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Concomitants of generalized order statistics from bivariate Lomax distribution

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Abstract. In this paper probability density function (*pdf*) for rth, $1 \leq r \leq n$ and the joint (*pdf*) of rth and sth, $1 \leq r < s \leq n$, concomitants of generalized order statistics from bivariate Lomax distribution is obtained. Also single and product moments are derived. Further the results are deduced for moments of kth upper record values and order statistics. Also their means and product moments are tabulated.

1. Introduction

The Lomax distribution introduced and studied by Lomax (1954). He used this distribution to analyze business failure data. Lomax distribution has been studied by several authors in literature. Balkema and de Haan (1974) showed that the (*df*) of Lomax distribution arises as a limit distribution of residual lifetime at great age. According to Arnold (1983), the Lomax distribution is well adapted for modeling reliability problems. Nayak (1987) used multivariate Lomax distribution in reliability theory. Balakrishnan and Ahsanullah (1994) derived the relations for single and product moments of record values from Lomax distribution. The Lomax distribution is also known as the Pareto distribution of second kind.

In this paper, we consider the bivariate Lomax distribution (Sankaran and Nair, 1993) with probability distribution function (*pdf*)

$$f(x,y) = \alpha_1\alpha_2 c(c+1)(1+\alpha_1 x+\alpha_2 y)^{-(c+2)},\ x,y,c,\alpha_1,\alpha_2 > 0, \tag{1}$$

and corresponding *df*

$$F(x,y) = 1-(1+\alpha_1 x+\alpha_2 y)^{-c},\ x,y,c,\alpha_1,\alpha_2 > 0.$$

The conditional *pdf* of Y given X is

$$f(y|x) = \frac{\alpha_2(c+1)(1+\alpha_1 x)^{(c+1)}}{(1+\alpha_1 x+\alpha_2 y)^{(c+2)}},\ y > 0. \tag{2}$$

The marginal *pdf* of X is

$$f(x) = \frac{c\alpha_1}{(1+\alpha_1 x)^{(c+1)}},\ x > 0, \tag{3}$$

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and the marginal *df* of X is

$$F(x) = 1 - (1+\alpha_1 x)^{-c},\ x > 0. \tag{4}$$

The concept of generalized order statistics (gos) was given by Kamps (1995). Several authors utilized the concept of (gos) in their work for detailed survey one may refer to Khan et al. (2006), Ahsanullah and beg (2006), Anwar et al. (2007), Beg and Ahsanullah (2008), Faizan and Athar (2008), Tavangar and Asadi (2008), Khan et al. (2009), Tahmasebi and Behboodian (2012), Athar et al. (2012), Athar et al. (2013), Athar and Nayabuddin (2013), Athar and Nayabuddin (2013), among others.

Let $n \in N$, $n \geq 2$, $k > 0$, $\tilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in \Re^{n-1}$, $M_r = \sum_{j=r}^{n-1} m_j$, such that $\gamma_r = k + n - r + M_r > 0$ for all $r \in \{1, 2, \ldots, n-1\}$. Then $X(r, n, \tilde{m}, k), r \in \{1, 2, \ldots, n\}$ are called *gos* if their joint *pdf* is given by

$$k\Big(\prod_{j=1}^{n-1}\gamma_j\Big)\Big(\prod_{i=1}^{n-1}\big[1-F(x_i)\big]^{m_i}f(x_i)\Big)\big[1-F(x_n)\big]^{k-1}f(x_n) \tag{5}$$

on the cone $F^{-1}(0) < x_1 \leq x_2 \leq \ldots \leq x_n < F^{-1}(1)$ of $\Re^n$.

Choosing the parameters appropriately, models such as ordinary order statistics ($\gamma_i = n - i + 1; i = 1, 2, \ldots, n$, i.e. $m_1 = m_2 = \ldots = m_{n-1} = 0, k = 1$), kth record values ($\gamma_i = k$, i.e. $m_1 = m_2 \ldots = m_{n-1} = -1, k \in N$), sequential order statistics ($\gamma_i = (n-i+1)\alpha_i; \alpha_1, \alpha_2, \ldots, \alpha_n > 0$), order statistics with non-integral sample size ($\gamma_i = (\alpha - i + 1); \alpha > 0$), Pfeifers record values ($\gamma_i = \beta_i; \beta_1, \beta_2, \ldots, \beta_n > 0$) and progressive type II censored order statistics ($m_i \in N_0, k \in N$) are obtained (Kamps, 1995, 2001).

