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A characterization of power function distribution based on lower records

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Abstract. Characterization of a probability distribution plays important role in probability and statistics. This paper discusses a new characterization of power function distribution based on lower records. It is hoped that the findings of the paper will be useful for researchers in different fields of applied sciences.

1. Introduction

An observation is called a record if its value is greater than (or analogously, less than) all the preceding observations. Record values arise naturally in many fields of studies such as climatology, sports, science, engineering, medicine, traffic, and industry, among others. As such study of their properties and applications play important roles in many areas of statistical research, for example, statistical inference, nonparametric statistics, among others. For more on record values, we refer the readers to Ahsanullah (2004), and, recently, to Su et al. (2008), among others. This paper discusses a new characterization of power function distribution based on lower records.

A random variable X is said to have the power function distribution if its cdf and pdf are, respectively, given by

$$G(x) = \left(\frac{x}{\lambda}\right)^{\alpha}, \tag{1}$$

and

$$g(x) = \frac{\alpha}{\lambda}\left(\frac{x}{\lambda}\right)^{\alpha-1}, \tag{2}$$

where $\alpha > 0$ and $0 < x < \lambda$. For statistical properties of power function distribution, see, for example, Johnson et al. (1994).

The organization of this paper is as follows. In Section 2, the exact distributions of record values are provided. Section 3 contains distribution of lower record values based on power function distribution. In Section 4, a new characterization of power function distribution based on lower record values is presented. The concluding remarks are provided in Section 5.

2. The exact distributions of record values

Here we provide the exact distributions of record values.

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2.1. Record values: definitions and notations

Here we provide the distribution of record values. For detailed treatment of record values, the interested readers are referred to Ahsanullah (2004) and references therein.

Definition 2.1 (Record values). Suppose that $(X_n)_{n\geq 1}$ is a sequence of independent and identically distributed random variables with cdf $F(x)$. Let $Y_n = \max(\min)\{X_j|1 \leq j \leq n\}$ for $n \geq 1$. We say that X_j is an upper (lower) record value of $\{X_n|n \geq 1\}$, if $Y_j > (<)Y_{j-1}$, $j > 1$. By definition X_1 is an upper as well as a lower record value.

Definition 2.2 (Lower Record Values). The indices at which the lower record values occur are given by the record times $\{L(n), n \geq 1\}$, where $L(n) = \min\{j|j > L(n-1), X_j < X_{L(n-1)}, n \geq 1\}$ and $L(1) = 1$. We will denote $L(n)$ as the indices where the lower record values occur. The nth lower record value will be denoted by $X_{L(n)}$. If we define $F_n(x)$ as the cdf of $X_L(n)$ for $n \geq 1$, then we have

$$F_n(x) = \int_{-\infty}^{x} \frac{(H(u))^{n-1}}{(n-1)!} dF(u), \quad -\infty < x < \infty, \tag{3}$$

where $H(x) = -\ln F(x)$, and $h(x) = -\frac{d}{dx}H(x) = f(x)(F(x))^{-1}$. The pdf of $X_{L(n)}$, denoted by f_n, is

$$f_n(x) = \frac{(H(x))^{n-1}}{\Gamma(n)} f(x), \quad -\infty < x < \infty. \tag{4}$$

Definition 2.3 (Upper Record Values). The indices at which the upper record values occur are given by the record times $\{U(n), n \geq 1\}$, where $U(n) = \min\{j|j > U(n-1), X_j > X_{U(n-1)}, n > 1\}$ and $U(1) = 1$. Many properties of the upper record value sequence can be expressed in terms of the cumulative hazard rate function $R(x) = -\ln\overline{F}(x)$, where $\overline{F}(x) = 1 - F(x)$, $0 < \overline{F}(x) < 1$. If we define $F_n(x)$ as the cdf of $X_{U(n)}$ for $n \geq 1$, then we have

$$F_n(x) = \int_{-\infty}^{x} \frac{(R(u))^{n-1}}{\Gamma(n)} dF(u), \ -\infty < x < \infty,$$

from which it is easy to see that

$$F_n(x) = 1 - \overline{F}(x) \sum_{j=0}^{n-1} \frac{(R(x))^j}{\Gamma(j+1)},$$

that is,

$$\overline{F}_n(x) = \overline{F}(x) \sum_{j=0}^{n-1} \frac{(R(x))^j}{\Gamma(j+1)}.$$

