ProbStat Forum, Volume 06, July 2013, Pages 42–59
ISSN 0974-3235

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Bayes estimation of the parameters of the inverse Rayleigh distribution for left censored data

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Abstract. The inverse Rayleigh distribution is an important lifetime distribution in survival analysis. In this paper, we considered the Bayesian estimation for the parameter of the inverse Rayleigh distribution when the data are left censored. We obtained the Bayes estimators and corresponding risks of the unknown parameter under different loss functions (symmetric and asymmetric), assuming different informative and non-informative priors. The credible and posterior predictive intervals have been constructed under all priors. The performance of different estimators has been compared through the analysis of simulated and real life data sets.

1. Introduction

The inverse Rayleigh (IR) distribution has many applications in the area of reliability studies. Voda (1972) has mentioned that the distribution of lifetimes of several types of experimental units can be approximated by the inverse Rayleigh distribution. The probability density function of the inverse Rayleigh distribution with scale parameter λ is

$$f(x;\lambda) = \frac{2\lambda}{x^3} \exp\left[-\left(\frac{\lambda}{x^2}\right)\right],\ x > 0, \lambda > 0.$$

The corresponding cumulative distribution function is

$$F(x) = \exp\left\{-\left(\frac{\lambda}{x^2}\right)\right\},\ x > 0, \lambda > 0.$$

El-Helbawy and Abdel-Monem (2005) have obtained Bayesian estimators of the parameter of the inverse Rayleigh distribution under four loss functions. Some recent contributions on IR distribution can be seen from Soliman et al. (2010), Shawky and Badr (2011), Dey (2012) and Feroze and Aslam (2012). Although several papers have already appeared on the estimation of the parameter of IR distribution for complete sample case, but not much attention has been paid in case of censored samples. The main aim of this paper is to consider the Bayesian estimation of the unknown parameters when the data are left censored from IR distribution. There is a widespread application and use of left-censoring or left-censored data in survival analysis and reliability theory. For example, in medical studies patients are subject to regular

Keywords. Left censoring, Inverse Rayleigh distribution, Bayesian Prediction, Loss Functions, Credible Intervals
Received: 06 March 2013; Revised: 30 April 2013; Re-revised: 04 June 2013; Accepted: 30 July 2013
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examinations. Discovery of a condition only tells us that the onset of sickness fell in the period since the previous examination and nothing about the exact date of the attack. Thus the time elapsed since onset has been left censored. Similarly, we have to handle left-censored data when estimating functions of exact policy duration without knowing the exact date of policy entry; or when estimating functions of exact age without knowing the exact date of birth. Coburn et al. (2001) considered this problem due to the incidence of a higher proportion of rural children whose spells were "left censored" in the sample (i.e. those children who entered the sample uninsured), and who remained uninsured throughout the sample. The job duration might be incomplete because the beginning of the job spells is not observed; it is an incidence of left censoring (Bagger, 2005). For some further examples, one may refer to Balakrishnan (1989), Balakrishnan and Varadan (1991), Lee et al. (1980), etc.

The rest of the paper is organized as follows. In Section 2, we derived posterior distribution under informative and non-informative priors in the presence of left censoring. In Section 3, we provided the Bayes estimator and corresponding posterior risks under different loss functions. Credible intervals have been discussed in Section 4. Method of Elicitation of the hyper-parameters via prior predictive approach has been discussed in Section 5. Posterior predictive distribution and posterior predictive intervals have been derived in Section 6. Simulation results and discussions have been provided in Sections 7 and 8 respectively.

2. Likelihood function and posterior distribution

In this section, the likelihood function of the inverse Rayleigh distribution has been derived in presence of left censored observations. Let $X_{(r+1)}, \ldots, X_{(n)}$ be the last $n-r$ order statistics from a random sample of size n following inverse Rayleigh distribution. Then the joint probability density function of $X_{(r+1)}, \ldots, X_{(n)}$ is given by

$$\begin{aligned} f\left(x_{(r+1)}, \ldots, x_{(n)}; \lambda\right) &= \frac{n!}{r!}\left[F\left(x_{(r+1)}\right)\right]^r f\left(x_{(r+1)}\right) \cdots f\left(x_{(n)}\right) \\ &\propto \left\{\exp\left(-\frac{\lambda}{x_{(r+1)}^2}\right)\right\}^r \lambda^{n-r} \exp\left\{-\lambda\left(\sum_{i=r+1}^{n} \frac{1}{x_{(i)}^2}\right)\right\} \\ &\propto \lambda^{n-r} \exp\left\{-\lambda\tau_{(ir)}\right\}, \end{aligned}$$

where

$$\tau_{(ir)} = r x_{(r+1)}^{-2} + \sum_{i=r+1}^{n} x_{(i)}^{-2}.$$

2.1. Prior and posterior distributions

The uniform prior for the parameter λ is assumed to be

$$p(\lambda) \propto k, \quad \lambda > 0,$$

where k is any constant. The posterior distribution under the uniform prior for the left censored data is

$$p(\lambda|\mathbf{x}) = \frac{\left\{\tau_{(ir)}\right\}^{n-r+1} \lambda^{n-r} \exp\left\{-\lambda\tau_{(ir)}\right\}}{\Gamma(n-r+1)}, \quad \lambda > 0.$$

The Jeffreys prior for the parameter λ is defined to be

$$p(\lambda) \propto \frac{1}{\lambda}, \quad \lambda > 0.$$

The posterior distribution under the Jeffreys prior for the left censored data is

$$p(\lambda|\mathbf{x}) = \frac{\left\{\tau_{(ir)}\right\}^{n-r} \lambda^{n-r-1} \exp\left\{-\lambda\tau_{(ir)}\right\}}{\Gamma(n-r)}, \quad \lambda > 0.$$

The informative prior for the parameter λ is assumed to be exponential distribution

$$p(\lambda) = w \exp(-\lambda w), \quad \lambda > 0,$$

where $w > 0$ is the hyper-parameter. The posterior distribution under the assumption of exponential prior is

$$p(\lambda|\mathbf{x}) = \frac{\left\{\tau_{(ir)}\right\}^{n-r+1} \lambda^{n-r} \exp\left\{-\lambda\left(w + \tau_{(ir)}\right)\right\}}{\Gamma(n-r+1)}, \quad \lambda > 0.$$

The informative prior for the parameter λ is assumed to be gamma distribution

$$p(\lambda) = \frac{b^a}{\Gamma(a)} \lambda^{a-1} \exp(-\lambda b), \quad \lambda > 0,$$

where a, $b > 0$ are the hyper-parameters. The posterior distribution under the assumption of gamma prior for the left censored data is

$$p(\lambda|\mathbf{x}) = \frac{\left\{b + \tau_{(ir)}\right\}^{n-r+a} \lambda^{n-r+a-1} \exp\left\{-\lambda\left(b + \tau_{(ir)}\right)\right\}}{\Gamma(n-r+a)}, \quad \lambda > 0.$$