In view of (5) with $m_i = m; i = 1, 2, \ldots, n-1$, the *pdf* of rth *gos*, $X(r, n, m, k)$ is

$$f_{X(r,n,m,k)} = \frac{C_{r-1}}{(r-1)!}[\bar{F}(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)) \tag{6}$$

and joint *pdf* of $X(s, n, m, k)$ and $X(r, n, m, k)$, $1 \leq r < s \leq n$, is

$$\begin{aligned} f_{X(r,s,n,m,k)}(x,y) &= \frac{C_{s-1}}{(r-1)!(s-r-1)!}[\bar{F}(x)]^m f(x) g_m^{r-1}\big(F(x)\big)[h_m\big(F(y)\big) - h_m\big(F(x)\big)]^{s-r-1} \\ &\quad \times [\bar{F}(y)]^{\gamma_s - 1} f(y),\ \alpha \leq x < y \leq \beta, \end{aligned} \tag{7}$$

where

$$C_{r-1} = \prod_{i=1}^{r}\gamma_i, \gamma_i = k + (n-i)(m+1),$$

$$h_m(x) = \begin{cases} -\dfrac{1}{m+1}(1-x)^{m+1} & , \ m \neq -1 \\ -\log(1-x) & , \ m = -1 \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(0),\ x \in (0, 1).$$

Let (X_i, Y_i), $i = 1, 2, \ldots, n$, be n pairs of independent random variables from some bivariate population with distribution function $F(x, y)$. If we arrange the X variates in ascending order as $X(1, n, m, k) \leq X(2, n, m, k) \leq \cdots \leq X(n, n, m, k)$, then Y variates paired (not necessarily in ascending order) with these generalized ordered statistics are called the concomitants of generalized order statistics and are denoted by $Y_{[1,n,m,k]}$, $Y_{[2,n,m,k]}$, $\ldots$, $Y_{[n,n,m,k]}$. The *pdf* of $Y_{[r,n,m,k]}$, the rth concomitant of generalized order statistics is given as

$$g_{[r,n,m,k]} = \int_{-\infty}^{\infty} f_{Y|X}(y|x) f_{X(r,n,m,k)}(x)dx \tag{8}$$

and the joint *pdf* of $Y_{[r,n,m,k]}$ and $Y_{[s,n,m,k]}$ is

$$g_{[r,s,n,m,k]}(y_1, y_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{x_2} f_{Y|X}(y_1|x_1) f_{Y|X}(y_2|x_2) f_{X(r,s,n,m,k)}(x_1, x_2) dx_1 dx_2. \tag{9}$$

It is well known that the distribution function of order statistics are connected by the relations (David, 1981)

$$F_{r:n}(x) = \sum_{i=n-r+1}^{n} (-1)^{i-n+r-1} \binom{i-1}{n-r} \binom{n}{i} F_{1:i}(x)$$

where $F_{r:n}(x)$ is the *df* of rth order statistics.

Thus the *pdf* of rth concomitants of order statistics $Y_{[r:n]}$ is

$$g_{[r:n]}(y) = \sum_{i=n-r+1}^{n} (-1)^{i-n+r-1} \binom{i-1}{n-r} \binom{n}{i} g_{[1:i]}(y)$$

and the kth moments of $Y_{[r:n]}$ is

$$\mu^k_{[r:n]}(y) = \sum_{i=n-r+1}^{n} (-1)^{i-n+r-1} \binom{i-1}{n-r} \binom{n}{i} \mu^{(k)}_{[1:i]}(y). \tag{10}$$

Here some important transformation and formulas are presented, which will be used in the subsequent sections (Prudnikov et al., 1986; Srivastava and Karlsson, 1985)

$$(\lambda)_{-n} = \frac{(-1)^n}{(1-\lambda)_n}, \ n = 1, 2, 3, \ldots, \ \lambda \neq 0, \pm 1, \pm 2, \ldots, \tag{11}$$

$$(1+z)^{-a} = \sum_{p=0}^{\infty} \frac{(-1)^p (a)_p z^p}{p!}, \tag{12}$$

$$(\lambda + m) = \frac{\lambda(\lambda+1)_m}{(\lambda)_m}, \tag{13}$$

where $(\lambda)_m = \frac{\Gamma(\lambda+m)}{\Gamma(\lambda)}$, $\lambda \neq 0, -1, -2, \ldots$

$$(\lambda + m + n) = \frac{\lambda(\lambda+1)_{m+n}}{(\lambda)_{m+n}}, \tag{14}$$

$$(\lambda)_{m+n} = (\lambda)_m (\lambda + m)_n. \tag{15}$$

Important identities/result in hypergeometric function are

$${}_2F_1 \left[\begin{matrix} a, & b \\ & & ; & -z \\ c & \end{matrix} \right] = \sum_{p=0}^{\infty} \frac{(a)_p (b)_p}{(c)_p} \frac{(-z)^p}{p!} \tag{16}$$

is conditionally convergent for $|z| = 1, z \neq -1$ if $-1 < Re(\omega) \leq 0$,

$${}_2F_1 \left[\begin{matrix} a, & b \\ & & ; & 1 \\ c & \end{matrix} \right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \ Re(c-a-b) > 0, \ c \neq, 0, -1, -2, \ldots, \tag{17}$$

$$\int_0^{\infty} x^{p-1} {}_2F_1 \left[\begin{matrix} a, & b \\ & & ; & -mx \\ c & \end{matrix} \right] dx = \frac{(m)^{-p}\Gamma(c)\Gamma(p)\Gamma(a-p)\Gamma(b-p)}{\Gamma(a)\Gamma(b)\Gamma(c-p)}, \tag{18}$$