The pdf of $X_{U(n)}$, denoted by f_n is

$$f_n(x) = \frac{(R(x))^{n-1}}{\Gamma(n)} f(x), \quad -\infty < x < \infty.$$

It is easy to see that

$$\overline{F}_n(x) - \overline{F}_{n-1}(x) = \overline{F}(x) \frac{f_n(x)}{f(x)}.$$

3. Distribution of lower record values based on power function distribution

The pdf f_n and cdf F_n of the nth lower record value $X_{L(n)}$ from power function distribution can easily be obtained by using Equations (1) and (2) in (3) and (4), respectively, as

$$f_n(x) = \frac{[-\ln(x/\lambda)^{\alpha}]^{n-1}\left[(\alpha/\lambda)(x/\lambda)^{\alpha-1}\right]}{\Gamma(n)},$$

and

$$F_n(x) = \frac{\Gamma(n, -\ln(x/\lambda)^{\alpha})}{\Gamma(n)},$$

where $n > 0$ (an integer), $0 < x < \lambda$, $\alpha > 0$, and $\Gamma(c,z) = \int\limits_z^{\infty} e^{-t} t^{c-1} dt$, $c > 0$, denotes incomplete gamma function. The kth moment of the nth lower record value $X_{L(n)}$ from power function distribution is easily given by

$$E\left[X_{L(n)}^k\right] = \int\limits_0^{\lambda} x^k \frac{[-\ln(x/\lambda)^{\alpha}]^{n-1}\left[(\alpha/\lambda)(x/\lambda)^{\alpha-1}\right]}{\Gamma(n)} dx.$$

Letting $-\ln(x/\lambda)^{\alpha} = u$ in the above equation and simplifying it, the expression for the kth moment of the nth lower record value $X_{L(n)}$ is easily obtained as

$$E\left[X_{L(n)}^k\right] = \frac{\lambda^k \alpha^n}{(\alpha+k)^n},$$

from which the single moment, the second single moment and variance of the nth lower record value $X_{L(n)}$ from power function distribution can be obtained respectively as follows

$$\begin{aligned} E\left[X_{L(n)}\right] &= \frac{\lambda \alpha^n}{(\alpha+1)^n} \\ E\left[X_{L(n)}^2\right] &= \frac{\lambda^2 \alpha^n}{(\alpha+2)^n} \\ Var\left[X_{L(n)}\right] &= \lambda^2 \alpha^n \left[\frac{1}{(\alpha+2)^n} - \frac{\alpha^n}{(\alpha+1)^{2n}}\right]. \end{aligned}$$

For a discussion on record values from power function distribution, one is referred to Ahsanullah (2004), where distributional properties of upper record values from a three parameter power function distribution, recurrence relation between moments, estimation of the parameters and prediction of record values have been considered.

4. A characterization of power function distribution

This section presents a new characterization of power function distribution based on lower records, as given below.

Theorem 4.1. *Let X be an absolutely continuous (with respect to Lebesgue measure) random variable with cumulative distribution function $F(x)$. Assume that $F(0) = 0$ and $F(1) = 1$. Then X has a power function distribution with $F(x) = x^{\delta}$, $0 < x < 1$, $\delta > 0$, if and only if*

$$(\delta + k)\, E(X_{L(n+1)}^k | X_{L(m)} = x) = \delta\, E(X_{L(n)}^k | X_{L(m)} = x).$$

Proof. Suppose that $F(x) = x^{\delta}$, $f(x) = \delta x^{\delta-1}$, $0 < x < 1$, $\delta > 0$. Then

$$\begin{aligned} \mathrm{l_{n+1,m}} = &\; E \quad (X_{L(n+1)}^k | X_{L(m)} = x) \\ = &\int_0^x \frac{y^k}{\Gamma(n-m+1)} (\delta(\ln x - \ln y))^{n-m} \frac{\delta\, y^{\delta-1}}{x^{\delta}} dy \\ = &\int_0^x \frac{y^k}{\Gamma(n-m+1)} (\delta)^{n-m+1} (-\ln \frac{y}{x}))^{n-m} \frac{y^{\delta-1}}{x^{\delta-1}} \frac{dy}{x} \end{aligned}$$