The informative prior for the parameter λ is assumed to be Inverse Levy distribution

$$p(\lambda) = \sqrt{\frac{c}{2\pi}} \lambda^{\frac{-1}{2}} \exp\left(-\frac{\lambda c}{2}\right), \quad \lambda > 0,$$

where $c > 0$ is the hyper-parameter. The posterior distribution under the assumption of Inverse Levy prior for the left censored data is

$$p(\lambda|\mathbf{x}) = \frac{\left\{\frac{c}{2} + \tau_{(ir)}\right\}^{n-r+\frac{1}{2}} \lambda^{n-r-\frac{1}{2}} \exp\left\{-\lambda\left(\frac{c}{2} + \tau_{(ir)}\right)\right\}}{\Gamma\left(n-r+\frac{1}{2}\right)}, \quad \lambda > 0.$$

3. Bayes estimators and posterior risks under different loss functions

This section discusses the derivation of the Bayes Estimator (BE) and corresponding Posterior Risks (PR) under different loss functions. The Bayes estimators are evaluated under Squared Error Loss Function (SELF), Precautionary Loss Function (PLF), Weighted Squared Error Loss Function (WSELF), Quasi-Quadratic Loss Function (QQLF), Squared-Log Error Loss Function (SLELF), and Entropy Loss Function (ELF). The Bayes Estimator (BE) and corresponding Posterior Risks (PR) under different loss functions are given in the following Table.

Table 1: Bayes Estimator and Posterior Risks under different Loss Functions

Loss Function $=L\left(\lambda,\ \hat{\lambda}\right)$	Bayes Estimator	Posterior Risk
SELF: $\left(\lambda - \hat{\lambda}\right)^2$	$E(\lambda\|\mathbf{x})$	$Var(\lambda\|\mathbf{x})$
PLF: $\frac{(\lambda-\hat{\lambda})^2}{\hat{\lambda}}$	$\sqrt{E(\lambda^2\|\mathbf{x})}$	$2\left\{\sqrt{E(\lambda^2\|\mathbf{x})} - E(\lambda\|\mathbf{x})\right\}$
WSELF: $\frac{(\lambda-\hat{\lambda})^2}{\lambda}$	$\left\{E\left(\lambda^{-1}\|\mathbf{x}\right)\right\}^{-1}$	$E(\lambda\|\mathbf{x}) - \left\{E\left(\lambda^{-1}\|\mathbf{x}\right)\right\}^{-1}$
QQLF: $\left(e^{-c\hat{\lambda}} - e^{-c\lambda}\right)^2$	$\frac{-1}{c}\ln\left\{E\left(e^{-c\lambda}\|\mathbf{x}\right)\right\}$	$E\left(e^{-2c\lambda}\|\mathbf{x}\right) - \left\{E\left(e^{-c\lambda}\|\mathbf{x}\right)\right\}^2$
SLELF: $\left(\ln\hat{\lambda} - \ln\lambda\right)^2$	$\exp\{E(\ln\lambda\|\mathbf{x})\}$	$E\{(\ln\lambda\|\mathbf{x})\}^2 - \{E(\ln\lambda\|\mathbf{x})\}^2$
ELF: $b\left\{\left(\frac{\hat{\lambda}}{\lambda}\right) - \ln\left(\frac{\hat{\lambda}}{\lambda}\right) - 1\right\}$	$\left\{E\left(\lambda^{-1}\|\mathbf{x}\right)\right\}^{-1}$	$\ln\left\{E\left(\lambda^{-1}\|\mathbf{x}\right)\right\} + E(\ln\lambda\|\mathbf{x})$

The Bayes Estimators and Posterior Risks under uniform prior are:

$$\hat{\lambda}_{SELF} = \frac{n-r+1}{\left(rx_{(r+1)}^{-2} + \sum\limits_{i=r+1}^{n} x_{(i)}^{-2}\right)}, \quad \rho\left(\hat{\lambda}_{SELF}\right) = \frac{n-r+1}{\left(rx_{(r+1)}^{-2} + \sum\limits_{i=r+1}^{n} x_{(i)}^{-2}\right)^2},$$

$$\hat{\lambda}_{PLF} = \frac{\sqrt{(n-r+1)(n-r+2)}}{\left\{rx_{(r+1)}^{-2} + \sum\limits_{i=r+1}^{n} x_{(i)}^{-2}\right\}},$$

$$\rho\left(\hat{\lambda}_{PLF}\right) = 2\left\{\frac{\sqrt{(n-r+1)(n-r+2)}}{\left(rx_{(r+1)}^{-2} + \sum\limits_{i=r+1}^{n} x_{(i)}^{-2}\right)} - \frac{n-r+1}{\left(rx_{(r+1)}^{-2} + \sum\limits_{i=r+1}^{n} x_{(i)}^{-2}\right)}\right\},$$

$$\hat{\lambda}_{WSELF} = \frac{n-r}{\left\{rx_{(r+1)}^{-2} + \sum\limits_{i=r+1}^{n} x_{(i)}^{-2}\right\}}, \quad \rho\left(\hat{\lambda}_{WSELF}\right) = \frac{1}{\left\{rx_{(r+1)}^{-2} + \sum\limits_{i=r+1}^{n} x_{(i)}^{-2}\right\}},$$

$$\hat{\lambda}_{QQLF} = -\frac{1}{c}\ln\left\{\frac{\tau_{(ir)}}{c+\tau_{(ir)}}\right\}^{(n-r+1)},$$

$$\rho\left(\hat{\lambda}_{QQLF}\right) = \left\{\frac{\tau_{(ir)}}{2c+\tau_{(ir)}}\right\}^{(n-r+1)} - \left\{\frac{\tau_{(ir)}}{c+\tau_{(ir)}}\right\}^{2(n-r+1)},$$

$$\hat{\lambda}_{SLELF} = \frac{\exp\{\psi(n-r+1)\}}{\left\{rx_{(r+1)}^{-2} + \sum\limits_{i=r+1}^{n} x_{(i)}^{-2}\right\}}, \quad \rho\left(\hat{\lambda}_{SLELF}\right) = \psi'(n-r+1),$$

$$\hat{\lambda}_{ELF} = \left\{\frac{\left(rx_{(r+1)}^{-2} + \sum\limits_{i=r+1}^{n} x_{(i)}^{-2}\right)\Gamma(n-r)}{\Gamma(n-r+1)}\right\}^{-1},$$

$$\rho\left(\hat{\lambda}_{ELF}\right) = \ln\left\{\frac{\left(rx_{(r+1)}^{-2} + \sum\limits_{i=r+1}^{n} x_{(i)}^{-2}\right)\Gamma(n-r)}{\Gamma(n-r+1)}\right\} + \left\{\psi(n-r+1) - \ln\left(rx_{(r+1)}^{-2} + \sum_{i=r+1}^{n} x_{(i)}^{-2}\right)\right\},$$

where $\psi(\cdot)$ and $\psi'(\cdot)$ are the digamma and polygamma functions respectively.

The Bayes Estimators and posterior Risks under the rest of priors can be obtained in a similar manner.