if $0 < Re\ p < a$, $Re\ b$; $|arg\ m| < \pi$

$$\int_0^\infty x^{s-1}{}_3F_2\left[\begin{array}{ccc} (a_1), & (a_2), & (a_3) \\ & & ; \quad -x \\ (b_1), & (b_2) & \end{array}\right] dx = \frac{\Gamma(b_1)\Gamma(b_2)\Gamma(s)\Gamma(a_1-s)\Gamma(a_2-s)\Gamma(a_3-s)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(b_1-s)\Gamma(b_2-s)}, \tag{19}$$

if $0 < Re < s < Re\ a_j$; $j = 1, 2, 3$

$$ {}_3F_2\left[\begin{array}{ccc} -N, & 1, & a \\ & & ,1 \\ b, & a-m, & \end{array}\right] = \frac{(b-1)(a-m-1)}{(b+N-1)(a-1)}{}_3F_2\left[\begin{array}{ccc} -m, & 1, & 2-b \\ & & ,1 \\ 2-b-N, & 2-a, & \end{array}\right] \tag{20}$$

$$ {}_3F_2\left[\begin{array}{ccc} -N, & 1, & 1 \\ & & ,1 \\ l, & m & \end{array}\right] = \frac{(l-1)}{(N+l-1)}{}_3F_2\left[\begin{array}{ccc} -N, & m-1, & 1 \\ & & ,1 \\ m, & 2-N-l & \end{array}\right], \tag{21}$$

for $m = 1, 2, \ldots,\ l = -N, -N-1, -N-1, \ldots$

For real positive k, c and a positive integer b

$$\sum_{a=0}^{b}(-1)^a\binom{b}{a}B(a+k,\ c) = B(k,\ c+b). \tag{22}$$

Note that (Erdélyi et al., 1954)

$$\int_y^\infty x^{-\lambda}(x-y)^{\mu-1}dx = \frac{\Gamma(\lambda-\mu)\Gamma\mu}{\Gamma\lambda}y^{(\mu-\lambda)},\ 0 < Re\mu < Re < \lambda \tag{23}$$

$$\int_0^\infty x^{v-1}(a+x)^{-\mu}(x+y)^{-\rho}dx = \frac{\Gamma v\Gamma(\mu-v+\rho)}{\Gamma(\mu+\rho)a^\mu}y^{(v-\varrho)}{}_2F_1\left[\begin{array}{cc} \mu, & v \\ & ; \quad 1-\frac{y}{a} \\ \mu+\rho & \end{array}\right] \tag{24}$$

if $|arg\, a| < \Pi, Rev > 0,\ |arg\, y| < \Pi,\ Re\rho > Re(v-\mu)$.

Note that (Srivastava and Karlsson, 1985)

$$F^{p:q;k}_{l:m;n}\left[\begin{array}{ccc} (a_p); & (b_q); & (c_k) \\ & & ; \quad x, y \\ (\alpha_l); & (\beta_m); & (\gamma_n); \end{array}\right] = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s}\frac{x^r}{r!}\frac{y^s}{s!} \tag{25}$$

is known as Kampé de Fériet's series.

Note that (Prudnikov et al., 1986)

$$ {}_pF_q\left[\begin{array}{cc} \alpha_1, & \alpha_2, \ldots, \alpha_p \\ & ; z \\ \beta_1 & \beta_2, \ldots, \beta_q \end{array}\right] = \sum_{k=0}^{\infty}\frac{(\alpha_1)_k(\alpha_2)_k\ldots(\alpha_p)_k}{(\beta_1)_k(\beta_2)_k\ldots(\beta_q)_k}\frac{(z)^k}{k!} \tag{26}$$

is known as generalized hypergeometric series.

2. Probability density function of $Y_{[r,n,m,k]}$

For the bivariate Lomax distribution as given in (1), using (2), (3), (4) and (6) in (8), the *pdf* of rth concomitants of *gos* $Y_{[r,n,m,k]}$ is given as

$$g_{[r,n,m,k]}(y) = \frac{\alpha_2 C_{r-1}c(c+1)}{(r-1)!(m+1)^{r-1}}\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\int_0^\infty\frac{\alpha_1}{(1+\alpha_1 x+\alpha_2 y)^{c+2}}\frac{1}{(1+\alpha_1 x)^{c(\gamma_{r-i}-1)}}dx. \tag{27}$$

Let $t=\alpha_1 x$, then the R.H.S. of (27) reduces to

$$= \frac{\alpha_2 C_{r-1}}{(r-1)!(m+1)^{r-1}} c(c+1)\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\int_0^\infty (1+t+\alpha_2 y)^{-(c+2)}(1+t)^{-c(\gamma_{r-i}-1)}dt. \tag{28}$$

Using relation (24) in (28), we get

$$\begin{aligned} g_{[r,n,m,k]}(y) &= \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} c(c+1)\frac{(\alpha_2)}{(1+\alpha_2 y)^{(c+1)}}\sum_{i=0}^{r-1}(-1)^i \\ &\quad \times\binom{r-1}{i}\frac{1}{(c\gamma_{r-i}+1)}\,{}_2F_1\left[\begin{matrix}(c\gamma_{r-i}-c), & 1 \\ (c\gamma_{r-i}+2) & \end{matrix} ; -\alpha_2 y\right]. \end{aligned} \tag{29}$$

