \beta(\alpha_\lambda)$$

implying that

$$\bar{F}(y) = e^{-\beta(\alpha_\lambda)y}$$

and the Theorem is proved. \square

Corollary 2.8. Let $X(r, n, m, k)$ be the r^{th} m -gos from a sample with absolutely continuous df $F(x)$ and pdf $f(x)$. Then for $1 \leq r < n_2 \leq n_1$,

$$X(n_1 - n_2 + r - j, n_1, m, k) \stackrel{d}{=} X(n_1 - n_2, n_1, m, k) X(r, n_2 - j, m, k), \quad j = 0, 1,$$

where $X(r, n_2 - j, m, k)$ is independent of $X(n_1 - n_2, n_1, m, k)$ if and only if X_1 has $Par(\beta(\alpha_\lambda))$ distribution.

Proof. Consider

$$\log X(n_1 - n_2 + r, n_1, m, k) \stackrel{d}{=} \log X(n_1 - n_2, n_1, m, k) + \log X(r, n_2, m, k)$$

implies

$$X(n_1 - n_2 + r, n_1, m, k) \stackrel{d}{=} X(n_1 - n_2, n_1, m, k) X(r, n_2, m, k)$$

and the proof follows in view of (2) and (4). \square

Remark 2.9. Let $X_{r:n}$ be the r^{th} order statistic from a sample with absolutely continuous the *df* $F(x)$ and *pdf* $f(x)$. Then for $1 \leq r < n_2 \leq n_1$, we have that

$$X_{n_1-n_2+r-j:n_1} \stackrel{d}{=} X_{n_1-n_2:n_1} X_{r:n_2-j}, \quad j = 0, 1,$$

where $X_{r:n_2-j}$ is independent of $X_{n_1-n_2:n_1}$ if and only if the random variable (rv) X_1 has $\text{Par}(\beta(\alpha_\lambda))$ distribution.

At $j = n_1 - n_2$ and $\beta(\alpha_\lambda) = \alpha$, we have that $X_{n_1-n_2+r:n_1} \stackrel{d}{=} X_{n_1-n_2:n_1} X_{r:n_2}$, that is $X_{s:n_1} \stackrel{d}{=} X_{r:n_1} X_{s-r:n_1-r}$ as obtained by Castaño-Martínez *et al.* (2012).

Corollary 2.10. Let $X^*(r, n, m, k)$ be the r^{th} m -dgos from a sample with absolutely continuous *df* $F(x)$ and *pdf* $f(x)$. Then for $1 \leq r < n_2 \leq n_1$,

$$X^*(n_1 - n_2 + r - j, n_1, m, k) \stackrel{d}{=} X^*(n_1 - n_2, n_1, m, k) X^*(r, n_2 - j, m, k), \quad j = 0, 1, \quad (14)$$

where $X^*(r, n_2 - j, m, k)$ and is independent of $X^*(n_1 - n_2, n_1, m, k)$ if and only if $X_1 \sim \text{pow}(\beta(\alpha_\lambda))$.

Proof. This can be shown by considering

$$-\log X(n_1 - n_2 + r, n_1, m, k) \stackrel{d}{=} -\log X(n_1 - n_2, n_1, m, k) - \log X(r, n_2, m, k)$$

implies

$$X^*(n_1 - n_2 + r, n_2, m, k) \stackrel{d}{=} X^*(n_1 - n_2, n_1, m, k) X^*(r, n_2, m, k)$$

and the result follows with an appeal to (3) and (5). \square

Remark 2.11. Let $X_{r:n}$ be the r^{th} order statistic from a sample with absolutely continuous the *df* $F(x)$ and *pdf* $f(x)$. Then for $1 \leq r < n_2 \leq n_1$, we have that

$$X_{n_2-r-j+1:n_1} \stackrel{d}{=} X_{n_2+1:n_1} \cdot X_{n_2-r-j+1:n_2-j}, \quad j = 0, 1,$$

where $X_{n_2-r-j+1:n_2-j}$ is independent of $X_{n_2+1:n_1}$ if and only if the random variable (rv) X_1 has $\text{pow}(\beta(\alpha_\lambda))$ distribution.

At $r = 1$ and $\beta(\alpha_\lambda) = \alpha$, this reduces $X_{n_1-j:n_1} \stackrel{d}{=} X_{n_2+1:n_1} X_{n_1-j:n_1-j}$ or $X_{r:n_1} \stackrel{d}{=} X_{r+1:n_1} X_{r:r}$, where $X_{r:r} \sim \text{pow}(r\alpha)$ as obtained by Navarro (2008) and Castaño-Martínez *et al.* (2012).

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