4. Bayes credible interval for the left censored data

The Bayesian credible intervals for type II left censored data under informative and non-informative priors, as discussed by Saleem and Aslam (2009), are presented in the following. The credible intervals for type II left censored data under all priors are:

$$\frac{\chi^2_{2(n-r+1)\frac{\alpha}{2}}}{2\left\{rx_{(r+1)}^{-2} + \sum\limits_{i=r+1}^{n} x_{(i)}^{-2}\right\}} < \lambda_{Uniform} < \frac{\chi^2_{2(n-r+1)\left(1-\frac{\alpha}{2}\right)}}{2\left\{rx_{(r+1)}^{-2} + \sum\limits_{i=r+1}^{n} x_{(i)}^{-2}\right\}},$$

$$\frac{\chi^2_{2(n-r)\frac{\alpha}{2}}}{2\left\{rx^{-2}_{(r+1)}+\sum\limits_{i=r+1}^{n}x^{-2}_{(i)}\right\}}<\lambda_{Jeffreys}<\frac{\chi^2_{2(n-r)\left(1-\frac{\alpha}{2}\right)}}{2\left\{rx^{-2}_{(r+1)}+\sum\limits_{i=r+1}^{n}x^{-2}_{(i)}\right\}},$$

$$\frac{\chi^2_{2(n-r+1)\frac{\alpha}{2}}}{2\left\{w+rx^{-2}_{(r+1)}+\sum\limits_{i=r+1}^{n}x^{-2}_{(i)}\right\}}<\lambda_{Exponential}<\frac{\chi^2_{2(n-r+1)\left(1-\frac{\alpha}{2}\right)}}{2\left\{w+rx^{-2}_{(r+1)}+\sum\limits_{i=r+1}^{n}x^{-2}_{(i)}\right\}},$$

$$\frac{\chi^2_{2(n-r+a)\frac{\alpha}{2}}}{2\left\{b+rx^{-2}_{(r+1)}+\sum\limits_{i=r+1}^{n}x^{-2}_{(i)}\right\}}<\lambda_{Gamma}<\frac{\chi^2_{2(n-r+a)\left(1-\frac{\alpha}{2}\right)}}{2\left\{b+rx^{-2}_{(r+1)}+\sum\limits_{i=r+1}^{n}x^{-2}_{(i)}\right\}},$$

and

$$\frac{\chi^2_{2(n-r+1/2)\frac{\alpha}{2}}}{2\left\{\frac{c}{2}+rx^{-2}_{(r+1)}+\sum\limits_{i=r+1}^{n}x^{-2}_{(i)}\right\}}<\lambda_{In-Levy}<\frac{\chi^2_{2(n-r+1/2)\left(1-\frac{\alpha}{2}\right)}}{2\left\{\frac{c}{2}+rx^{-2}_{(r+1)}+\sum\limits_{i=r+1}^{n}x^{-2}_{(i)}\right\}}.$$

5. Elicitation

In Bayesian analysis the elicitation of opinion is a crucial step. It helps to make it easy for us to understand what the experts believe in and what their opinions are. In statistical inference the characteristics of a certain predictive distribution proposed by an expert determine the hyperparameters of a prior distribution.

In this article, we focus on a probability elicitation method known as prior predictive elicitation. Predictive elicitation is a method for estimating hyperparameters of prior distributions by inverting corresponding prior predictive distributions. Elicitation of hyperparameter from the prior $p(\lambda)$ is conceptually difficult task because we first have to identify prior distribution and then its hyperparameters. The prior predictive distribution is used for the elicitation of the hyperparameters which is compared with the experts' judgment about this distribution and then the hyperparameters are chosen in such a way so as to make the judgment agree closely as possible with the given distribution (for more details see the works of Grimshaw et al. (2001), Kadane (1980), O'Hagan et al. (2006), Jenkinson (2005) and Leon et al. (2003)). According to Aslam (2003), the method of assessment is to compare the predictive distribution with experts' assessment about this distribution and then to choose the hyperparameters that make the assessment agree closely with the member of the family. He discusses three important methods to elicit the hyperparameters: (i) Via the Prior Predictive Probabilities (ii) Via Elicitation of the Confidence Levels (iii) Via the Predictive Mode and Confidence Level.

5.1. Prior predictive distribution

The prior predictive distribution is:

$$p(y)=\int\limits_0^\infty p(y|\lambda)\,p(\lambda)\,d\lambda \tag{1}$$

According to (1), the predictive distribution under exponential prior is:

$$p(y)=\frac{2w}{y^3\left\{w+y^{-2}\right\}^2},\quad 0<y<\infty.$$

According to (1), the predictive distribution under gamma prior is:

$$p(y)=\frac{2ab^a}{y^3\left\{b+y^{-2}\right\}^{a+1}},\quad 0<y<\infty.$$

According to (1), the predictive distribution under inverse Levy prior is:

$$p(y) = \frac{\sqrt{c}}{\sqrt{2}y^3\left\{c/2 + y^{-2}\right\}^{3/2}}, \quad 0 < y < \infty.$$

By using the method of elicitation defined by Aslam (2003), we obtained the following values of the hyper-parameters $w = 0.25163, a = 2.25163, b = 0.96852$ and $c = 2.00163$.

6. Posterior predictive distribution

The predictive distribution contains the information about the independent future random observation given preceding observations. For more details see the works of Bolstad (2004) and Bansal (2007).

6.1. *Posterior predictive distribution and predictive interval*

The posterior predictive distribution of the future observation $y = x_{n+1}$ is

$$p(y|\mathbf{x}) = \int_0^\infty p(\lambda|\mathbf{x})\, p(y|\lambda)\, d\lambda,$$

where $p(y|\lambda) = \frac{2\lambda}{y^3}\exp\left\{-\left(\frac{\lambda}{y^2}\right)\right\}$ is the future observation density and $p(\lambda|\mathbf{x})$ is the posterior distribution obtained by incorporating the likelihood with the respective prior distributions.

A $(1-\alpha)\,100\%$ Bayesian interval (L, U) can be obtained by solving the following two equations simultaneously

$$\int_{-\infty}^{L} p(y|\mathbf{x})\, dy = \frac{k}{2} = \int_{U}^{\infty} p(y|\mathbf{x})\, dy.$$

The posterior predictive distribution of the future observation $y = x_{n+1}$ under uniform prior is

$$p(y|\mathbf{x}) = \frac{2(n-r+1)\left\{rx_{(r+1)}^{-2} + \sum\limits_{i=r+1}^{n} x_{(i)}^{-2}\right\}^{(n-r+1)}}{\left\{y^{-2} + rx_{(r+1)}^{-2} + \sum\limits_{i=r+1}^{n} x_{(i)}^{-2}\right\}^{(n-r+2)}}, \quad y > 0,$$

and predictive interval is:

$$\begin{aligned}
\left\{\frac{rx_{(r+1)}^{-2} + \sum\limits_{i=r+1}^{n} x_{(i)}^{-2}}{L^{-2} + rx_{(r+1)}^{-2} + \sum\limits_{i=r+1}^{n} x_{(i)}^{-2}}\right\}^{(n-r+1)} &= \frac{k}{2},\\
\left\{\frac{rx_{(r+1)}^{-2} + \sum\limits_{i=r+1}^{n} x_{(i)}^{-2}}{U^{-2} + rx_{(r+1)}^{-2} + \sum\limits_{i=r+1}^{n} x_{(i)}^{-2}}\right\}^{(n-r+1)} &= 1 - \frac{k}{2}.
\end{aligned}$$

The posterior predictive distributions and posterior predictive intervals under remaining priors can be derived in the similar manner.