We now prove that $\int g_{[r,n,m,k]}(y)dy=1$. We have,

$$\begin{aligned} \int_0^\infty g_{[r,n,m,k]}(y)dy &= \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} c(c+1)\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\frac{1}{c\gamma_{r-i}+1} \\ &\quad \times\int_0^\infty \frac{\alpha_2}{(1+\alpha_2 y)^{(c+1)}}\,{}_2F_1\left[\begin{matrix}(c\gamma_{r-i}-c), & 1 \\ (c\gamma_{r-i}+2) & \end{matrix} ; -\alpha_2 y\right]dy. \end{aligned} \tag{30}$$

Using relation (16) in (30), we get

$$= \frac{\alpha_2 C_{r-1} c(c+1)}{(r-1)!(m+1)^{r-1}}\sum_{i=0}^{r-1}\binom{r-1}{i}\frac{(-1)^i}{c\gamma_{r-i}+1}\sum_{p=0}^{\infty}\frac{(c\gamma_{r-i}-c)_p(1)_p(-1)^p}{(c\gamma_{r-i}+2)_p\ p!}\int_0^\infty \frac{1}{(1+\alpha_2 y)^{(c+1)}}(\alpha_2 y)^p dy. \tag{31}$$

Let $t=1+\alpha_2 y$, then R.H.S. of (31) becomes

$$= \frac{C_{r-1} c(c+1)}{(r-1)!(m+1)^{r-1}}\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\frac{1}{c\gamma_{r-i}+1}\sum_{p=0}^{\infty}\frac{(c\gamma_{r-i}-c)_p(1)_p(-1)^p}{(c\gamma_{r-i}+2)_p\ p!}\int_1^\infty t^{-(c+1)}(t-1)^p dt. \tag{32}$$

Now using relation (23) in (32), we get

$$= \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} c(c+1)\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\frac{1}{c\gamma_{r-i}+1}\sum_{p=0}^{\infty}\frac{(c\gamma_{r-i}-c)_p(1)_p(-1)^p\Gamma(c-p)}{(c\gamma_{r-i}+2)_p\Gamma(c+1)}.$$

In view of (11), we have

$$= \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}}(c+1)\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\frac{1}{c\gamma_{r-i}+1}\,{}_3F_2\left[\begin{matrix}(c\gamma_{r-i}-c), & 1, & 1 \\ (c\gamma_{r-i}+2), & (1-c) & \end{matrix} ,1\right]. \tag{33}$$

Using relation (21) in (33), we get

$$= \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}}\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\,{}_2F_1\left[\begin{matrix}(c\gamma_{r-i}-c), & 1 \\ (1-c) & \end{matrix} ; 1\right]. \tag{34}$$

Now in view of (17) in (34), we have

$$\int_0^\infty g_{[r,n,m,k]}(y)dy = \frac{C_{r-1}}{(r-1)!(m+1)^{r}}\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}B\Big(\frac{k}{m+1}+(n-r)+i,\ 1\Big)=1$$

in view of (22).

3. Moment of $Y_{[r,n,m,k]}$

In view of (29), we have

$$
\begin{aligned}
E\Big(Y^{(a)}_{[r,n,m,k]}\Big) &= \int y^a g_{[r,n,m,k]}(y)dy \\
&= \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} c(c+1) \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \frac{1}{c\gamma_{r-i}+1} \\
&\quad \times \int_0^\infty y^a \frac{\alpha_2}{(1+\alpha_2 y)^{(c+1)}} {}_2F_1\left[\begin{matrix} (c\gamma_{r-i}-c), & 1 \\ (c\gamma_{r-i}+2) & \end{matrix} ; -\alpha_2 y\right] dy \\
&= \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} c(c+1) \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \frac{1}{c\gamma_{r-i}+1} \\
&\quad \times \sum_{p=0}^{\infty} \frac{(c\gamma_{r-i}-c)_p (1)_p}{(c\gamma_{r-i}+2)p!} \int_0^\infty y^a \frac{(\alpha_2)}{(1+\alpha_2 y)^{(c+1)}} (-\alpha_2 y)^p \, dy. \qquad (35)
\end{aligned}
$$

Letting $t = 1+\alpha_2 y$ in (35) we have

$$
= \frac{1}{(\alpha_2)^a} \frac{C_{r-1}c(c+1)}{(r-1)!(m+1)^{r-1}} \sum_{i=0}^{r-1} \binom{r-1}{i} \frac{(-1)^i}{c\gamma_{r-i}+1} \sum_{p=0}^{\infty} \frac{(c\gamma_{r-i}-c)_p(1)_p(-1)^p}{(c\gamma_{r-i}+2)p!} \int_1^\infty t^{-(c+1)}(1-t)^{(p+a)} dt. \qquad (36)
$$