Let $-\ln \frac{y}{x} = u$,then $y = xe^{-u}$, $\frac{dy}{x} = -e^{-u} du$, and

$$l_{n+1,m} = \int_0^{\infty} \frac{(xe^{-u})^k}{\Gamma(n-m+1)} (\delta)^{n-m+1} (u)^{n-m} (e^{-u})^{\delta-1} e^{-u} du$$

$$
\begin{aligned}
&= \frac{x^k \delta^{n-m+1}}{\Gamma(n-m+1)} \int_0^\infty e^{-(k+\delta)u} u^{n-m} (e^{-u})^{\delta-1} du \\
&= x^k \left(\frac{\delta}{k+\delta}\right)^{n-m+1}.
\end{aligned}
$$

Thus

$$(k+\delta)\, E(X^k_{L(n+1)} | X_{L(m)} = x) \;=\; \delta\, x^k \left(\frac{\delta}{k\,+\,\delta}\right)^{n\,-\,m}.$$

Putting $n+1 = n$, we obtain

$$l_{n,m} = E(X^k_{L(n)} | X_{L(m)} = x) = x^k \left(\frac{\delta}{k+\delta}\right)^{n-m}.$$

and

$$(\delta + k) E(X^k_{L(n)} | X_{L(m)} = x) = \delta\, x^k \left(\frac{\delta}{k+\delta}\right)^{n-m-1}.$$

Thus

$$(k+\delta)\, E(X^k_{L(n+1)} | X_{L(m)} = x) = \frac{\delta}{\delta+k} \delta\, x^k \left(\frac{\delta}{k\,+\,\delta}\right)^{n-m-1} = \delta\, E(X^k_{L(n)} | X_{L(m)} = x).$$

Now, suppose that

$$(\delta + k)\, E(X^k_{L(n+1)} | X_{L(m)} = x) = \delta\, E(X^k_{L(n)} | X_{L(m)} = x),$$

then

$$(k+\delta) \int_0^x \frac{y^k}{\Gamma(n-m+1)} (\ln F(x) - \ln F(y))^{n-m} \frac{f(y)}{F(x)} dy = \delta \int_0^x \frac{y^k}{\Gamma(n-m)} (\ln F(x) - \ln F(y))^{n-m-1} \frac{f(y)}{F(x)} dy.$$

Canceling $F(x)$ from both sides, we obtain from the above equation

$$(k+\delta) \int_0^x \frac{y^k}{\Gamma(n-m+1)} (\ln F(x) - \ln F(y))^{n-m} f(y) dy = \delta \int_0^x \frac{y^k}{\Gamma(n-m)} (\ln F(x) - \ln F(y))^{n-m-1} f(y) dy.$$

Differentiating both sides of the equation with respect to x and simplifying for n-m times, we obtain

$$(k+\delta) h(x) \int_0^x y^k f(y) dy = \delta\, x^k f(x), h(x) = \frac{f(x)}{F(x)}$$

Thus

$$(k+\delta) \int_0^x y^k f(y) dy = \delta\, x^k F(x).$$

Differentiating the above equation with respect to x, we get

$$(k+\delta) x^k f(x) \;=\; \delta\, x^k f(x) + \delta\, k\, x^{k-1} F(x),$$

that is,

$$\frac{f(x)}{F(x)} = \frac{\delta}{x}.$$

On integrating the above equation with respect to x and using the boundary condition $F(0) = 0$ and $F(1) = 1$, we obtain

$$F(x) \;=\; x^\delta, \quad 0 < x < 1,\; \delta > 0.$$

This completes the proof of Theorem 4.1. □

5. Concluding remarks

The purpose of this research paper was to obtain a new characterization of power function distribution based on lower records. For the sake of completeness, some distributional properties of record values from power function distribution are also provided. We hope that the findings of this paper will be useful for the practitioners in various fields of studies and further enhancement of research in distribution theory, record value theory, and their applications.

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