7. Simulation study

Monte Carlo simulation techniques are widely used in statistical research. Since real-world data sets can often be radically non-normal, it is essential that statisticians have a variety of techniques available for univariate or multivariate non-normal data generation. This section shows how simulation can be helpful and illuminating way to approach problems in Bayesian analysis. Bayesian problems of updating estimates can be handled easily and straight forwardly with simulation. Here, the inverse transformation method of simulation is used to compare the performance of different estimators. The study has been carried out for different values of $(n,\ r)$ using $\lambda \in (1.5,\ 3)$. Censoring rate is assumed to be 20 %. These samples have been drawn by following steps:

Step 1: Draw samples of size 'n' from the inverse Rayleigh model using inverse transformation technique, from the random number generator $X = \left\{-\frac{\lambda}{\ln U}\right\}^{1/2}$, where U is uniformly distributed random variable.

Step 2: Determine the test termination points on left, that is, determine the values of x_r.

Step 3: The observations which are less than or equal to x_r have been considered to be censored.

Step 4: Use the remaining observations for the analysis.

Step 5: Repeat 1000 times steps 1 to 5.

Sample size is varied to observe the effect of small and large samples on the estimators. Changes in the estimators and their risks have been investigated under different loss functions and the prior distributions of λ while keeping the sample size fixed. All these results are based on 1000 repetitions. In the tables, the estimators for the parameter and the risks have been averaged over the total number of repetitions. Mathematica 8.0 has been used to carry out the results. The results are summarized in the following Tables.

Table 2: Bayes Estimates and the Posterior Risks under uniform prior for $\lambda = 1.5$

(n, r)	SELF	PLF	WSELF	QQLF	SLELF	ELF
$20, 4$	1.65438	1.68885	1.55962	1.56275	1.61307	1.56351
	(0.16691)	(0.09517)	(0.09747)	(0.00611)	(0.06059)	(0.03092)
$40, 8$	1.57084	1.61803	1.53742	1.52698	1.54735	1.53151
	(0.07618)	(0.04794)	(0.04804)	(0.00327)	(0.03077)	(0.01554)
$60, 12$	1.54745	1.56902	1.52361	1.5204	1.54264	1.51773
	(0.04945)	(0.03154)	(0.03174)	(0.00223)	(0.02062)	(0.01038)
$80, 16$	1.53326	1.55514	1.51352	1.51698	1.53260	1.51230
	(0.03652)	(0.02365)	(0.02365)	(0.00169)	(0.01550)	(0.00780)
$100, 20$	1.52527	1.54706	1.50741	1.50565	1.52815	1.50905
	(0.02892)	(0.01892)	(0.01884)	(0.00137)	(0.01242)	(0.00624)

Table 3: Bayes Estimates and the Posterior Risks under uniform prior for $\lambda = 3$

(n, r)	SELF	PLF	WSELF	QQLF	SLELF	ELF
$20, 4$	3.28127	3.38036	3.13284	3.04118	3.21141	3.15760
	(0.65885)	(0.19048)	(0.19580)	(0.00161)	(0.06059)	(0.03092)
$40, 8$	3.14288	3.21926	3.05887	3.02651	3.13157	3.09851
	(0.30538)	(0.09539)	(0.09559)	(0.00075)	(0.03077)	(0.01554)
$60, 12$	3.10293	3.14484	3.03532	3.02517	3.05739	3.06246
	(0.19899)	(0.06321)	(0.06324)	(0.00049)	(0.02062)	(0.01038)
$80, 16$	3.07287	3.09768	3.02827	3.01907	3.0475	3.03442
	(0.14662)	(0.04711)	(0.04732)	(0.00036)	(0.01550)	(0.00780)
$100, 20$	3.05216	3.05391	3.02625	3.00844	3.03475	3.01936
	(0.11592)	(0.03736)	(0.03783)	(0.00029)	(0.01242)	(0.00624)

Table 4: Bayes Estimates and the Posterior Risks under Jeffreys prior for $\lambda = 1.5$

(n, r)	SELF	PLF	WSELF	QQLF	SLELF	ELF
20, 4	1.54592	1.58494	1.46501	1.46866	1.51242	1.46208
	(0.15502)	(0.09464)	(0.09717)	(0.00686)	(0.06449)	(0.03296)
40, 8	1.53108	1.56548	1.47425	1.48178	1.50195	1.48275
	(0.07465)	(0.04780)	(0.04756)	(0.00347)	(0.03174)	(0.01604)
60, 12	1.52521	1.52386	1.48231	1.49651	1.50092	1.48749
	(0.04907)	(0.03126)	(0.03154)	(0.00231)	(0.02105)	(0.01060)
80, 16	1.51061	1.51504	1.48574	1.49865	1.50008	1.48941
	(0.03601)	(0.02340)	(0.02358)	(0.00173)	(0.01575)	(0.00792)
100, 20	1.50936	1.51084	1.49553	1.50282	1.49296	1.49240
	(0.02868)	(0.01871)	(0.01875)	(0.00139)	(0.01258)	(0.00632)

Table 5: Bayes Estimates and the Posterior Risks under Jeffreys prior for $\lambda = 3$

(n, r)	SELF	PLF	WSELF	QQLF	SLELF	ELF
20, 4	3.12850	3.25997	2.94141	4.09649	3.04835	2.92104
	(0.63663)	(0.19467)	(0.19509)	(0.00052)	(0.06449)	(0.03296)
40, 8	3.06775	3.13214	2.95996	4.30067	3.02486	2.96144
	(0.29983)	(0.09564)	(0.09548)	(0.00016)	(0.03174)	(0.01604)
60, 12	3.04759	3.06371	2.97743	4.36768	3.01658	2.97633
	(0.19580)	(0.06285)	(0.06322)	(0.00009)	(0.02105)	(0.01060)
80, 16	3.04125	3.05280	2.99203	4.41342	3.00634	2.97724
	(0.14587)	(0.04715)	(0.04712)	(0.00006)	(0.01570)	(0.00792)
100, 20	3.00671	3.03392	2.99604	4.42544	2.99188	2.99068
	(0.11384)	(0.03757)	(0.03747)	(0.00004)	(0.01258)	(0.00632)

Table 6: Bayes Estimates and the Posterior Risks under Exponential prior for $\lambda = 1.5$

(n, r)	SELF	PLF	WSELF	QQLF	SLELF	ELF
20, 4	1.6106	1.65455	1.53696	1.54073	1.57264	1.53441
	(0.15814)	(0.09323)	(0.09606)	(0.00620)	(0.06059)	(0.03092)
40, 8	1.56221	1.57347	1.51859	1.51702	1.53233	1.52564
	(0.07531)	(0.04662)	(0.04745)	(0.00329)	(0.03077)	(0.01554)
60, 12	1.53146	1.55327	1.51816	1.51691	1.51685	1.51188
	(0.04844)	(0.03122)	(0.03163)	(0.00223)	(0.02062)	(0.01038)
80, 16	1.52590	1.54222	1.50248	1.50995	1.50785	1.50765
	(0.03617)	(0.02346)	(0.02348)	(0.00170)	(0.01550)	(0.00780)
100, 20	1.51982	1.53387	1.50210	1.50579	1.50498	1.50085
	(0.02873)	(0.01876)	(0.01878)	(0.00137)	(0.01242)	(0.00624)