In view of relation (23), (36), becomes

$$
= \frac{(\alpha_2)^{-a} C_{r-1} c(c+1)}{(r-1)!(m+1)^{r-1}} \sum_{i=0}^{r-1} \binom{r-1}{i} \frac{(-1)^i}{c\gamma_{r-i}+1} \sum_{p=0}^{\infty} \frac{(c\gamma_{r-i}-c)_p(1)_p(-1)^p}{(c\gamma_{r-i}+2)_p p!} \frac{\Gamma(c-a-p)\Gamma(p+1+a)}{\Gamma(c+1)}. \qquad (37)
$$

Now on using (11) in (37), we get after simplification

$$
= \frac{\alpha_2^{-a} C_{r-1} c(c+1)}{(r-1)!(m+1)^{r-1}} \sum_{i=0}^{r-1} \frac{\binom{r-1}{i}(-1)^i}{c\gamma_{r-i}+1} \frac{\Gamma(c-a)\Gamma(1+a)}{\Gamma(c+1)} {}_3F_2\left[\begin{matrix} (c\gamma_{r-i}-c), & 1, & (1+a) \\ (c\gamma_{r-i}+2), & (1-c+a) & \end{matrix} ,1\right]. \qquad (38)
$$

After application of (20) in (38), we have

$$
= \frac{c\alpha_2^{-a} C_{r-1}}{(r-1)!(m+1)^{r-1}} \frac{(c-a)\Gamma(c-a)}{\Gamma(c+1)} \frac{\Gamma(1+a)}{a} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} {}_2F_1\left[\begin{matrix} 1, & (-c\gamma_{r-i}) \\ (1-a) & \end{matrix} ; 1\right]. \qquad (39)
$$

Applying (17) in (39), we get

$$
\begin{aligned}
&= \frac{1}{(\alpha_2)^a} \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \frac{\Gamma(c+1-a)\Gamma(1+a)}{c\Gamma(c)} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \frac{1}{\gamma_{r-i}-\frac{a}{c}} \\
&= \frac{\alpha_2^{-a} C_{r-1}}{(r-1)!(m+1)^r} \frac{\Gamma(c+1-a)\Gamma(1+a)}{\Gamma(c+1)} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} sB\Big(\frac{k}{m+1} + (n-r) + i - \frac{a}{c(m+1)} \ , \ 1\Big),
\end{aligned}
$$

which after application of (22), yields

$$
\begin{aligned}
E\Big(y^{(a)}_{[r,n,m,k]}\Big) &= \frac{1}{(\alpha_2)^a} \frac{C_{r-1}}{(m+1)^r} \frac{\Gamma(c+1-a)\ \Gamma(1+a)}{c\Gamma(c)} \frac{\Gamma(\frac{k+(n-r)(m+1)-\frac{a}{c}}{m+1})}{\Gamma(\frac{k+n(m+1)-\frac{a}{c}}{m+1})} \qquad (40) \\
&= \frac{1}{(\alpha_2)^a} \frac{\Gamma(c+1-a)\ \Gamma(1+a)}{c\Gamma(c)} \frac{1}{\prod_{i=1}^{r}(1-\frac{a}{c\gamma_i})} \qquad (41)
\end{aligned}
$$

Remark 3.1. Set $m = 0$, $k = 1$ in (40), to get moments of concomitants of order statistics from bivariate Lomax distribution

$$E\Big(y^{(a)}_{[r:n]}\Big) = \frac{1}{(\alpha_2)^a}\frac{n!}{(n-r)!}\frac{\Gamma(c+1-a)\ \Gamma(1+a)}{c\Gamma(c)}\frac{\Gamma(n-r+1-\frac{a}{c})}{\Gamma(n+1-\frac{a}{c})}$$

and at $r = 1$, we get

$$E\Big(y^{(a)}_{[1:n]}\Big) = \frac{1}{(\alpha_2)^a}\frac{n}{(nc-a)}\frac{\Gamma(c+1-a)\ \Gamma(1+a)}{\Gamma(c)}. \tag{42}$$

Further, in view of (10), (42) becomes

$$E\Big(y^{(a)}_{[r:n]}\Big) = \sum_{i=n-r+1}^{n}(-1)^{i-n+r-1}\binom{i-1}{n-r}\binom{n}{i}\frac{i}{(ic-a)(\alpha_2)^a}\ \frac{\Gamma(c+1-a)\ \Gamma(1+a)}{\Gamma(c)}.$$

Remark 3.2. At $m = -1$ in (41), we get moment of concomitants of kth upper record value from bivariate Lomax distribution

$$E\Big(y^{(a)}_{[r,n,-1,k]}\Big) = \frac{1}{(\alpha_2)^a}\frac{\Gamma(c+1-a)\ \Gamma(1+a)}{c\Gamma(c)}\frac{1}{(1-\frac{a}{ck})^r}\ .$$

Here in Table 3, it may be noted that the well known property of order statistics $\sum_{i=1}^{n} E(X_{i:n}) = nE(X)$ (David and Nagaraja, 2003) is satisfied.

4. Joint probability density function of $Y_{[r,n,m,k]}$ and $Y_{[s,n,m,k]}$

For the bivariate Lomax distribution as given in (1), using (2), (3), (4) and (7) in (9), the joint *pdf* of rth and sth concomitants of *gos* $Y_{[r,n,m,k]}$ and $Y_{[s,n,m,k]}$ is given as

$$\begin{aligned} g_{[r,s,n,m,k]}(y_1,y_2) \quad &= \quad \frac{C_{s-1}c^2(c+1)^2(\alpha_1)(\alpha_2)^2}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j} \\ &\times\int_0^\infty \frac{1}{(1+\alpha_1 x_2)^{c(\gamma_{s-j}-1)}}\frac{1}{(1+\alpha_1x_2+\alpha_2y_2)^{c+2}}I(x_2,y_1)dx_2 \end{aligned} \tag{43}$$

where

$$I(x_2,y_1) = \int_0^{x_2}\frac{\alpha_1}{(1+\alpha_1x_1)^{c(s-r+i-j)(m+1)-c}}\frac{1}{(1+\alpha_1x_1+\alpha_2y_1)^{c+2}}dx_1. \tag{44}$$

Let $t_1 = (1 + \alpha_1 x_1)$, then the R.H.S. of (44) reduces to

$$I(x_2,y_1) = (\lambda)^{-\beta}\int_1^{1+\alpha_1x_2} t_1^{-\alpha}(1+\frac{t_1}{\lambda})^{-\beta}dt_1, \tag{45}$$

where $\alpha = c(s-r+i-j)(m+1)-c$, $\beta = (c+2)$ and $\lambda = \alpha_2 y_1$. Now on using (12) in (45), and simplifying, we get

$$I(x_2,y_1) = (\lambda)^{-\beta}\sum_{p=0}^{\infty}(-1)^p\frac{(\beta)_p(\frac{1}{\lambda})^p}{p!}\frac{1}{(-\alpha+p+1)}\Big[(1+\alpha_1x_2)^{-(\alpha-p-1)}-1\Big].$$

Table 1: Mean of the concomitant of order statistics

n	r	$\alpha_2 = 1,\ c = 2$	$\alpha_2 = 2,\ c = 3$	$\alpha_2 = 3,\ c = 4$
1	1	1.0000	0.2500	0.1111
2	1	0.6667	0.2000	0.0952
	2	1.3333	0.3000	0.1270
3	1	0.6000	0.1875	0.0909
	2	0.8000	0.2250	0.1039
	3	1.6000	0.3375	0.1385
4	1	0.5714	0.1818	0.0889
	2	0.6857	0.2045	0.0969
	3	0.9143	0.2455	0.1108
	4	1.8286	0.3682	0.1478
5	1	0.5556	0.1786	0.0877
	2	0.6349	0.1948	0.0936
	3	0.7619	0.2192	0.1021
	4	1.0159	0.2629	0.1166
	5	2.0317	0.3945	0.1555
6	1	0.5454	0.1765	0.0870
	2	0.6061	0.1890	0.0915
	3	0.6926	0.2063	0.0976
	4	0.8312	0.2320	0.1065
	5	1.1082	0.2785	0.1217
	6	2.2165	0.4177	0.1623
7	1	0.5385	0.1750	0.0864
	2	0.5874	0.1853	0.0902
	3	0.6527	0.1985	0.0949
	4	0.7459	0.2166	0.1013
	5	0.8950	0.2436	0.1105
	6	1.1935	0.2924	0.1262
	7	2.3870	0.4386	0.1683

Table 2: Mean of the concomitant of record statistics

r	$\alpha_2 = 1,\ c = 2,\ k = 1$	$\alpha_2 = 2,\ c = 3,\ k = 2$	$\alpha_2 = 3,\ c = 4,\ k = 3$
1	1.0000	0.2000	0.0909
2	2.0000	0.2400	0.0992
3	4.0000	0.2880	0.1081
4	8.0000	0.3456	0.1180
5	16.0000	0.4147	0.1287
6	32.0000	0.4977	0.1404
7	64.0000	0.5972	0.1532
8	128.0000	0.7166	0.1671
9	256.0000	0.8599	0.1823
10	512.0000	1.0319	0.1989

Therefore, in view of (43), we have

$$\begin{aligned} g_{[r,s,n,m,k]}(y_1,y_2) &= \frac{c^2(c+1)^2(\alpha_1)(\alpha_2)^2 C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \\ &\times \sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j}(\lambda)^{-\beta}\sum_{p=0}^{\infty}\frac{(-1)^p}{(-\alpha+p+1)}\frac{(\beta)_p(\frac{1}{\lambda})^p}{p!} \\ &\times \int_0^{\infty}\frac{\alpha_1(1+\alpha_1x_2)^{-(c\gamma_{s-j}-c)}}{(1+\alpha_1x_2+\alpha_2y_2)^{c+2}}\Big[(1+\alpha_1x_2)^{-(\alpha-p-1)}-1\Big]dx_1. \end{aligned} \tag{46}$$

Letting $t_2=(1+\alpha_1x_2)$ in (46), and using relation (12), we get

$$\begin{aligned} g_{[r,s,n,m,k]}(y_1,y_2) &= \frac{c^2(c+1)^2(\alpha_2)^2 C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j} \\ &\times(\lambda)^{-\beta}(\delta)^{-\beta}\sum_{p=0}^{\infty}\frac{(\beta)_p(-\frac{1}{\lambda})^p}{p!}\ \sum_{l=0}^{\infty}\frac{(\beta)_l(-\frac{1}{\delta})^l}{l!}\frac{1}{(1-\theta+l)(2-\theta-\alpha+l+p)}, \end{aligned} \tag{47}$$

where $\delta=\alpha_2y_2$ and $\theta=c\gamma_{s-j}-c$.