Table 7: Bayes Estimates and the Posterior Risks under Exponential prior for $\lambda = 3$

(n, r)	SELF	PLF	WSELF	QQLF	SLELF	ELF
$20, 4$	3.15586	3.22347	2.93299	2.87923	3.07895	2.95351
	(0.60628)	(0.18164)	(0.18331)	(0.00188)	(0.06059)	(0.03092)
$40, 8$	3.09826	3.14776	2.95927	2.93029	3.04844	2.97951
	(0.29615)	(0.09327)	(0.09248)	(0.00084)	(0.03077)	(0.01554)
$60, 12$	3.03718	3.08136	2.98993	2.96242	3.01869	2.98232
	(0.19052)	(0.06194)	(0.06229)	(0.00053)	(0.02062)	(0.01038)
$80, 16$	3.03207	3.07040	2.99373	2.97654	3.00918	2.99233
	(0.14282)	(0.04670)	(0.04678)	(0.00038)	(0.01550)	(0.00780)
$100, 20$	3.01948	3.06507	2.99631	2.99538	3.00560	2.99595
	(0.11341)	(0.03749)	(0.03745)	(0.00030)	(0.01242)	(0.00624)

Table 8: Bayes Estimates and the Posterior Risks under Gamma prior for $\lambda = 1.5$

(n, r)	SELF	PLF	WSELF	QQLF	SLELF	ELF
$20, 4$	1.61114	1.65071	1.52843	1.53665	1.59344	1.53507
	(0.14689)	(0.08689)	(0.08859)	(0.00580)	(0.05632)	(0.02870)
$40, 8$	1.56028	1.59652	1.52167	1.53033	1.54073	1.52177
	(0.07238)	(0.04562)	(0.04576)	(0.00315)	(0.02963)	(0.01496)
$60, 12$	1.55210	1.55784	1.51434	1.52091	1.52122	1.51731
	(0.04857)	(0.03055)	(0.03075)	(0.00217)	(0.02010)	(0.01012)
$80, 16$	1.54299	1.54204	1.50799	1.51413	1.51736	1.51167
	(0.03639)	(0.02302)	(0.02311)	(0.00166)	(0.01521)	(0.00764)
$100, 20$	1.52934	1.53918	1.50351	1.51078	1.51279	1.50462
	(0.02864)	(0.01854)	(0.01850)	(0.00134)	(0.01223)	(0.00614)

Table 9: Bayes Estimates and the Posterior Risks under Gamma prior for $\lambda = 3$

(n, r)	SELF	PLF	WSELF	QQLF	SLELF	ELF
$20, 4$	2.98550	3.07935	2.81061	2.705575	2.89848	2.80147
	(0.50252)	(0.16209)	(0.16292)	(0.00194)	(0.05632)	(0.02870)
$40, 8$	2.98639	3.03123	2.88127	2.86840	2.94528	2.91509
	(0.26478)	(0.08661)	(0.08665)	(0.00086)	(0.02963)	(0.01496)
$60, 12$	2.9885	3.02076	2.94717	2.91546	2.97597	2.92674
	(0.17958)	(0.05923)	(0.05984)	(0.00054)	(0.02010)	(0.01012)
$80, 16$	2.99289	3.01938	2.96630	2.96747	2.99081	2.95949
	(0.13630)	(0.04506)	(0.04546)	(0.00039)	(0.01521)	(0.00764)
$100, 20$	3.00225	3.01149	2.97512	2.96775	2.99704	2.96233
	(0.11029)	(0.03628)	(0.03661)	(0.00030)	(0.01223)	(0.00614)

Table 10: Bayes Estimates and the Posterior Risks under Inverse Levy prior for $\lambda = 1.5$

(n, r)	SELF	PLF	WSELF	QQLF	SLELF	ELF
20, 4	1.52733	1.59066	1.42438	1.45927	1.47818	1.36727
	(0.14257)	(0.09223)	(0.09189)	(0.00670)	(0.06280)	(0.03191)
40, 8	1.52690	1.53887	1.47429	1.48728	1.48149	1.43716
	(0.07115)	(0.04628)	(0.04673)	(0.00340)	(0.03125)	(0.01579)
60, 12	1.51715	1.52616	1.48734	1.47513	1.4927	1.46167
	(0.04805)	(0.03099)	(0.03131)	(0.00232)	(0.02083)	(0.01049)
80, 16	1.51253	1.52309	1.48873	1.49207	1.49302	1.46576
	(0.03580)	(0.02334)	(0.02344)	(0.00173)	(0.01562)	(0.00785)
100, 20	1.50978	1.51239	1.49052	0.49894	1.50814	1.47801
	(0.02853)	(0.01861)	(0.01875)	(0.00138)	(0.01250)	(0.00628)

Table 11: Bayes Estimates and the Posterior Risks under Inverse Levy prior for $\lambda = 3$

(n, r)	SELF	PLF	WSELF	QQLF	SLELF	ELF
20, 4	2.90203	2.95439	2.72146	2.67184	2.84175	2.53566
	(0.52572)	(0.17131)	(0.17558)	(0.00238)	(0.06280)	(0.03191)
40, 8	2.95368	2.98431	2.88166	2.85057	2.90273	2.74453
	(0.27331)	(0.08976)	(0.09148)	(0.00093)	(0.03125)	(0.01579)
60, 12	2.98041	2.99114	2.89472	2.89688	2.95163	2.81328
	(0.18523)	(0.06088)	(0.06094)	(0.00057)	(0.02083)	(0.01049)
80, 16	2.98657	2.99563	2.93044	2.92831	2.96835	2.86574
	(0.13956)	(0.04591)	(0.04615)	(0.00041)	(0.01562)	(0.00785)
100, 20	3.00038	3.00845	2.94196	2.93896	2.98091	2.90752
	(0.11266)	(0.03703)	(0.03701)	(0.00032)	(0.01250)	(0.00628)

Table 12: The lower limit (LL), the upper limit (UL) and the width of the 95% credible intervals under uniform prior

	$\lambda = 1.5$			$\lambda = 3$		
n, r	LL	UL	Width	LL	UL	Width
20, 4	0.91651	2.40465	1.48814	1.87279	4.91366	3.04087
40, 8	1.06531	2.11858	1.05327	2.13053	4.23696	2.10643
60, 12	1.12568	1.97623	0.85055	2.25301	3.95537	1.70236
80, 16	1.17637	1.91647	0.74010	2.36372	3.85081	1.48709
100, 20	1.20426	1.86402	0.65976	2.41714	3.74139	1.32425

Table 13: The lower limit (LL), the upper limit (UL) and the width of the 95% credible intervals under Jeffreys prior

	$\lambda = 1.5$			$\lambda = 3$		
n, r	LL	UL	Width	LL	UL	Width
$20, 4$	0.84638	2.28964	1.44326	1.72949	4.67864	2.94915
$40, 8$	1.0265	2.06359	1.03709	2.05290	4.12700	2.07410
$60, 12$	1.09900	1.94079	0.84179	2.19962	3.88445	1.68483
$80, 16$	1.15579	1.89016	0.73437	2.32235	3.79794	1.47559
$100, 20$	1.18759	1.84325	0.65566	2.38367	3.69970	1.31603

Table 14: The lower limit (LL), the upper limit (UL) and the width of the 95% credible intervals under exponential prior