Set $d=1-\theta$ and $g=2-\theta-\alpha$ in (47), to get

$$= \frac{c^2(c+1)^2(\alpha_2)^2 C_{s-1}\lambda^{-\beta}\delta^{-\beta}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j}\sum_{p=0}^{\infty}\frac{(\beta)_p(-\frac{1}{\lambda})^p}{(g+p+l)p!}\ \sum_{l=0}^{\infty}\frac{(\beta)_l(-\frac{1}{\delta})^l}{(d+l)l!}. \tag{48}$$

After substituting the value of λ and δ in (48), we get

$$\begin{aligned} &= \frac{c^2(c+1)^2(\alpha_2)^2 C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j}(\alpha_2y_2)^{-\beta}(\alpha_2y_1)^{-\beta} \\ &\times\sum_{l=0}^{\infty}\frac{(\beta)_l(\frac{-1}{\alpha_2y_2})^l}{(d+l)l!}\sum_{p=0}^{\infty}\frac{(\beta)_p(\frac{-1}{\alpha_2y_1})^p}{(g+p+l)p!}. \end{aligned} \tag{49}$$

Using relation (13) and (14) in (49), it becomes

$$\begin{aligned} &= \frac{c^2(c+1)^2 C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j}\frac{1}{dg} \\ &\times\frac{(\alpha_2)}{(\alpha_2y_1)^{\beta}}\frac{(\alpha_2)}{(\alpha_2y_2)^{\beta}}\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\frac{(g)_{p+l}}{(g+1)_{p+l}}\frac{(\beta)_l(d)_l(\beta)_p}{(d+1)_l}\frac{(\frac{-1}{\alpha_2y_1})^p}{p!}\frac{(\frac{-1}{\alpha_2y_2})^l}{l!}. \end{aligned}$$

Therefore

$$\begin{aligned} g_{[r,s,n,m,k]}(y_1,y_2) &= \frac{c^2(c+1)^2 C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j}\frac{1}{dg} \\ &\times\frac{(\alpha_2)}{(\alpha_2y_1)^{\beta}}\frac{(\alpha_2)}{(\alpha_2y_2)^{\beta}}F^{1:1;0}_{1:2;1}\left[\begin{matrix}(g); & (\beta); & (d); & (\beta); & \\ & & & ; & \frac{-1}{\alpha_2y_2},\frac{-1}{\alpha_2y_1} \\ (g+1); & (d+1); & & & \end{matrix}\right] \end{aligned} \tag{50}$$

where $F^{1:1;0}_{1:2;1}[\ \ ]$ is as defined in (25).

We now prove that $\int_0^\infty\int_0^\infty g_{[r,s,n,m,k]}(y_1,y_2)dy_1dy_2 = 1$. We have

$$\int_0^\infty\int_0^\infty g_{[r,s,n,m,k]}(y_1,y_2)dy_1dy_2 = A\int_0^\infty\int_0^\infty \frac{(\alpha_2)}{(\alpha_2 y_1)^\beta}\frac{(\alpha_2)}{(\alpha_2 y_2)^\beta} \times F^{1:1;0}_{1:2;1}\left[\begin{matrix} (g); & (\beta); & (d); & (\beta); & \\ & & & ; & \frac{-1}{\alpha_2 y_2}, \frac{-1}{\alpha_2 y_1} \\ (g+1); & (d+1); & & & \end{matrix}\right] dy_1dy_2$$

where

$$A = \frac{c^2(c+1)^2 C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j}\frac{1}{dg}.$$

Then,

$$\int_0^\infty\int_0^\infty g_{[r,s,n,m,k]}(y_1,y_2)dy_1dy_2 = A\int_0^\infty\int_0^\infty \frac{(\alpha_2)}{(\alpha_2 y_1)^\beta}\frac{(\alpha_2)}{(\alpha_2 y_2)^\beta} \times\sum_{p=0}^\infty\sum_{l=0}^\infty \frac{(g)_{l+p}}{(g+1)_{l+p}}\frac{(\beta)_l(d)_l(\beta)_p}{(d+1)_l}\frac{(\frac{-1}{\alpha_2 y_1})^p}{p!}\frac{(\frac{-1}{\alpha_2 y_2})^l}{l!}dy_1dy_2. \quad (51)$$

On applying (15) in (51), we can write

$$= A\int_0^\infty \frac{(\alpha_2)}{(\alpha_2 y_2)^\beta}\sum_{l=0}^\infty \frac{(g)_l(\beta)_l(d)_l}{(g+1)_l(d+1)_l}\frac{(\frac{-1}{\alpha_2 y_2})^l}{l!}\left\{\int_0^\infty \frac{(\alpha_2)}{(\alpha_2 y_1)^\beta}\sum_{p=0}^\infty \frac{(g+l)_p(\beta)_p}{(g+l+1)_p}\frac{(\frac{-1}{\alpha_2 y_1})^p}{p!}dy_1\right\}dy_2. \quad (52)$$