	$\lambda = 1.5$			$\lambda = 3$		
n, r	LL	UL	Width	LL	UL	Width
$20, 4$	0.89565	2.34993	1.45428	1.78772	4.69046	2.90274
$40, 8$	1.05289	2.09387	1.04098	2.08140	4.13927	2.05787
$60, 12$	1.11695	1.96091	0.84396	2.21832	3.89446	1.67614
$80, 16$	1.16947	1.90523	0.73576	2.33602	3.80569	1.46967
$100, 20$	1.19861	1.85528	0.65667	2.39450	3.70634	1.31184

Table 15: The lower limit (LL), the upper limit (UL) and the width of the 95% credible intervals under gamma prior

	$\lambda = 1.5$			$\lambda = 3$		
n, r	LL	UL	Width	LL	UL	Width
$20, 4$	0.92251	2.33811	1.41560	1.73604	4.40004	2.66436
$40, 8$	1.06567	2.09220	1.02653	2.0425	4.00999	1.96749
$60, 12$	1.12528	1.96152	0.83624	2.18824	3.81439	1.62615
$80, 16$	1.17547	1.90609	0.73062	2.31013	3.74598	1.43585
$100, 20$	1.20333	1.85634	0.65301	2.37272	3.66031	1.28759

Table 16: The lower limit (LL), the upper limit (UL) and the width of the 95% credible intervals under inverse levy prior

	$\lambda = 1.5$			$\lambda = 3$		
n, r	LL	UL	Width	LL	UL	Width
$20, 4$	0.80664	2.14825	1.34161	1.51435	4.03302	2.51867
$40, 8$	0.99900	1.99735	0.99835	1.91219	3.82314	1.91095
$60, 12$	1.07880	1.89949	0.82069	2.09593	3.69037	1.59444
$80, 16$	1.13934	1.85967	0.72033	2.23752	3.65216	1.41464
$100, 20$	1.17393	1.81955	0.64562	2.31340	3.58 57	1.27230

Table 17: The lower limit (LL), the upper limit (UL) of the 95% predictive intervals

	Uniform			Jeffreys			Exponential		
n, r	LL	UL	Width	LL	UL	Width	LL	UL	Width
20, 4	0.61798	7.88009	7.26211	0.54742	7.64463	7.09721	0.61091	7.78991	7.17900
40, 8	0.62970	7.81693	7.18723	0.61954	7.69754	7.07800	0.62602	7.77120	7.14518
60, 12	0.63020	7.75137	7.12117	0.62349	7.67185	7.04845	0.62775	7.72127	7.09352
80, 16	0.63371	7.75038	7.11667	0.63867	7.68045	7.04178	0.63185	7.72159	7.08974
100, 20	0.63387	7.73862	7.10475	0.64985	7.69069	7.04084	0.63238	7.72048	7.08810

Table 18: The lower limit (LL), the upper limit (UL) of the 95% predictive intervals.

	Gamma			Inverse Levy		
n, r	LL	UL	Width	LL	UL	Width
20, 4	0.61579	7.82219	7.20640	0.58145	7.42692	6.84547
40, 8	0.62810	7.78892	7.16082	0.61048	7.58158	6.97110
60, 12	0.62911	7.73433	7.10522	0.61733	7.59460	6.97727
80, 16	0.63281	7.73527	7.10246	0.62392	7.63935	7.01543
100, 20	0.63315	7.72844	7.09529	0.62603	7.64342	7.01739

Example 7.1. To illustrate the applicability of the estimation techniques developed in previous sections, we considered the analysis of the real data set which was also used by Lawless (1982). These data are from Nelson (1982), concerning the data on time to breakdown of an insulating fluid between electrodes at a voltage of 34 kV (minutes). The sample characteristics required to evaluate the estimates of scale parameter of inverse Rayleigh distribution are as follows $r = 4$, $\sum_{i=5}^{19} x_{(i)}^{-2} = 0.458966$, $x_{(5)}^{-2} = 0.129393$.

Table 19: Bayes estimates and the corresponding posterior risks assuming the real life data set

	SELF		PLF		WSELF	
Prior	BEs	PRs	BEs	PRs	BEs	PRs
Uniform	16.3844	16.7781	16.8887	1.00851	15.3604	1.02403
Jeffreys	15.3604	15.7294	15.8641	1.00750	14.3364	1.02403
Exponential	13.0275	10.6073	13.4285	0.80188	12.2133	0.81422
Gamma	8.86947	4.56000	9.12291	0.49782	8.35534	0.51412
Inverse-Levy	7.83876	3.96427	8.08767	0.506882	7.33304	0.50572

Table 20: Bayes estimates and the corresponding posterior risks assuming the real life data set

	QQLF		SLELF		ELF	
Prior	BEs	PRs	BEs	PRs	BEs	PRs
Uniform	11.2814	1.7857×10^{-8}	15.8751	0.06449	15.3604	0.03296
Jeffreys	10.5763	5.4261×10^{-8}	14.8513	0.06894	14.3364	0.03529
Exponential	9.53050	1.8740×10^{-7}	12.6226	0.06449	12.2133	0.03296
Gamma	7.15661	4.4232×10^{-6}	8.61368	0.05968	8.35534	0.03045
Inverse-Levy	6.34377	1.6662×10^{-5}	7.5873	0.06664	7.33304	0.03409

8. Conclusions

The simulation study has displayed some interesting properties of the Bayes estimates. The risks of the estimates seem to be larger for the larger values of the parameter and vice versa, except under quasi-quadratic loss function. However, the risks under the said loss functions reduce as the sample size increases. Another interesting remark concerning the risks of the estimates is that increasing (decreasing) the value of the parameter reduces (increases) the risks of the estimates under quasi-quadratic loss function. The performance of squared-log error loss function and entropy loss function is independent of choice of parametric value. The increased values of the parameter result in higher levels of overestimation/underestimation of the parameters. In comparison of non-informative priors, the Jeffreys prior provides the better estimates as the corresponding risks are smaller under WSELF, PLF and SELF. While the uniform prior turns out to perform better under QQLF, SLELF and ELF. On the other hand, in comparison of informative and non-informative priors, gamma prior turns out to perform better under said loss functions except QQLF and SELF; therefore it produces more efficient estimates as compared to other informative or non-informative priors. In addition, estimates under quasi-quadratic loss function give the minimum risks among all loss functions for each prior. It can also be observed that the performance of estimates under informative priors is better than those under non-informative priors. The credible intervals are in accordance with the point estimates, that is, the width of credible interval is inversely proportional to sample size while, it is directly proportional to the parametric value. Tables 11-15, appended above, reveal that the effect of the bigger parametric values is in the form of larger width of interval. The credible intervals assuming inverse Levy prior are much narrower than the credible intervals assuming rest of informative and non-informative priors. It is the use of prior information that makes a difference in terms of gain in precision. The Bayesian predictive intervals of the future observation assuming informative and non-informative priors have also been constructed. The Inverse Levy prior can produce more precise predictive intervals than (its competitor) other informative and non-informative priors. The results of above data set clearly show that the gamma prior has the least posterior risk as compared to its competitor priors under majority of the loss functions except QQLF and SELF. The real life example dipicts the similar behaviour of the estimates of parameter under study, which we have observed under simmulation study. So result of simmulation study and real data set are similar in all aspects. In future this study can be extended under different informative priors, using some other loss functions and under different censoring schemes and also for mixture distribution.

Appendix 1. Derivation of formulae for Bayes estimators and posterior risks under WSELF and ELF

1.1. Derivation of Bayes estimator under the WSELF

The expression for WSELF is given as

$$L\left(\lambda,\ \hat{\lambda}\right) = \frac{\left(\lambda\ -\ \hat{\lambda}\right)^2}{\lambda}. \tag{2}$$

Taking expectation, we have

$$E\left\{L\left(\lambda,\ \hat{\lambda}\right)\right\} = E\left\{\frac{\left(\lambda\ -\ \hat{\lambda}\right)^2}{\lambda}\right\}.$$

Taking partial derivative with respect to $\hat{\lambda}$ and equating to zero, we have

$$\frac{\partial E\left\{L\left(\lambda,\ \hat{\lambda}\right)\right\}}{\partial\hat{\lambda}} = \frac{2}{\lambda}E\left(\lambda\ -\ \hat{\lambda}\right)(-1) = 0$$

which implies

$$\frac{2}{\lambda}E\left(\lambda\ -\ \hat{\lambda}\right)(-1) = 0.$$

After simplifications, the Bayes estimator under WSELF is derived as

$$\hat{\lambda}_{WSELF} = \left\{E\left(\lambda^{-1}\middle|\,\mathbf{x}\right)\right\}^{-1}. \tag{3}$$

1.2. Derivation of posterior risk under WSELF

Taking expectation on (2), we have

$$\begin{aligned} E\left\{L\left(\lambda,\ \hat{\lambda}\right)\right\} &= E\left\{\frac{\left(\lambda\ -\ \hat{\lambda}\right)^2}{\lambda}\right\}, \\ E\left\{L\left(\lambda,\ \hat{\lambda}\right)\right\} &= E\left(\lambda+\hat{\lambda}^2\lambda^{-1}-2\hat{\lambda}\right), \\ E\left\{L\left(\lambda,\ \hat{\lambda}\right)\right\} &= E\left(\lambda|\,\mathbf{x}\right)+\hat{\lambda}^2 E\left(\lambda^{-1}\middle|\,\mathbf{x}\right)-2\hat{\lambda}. \end{aligned} \tag{4}$$

From (3), we know that $\hat{\lambda}=\left\{E\left(\lambda^{-1}\middle|\,\mathbf{x}\right)\right\}^{-1}$, putting this in (4), we have

$$\begin{aligned} E\left\{L\left(\lambda,\ \hat{\lambda}\right)\right\} &= E\left(\lambda|\,\mathbf{x}\right)+\left\{E\left(\lambda^{-1}\middle|\,\mathbf{x}\right)\right\}^{-2}E\left(\lambda^{-1}\middle|\,\mathbf{x}\right)-2\left\{E\left(\lambda^{-1}\middle|\,\mathbf{x}\right)\right\}^{-1}, \\ E\left\{L\left(\lambda,\ \hat{\lambda}\right)\right\} &= E\left(\lambda|\,\mathbf{x}\right)+\left\{E\left(\lambda^{-1}\middle|\,\mathbf{x}\right)\right\}^{-1}-2\left\{E\left(\lambda^{-1}\middle|\,\mathbf{x}\right)\right\}^{-1}. \end{aligned}$$

Hence, the posterior risk under WSELF is:

$$E\left\{L\left(\lambda,\ \hat{\lambda}\right)\right\}=E\left(\lambda|\,\mathbf{x}\right)-\left\{E\left(\lambda^{-1}\middle|\,\mathbf{x}\right)\right\}^{-1}.$$

1.3. Derivation of the Bayes estimator under ELF

The expression for ELF is given as:

$$L\left(\lambda,\ \hat{\lambda}\right)=b\left\{\left(\frac{\hat{\lambda}}{\lambda}\right)-\ln\left(\frac{\hat{\lambda}}{\lambda}\right)-1\right\}. \tag{5}$$

Without loss of generality, we can assume $b=1$, therefore (5) becomes

$$L\left(\lambda,\ \hat{\lambda}\right)=\left\{\left(\frac{\hat{\lambda}}{\lambda}\right)-\ln\left(\frac{\hat{\lambda}}{\lambda}\right)-1\right\}. \tag{6}$$

Taking expectation, we have

$$E\left\{L\left(\lambda,\ \hat{\lambda}\right)\right\}=E\left\{\left(\frac{\hat{\lambda}}{\lambda}\right)-\ln\left(\frac{\hat{\lambda}}{\lambda}\right)-1\right\}$$

Taking partial derivative with respect to $\hat{\lambda}$ and equating to zero, we have

$$\frac{\partial E\left\{L\left(\lambda,\ \hat{\lambda}\right)\right\}}{\partial\hat{\lambda}}=E\left\{\left(\frac{1}{\lambda}\right)-\left(\frac{1}{\lambda}\right)\left(\frac{\lambda}{\hat{\lambda}}\right)\right\}=0$$

which implies

$$E\left\{\left(\frac{1}{\lambda}\right)-\left(\frac{1}{\lambda}\right)\left(\frac{\lambda}{\hat{\lambda}}\right)\right\}=0.$$

After simplifications, the Bayes estimator under ELF is derived as

$$\hat{\lambda}_{ELF}=\left\{E\left(\lambda^{-1}\middle|\,\mathbf{x}\right)\right\}^{-1}. \tag{7}$$

1.4. Derivation of the posterior risk under ELF

Taking expectation on (6), we have

$$E\left\{L\left(\lambda,\ \hat{\lambda}\right)\right\} = E\left\{\left(\frac{\hat{\lambda}}{\lambda}\right) - \ln\left(\frac{\hat{\lambda}}{\lambda}\right) - 1\right\}$$

$$= \hat{\lambda}E\left(\lambda^{-1}\middle|\,\mathbf{x}\right) - \ln\hat{\lambda} + E\left(\ln\lambda|\,\mathbf{x}\right) - 1. \tag{8}$$

From (7), we know that $\hat{\lambda} = \left\{E\left(\lambda^{-1}\middle|\,\mathbf{x}\right)\right\}^{-1}$, putting this in (8), we have

$$E\left\{L\left(\lambda,\ \hat{\lambda}\right)\right\} = 1 + \ln E\left(\lambda^{-1}\middle|\,\mathbf{x}\right) + E\left(\ln\lambda|\,\mathbf{x}\right) - 1.$$

Hence, the posterior risk under ELF is:

$$E\left\{L\left(\lambda,\ \hat{\lambda}\right)\right\} = \ln E\left(\lambda^{-1}\middle|\,\mathbf{x}\right) + E\left(\ln\lambda|\,\mathbf{x}\right). \tag{9}$$

The Bayes estimators and posterior risks under rest of the loss functions can be derived in a similar manner.

Appendix 2. Derivation of expression for Bayes estimators and posterior risks using uniform prior under WSELF and ELF

2.1. Derivation of Bayes estimator and posterior risk under WSELF using uniform prior

The formula for the Bayes estimator under WSELF is:

$$\hat{\lambda}_{WSELF} = \left\{E\left(\lambda^{-1}\middle|\,\mathbf{x}\right)\right\}^{-1} \tag{10}$$

where $E\left(\lambda^{-1}\middle|\,\mathbf{x}\right)$ can be find out as

$$E\left(\lambda^{-1}\middle|\,\mathbf{x}\right) = \int_0^\infty \lambda^{-1} p\left(\lambda|\,\mathbf{x}\right) d\lambda = \int_0^\infty \lambda^{-1} \frac{\left\{\tau_{(ir)}\right\}^{n-r+1} \lambda^{n-r} \exp\left\{-\lambda\tau_{(ir)}\right\}}{\Gamma\left(n-r+1\right)} d\lambda,$$

$$= \frac{\left\{\tau_{(ir)}\right\}^{n-r+1}}{\Gamma\left(n-r+1\right)} \int_0^\infty \lambda^{n-r-1} \exp\left\{-\lambda\tau_{(ir)}\right\} d\lambda.$$

Let $\lambda\tau_{(ir)} = t$, $\Rightarrow \tau_{(ir)} d\lambda = dt$

$$= \frac{\left\{\tau_{(ir)}\right\}^{n-r+1}}{\Gamma\left(n-r+1\right)} \int_0^\infty \left(\frac{t}{\tau_{(ir)}}\right)^{(n-r-1)} \exp\left(-t\right) \left(\frac{1}{\tau_{(ir)}}\right) dt,$$

$$= \frac{\left\{\tau_{(ir)}\right\}^{n-r+1}}{\Gamma\left(n-r+1\right)} \frac{\Gamma\left(n-r\right)}{\left\{\tau_{(ir)}\right\}^{n-r}} = \frac{\tau_{(ir)}}{n-r} = \frac{r x_{(r+1)}^{-2} + \sum\limits_{i=r+1}^{n} x_{(i)}^{-2}.}{n-r}$$

Therefore (10) implies

$$\hat{\lambda}_{WSELF} = \left\{E\left(\lambda^{-1}\middle|\,\mathbf{x}\right)\right\}^{-1} = \frac{n-r}{r x_{(r+1)}^{-2} + \sum\limits_{i=r+1}^{n} x_{(i)}^{-2}}. \tag{11}$$

2.2. Derivation of posterior risk under WSELF using uniform prior

The formula for the posterior risk under WSELF is:

$$\rho\left(\hat{\lambda}_{WSELF}\right) = E\left(\lambda|\,\mathbf{x}\right) - \left\{E\left(\lambda^{-1}\middle|\,\mathbf{x}\right)\right\}^{-1} and E\left(\lambda|\,\mathbf{x}\right) \tag{12}$$

$$\begin{aligned} E\left(\lambda|\,\mathbf{x}\right) &= \int_0^\infty \lambda p\left(\lambda|\,\mathbf{x}\right) d\lambda = \int_0^\infty \lambda \frac{\left\{\tau_{(ir)}\right\}^{n-r+1} \lambda^{n-r} \exp\left\{-\lambda\tau_{(ir)}\right\}}{\Gamma\left(n-r+1\right)} d\lambda, \\ &= \frac{\left\{\tau_{(ir)}\right\}^{n-r+1} \Gamma\left(n-r+2\right)}{\Gamma\left(n-r+1\right)\left\{\tau_{(ir)}\right\}^{n-r+2}} = \frac{n-r+1}{\tau_{(ir)}} \\ &= \frac{n-r+1}{rx_{(r+1)}^{-2} + \sum\limits_{i=r+1}^{n} x_{(i)}^{-2}}. \end{aligned} \tag{13}$$

Putting (11) and (13) in (12), we have

$$\begin{aligned} \rho\left(\hat{\lambda}_{WSELF}\right) &= \frac{n-r+1}{\left\{rx_{(r+1)}^{-2} + \sum\limits_{i=r+1}^{n} x_{(i)}^{-2}\right\}} - \frac{n-r}{\left\{rx_{(r+1)}^{-2} + \sum\limits_{i=r+1}^{n} x_{(i)}^{-2}\right\}} \\ &= \frac{1}{\left\{rx_{(r+1)}^{-2} + \sum\limits_{i=r+1}^{n} x_{(i)}^{-2}\right\}}. \end{aligned}$$

2.3. Derivation of the Bayes estimator under ELF using uniform prior

The formula for the Bayes estimator under ELF is:

$$\hat{\lambda}_{ELF} = \left\{E\left(\lambda^{-1}\middle|\,\mathbf{x}\right)\right\}^{-1} = \frac{n-r}{\left\{rx_{(r+1)}^{-2} + \sum\limits_{i=r+1}^{n} x_{(i)}^{-2}\right\}}.$$

2.4. Derivation of the posterior risk under ELF using uniform prior

The formula for the posterior risk under ELF is:

$$\rho\left(\hat{\lambda}_{ELF}\right) = \ln\left\{E\left(\lambda^{-1}\middle|\,\mathbf{x}\right)\right\} + E\left(\ln\lambda|\,\mathbf{x}\right) \tag{14}$$

here $\ln\left\{E\left(\lambda^{-1}\middle|\,\mathbf{x}\right)\right\}$is evaluated as

$$\ln\left\{E\left(\lambda^{-1}\middle|\,\mathbf{x}\right)\right\} = \ln\left\{\frac{rx_{(r+1)}^{-2} + \sum\limits_{i=r+1}^{n} x_{(i)}^{-2}}{n-r}\right\} \tag{15}$$

and

$$E\left(\ln\lambda|\,\mathbf{x}\right) = \frac{\left\{\tau_{(ir)}\right\}^{n-r+1}}{\Gamma\left(n-r+1\right)} \int_0^\infty \left(\ln\lambda\right) \lambda^{n-r} \exp\left\{-\lambda\tau_{(ir)}\right\} d\lambda, \tag{16}$$

since

$$\int_0^\infty (\ln x)\, x^{a-1} \exp\{-bx\}\, dx = \frac{\Gamma(a)}{b^a}\{\psi(a) - \ln b\},$$

therefore (16) becomes

$$\begin{aligned} &= \frac{\left\{\tau_{(ir)}\right\}^{n-r+1}}{\Gamma(n-r+1)}\left[\frac{\Gamma(n-r+1)}{\left\{\tau_{(ir)}\right\}^{n-r+1}}\left\{\psi(n-r+1) - \ln\left(rx_{(r+1)}^{-2} + \sum_{i=r+1}^{n} x_{(i)}^{-2}\right)\right\}\right], \\ &= \left\{\psi(n-r+1) - \ln\left(rx_{(r+1)}^{-2} + \sum_{i=r+1}^{n} x_{(i)}^{-2}\right)\right\}. \qquad (17) \end{aligned}$$

Putting (15) and (17) in (14), we have

$$\rho\left(\hat{\lambda}_{ELF}\right) = \ln\left\{\frac{rx_{(r+1)}^{-2} + \sum_{i=r+1}^{n} x_{(i)}^{-2}}{n-r}\right\} + \left\{\psi(n-r+1) - \ln\left(rx_{(r+1)}^{-2} + \sum_{i=r+1}^{n} x_{(i)}^{-2}\right)\right\}.$$

The Bayes estimators and posterior risks under rest of the loss functions can be derived in a similar manner.

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