Now using relation (26) in (52), we have

$$= A\int_0^\infty \frac{\alpha_2}{(\alpha_2 y_2)^\beta}\sum_{l=0}^\infty \frac{(g)_l(\beta)_l(d)_l}{(g+1)_l(d+1)_l}\frac{(\frac{-1}{\alpha_2 y_2})^l}{l!}\int_0^\infty \frac{(\alpha_2)}{(\alpha_2 y_1)^\beta}{}_2F_1\left[\begin{matrix} (\beta); & (g+l) & \\ & & ; \frac{-1}{\alpha_2 y_1} \\ (g+1+l) & & \end{matrix}\right] dy_1dy_2. \quad (53)$$

Now letting $t_1 = \frac{1}{\alpha_2 y_1}$ in (53), to get

$$= A\int_0^\infty \frac{(\alpha_2 y_2)}{(\alpha_2 y_2)^\beta}\sum_{l=0}^\infty \frac{(g)_l(\beta)_l(d)_l}{(g+1)_l(d+1)_l}\frac{(\frac{-1}{\alpha_2 y_2})^l}{l!}\int_0^\infty t^{(\beta-1)-1}{}_2F_1\left[\begin{matrix} (\beta); & (g+l) & \\ & & ; -t_1 \\ (g+1+l) & & \end{matrix}\right] dt_1dy_2.$$

Now in view of relation (18), we have

$$= A\int_0^\infty \frac{(\alpha_2)}{(\alpha_2 y_2)^\beta}\sum_{l=0}^\infty \frac{(g)_l(\beta)_l(d)_l}{(g+1)_l(d+1)_l}\frac{(\frac{-1}{\alpha_2 y_2})^l}{l!}\frac{(g+l)}{(\beta-1)(g+l+1-\beta)}dy_2. \quad (54)$$

Further, using relation (13) in (54), to get

$$= \frac{gA}{(g-\beta+1)}\int_0^\infty \frac{(\alpha_2)}{(\alpha_2 y_2)^\beta}\sum_{l=0}^\infty \frac{(g-\beta+1)_l(\beta)_l(d)_l}{(g-\beta+2)_l(d+1)_l}\frac{(\frac{-1}{\alpha_2 y_2})^l}{l!}dy_2. \quad (55)$$

Now using relation (26) in (55), we have

$$= \frac{gA}{(g-\beta+1)}\int_0^\infty \frac{(\alpha_2)}{(\alpha_2 y_2)^\beta}{}_3F_2\left[\begin{matrix} (g+1-\beta); & (\beta); & (d) & \\ & & & ; -\frac{1}{\alpha_2 y_2} \\ (g+2-\beta); & (d+1) & & \end{matrix}\right] dy_2. \quad (56)$$

Set $t_2 = \frac{1}{\alpha_2 y_2}$ in (56), to get

$$= \frac{gA}{(g-\beta+1)} \int_0^\infty (t_2)^{(\beta-1)-1} {}_3F_2 \left[\begin{matrix} (g+1-\beta); & (\beta); & (d) & \\ & & ; & -t_2 \\ (g+2-\beta); & (d+1) & & \end{matrix} \right] dt_2. \tag{57}$$

Now on using relation (19) in (57), we have

$$= \frac{Agd}{(\beta-1)^2(g+2-2\beta)(d+1-\beta)}. \tag{58}$$

Now after putting the value of g, d, β, A in (58), we get

$$= \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^s} \sum_{i=0}^{r-1} \sum_{j=0}^{s-r-1} (-1)^{i+j} \binom{r-1}{i} \binom{s-r-1}{j} \frac{[\frac{k}{m+1} + (n-s) + j]^{-1}}{[\frac{k}{m+1} + (n-r) + i]}.$$

Therefore,

$$\int_0^\infty \int_0^\infty g_{[r,s,n,m,k]}(y_1, y_2) dy_1 dy_2 = \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^s} \sum_{i=0}^{r-1} \binom{s-r-1}{j}$$
$$\times \sum_{j=0}^{s-r-1} (-1)^{i+j} \binom{r-1}{i} B\Big(\frac{k}{m+1} + (n-r+i),\ 1\Big) B\Big(\frac{k}{m+1} + (n-s+j),\ 1\Big) = 1.$$

in view of (22).

5. Product moments of two concomitants $Y_{[r,n,m,k]}$ and $Y_{[s,n,m,k]}$

The product moments of two concomitants $Y_{[r,n,m,k]}$ and $Y_{[s,n,m,k]}$ is given by

$$E\Big(Y^{(a)}_{[r,n,m,k]} Y^{(b)}_{[s,n,m,k]}\Big) = \int_0^\infty \int_0^\infty y_1^a y_2^b \ g_{[r,s,n,m,k]}(y_1, y_2) dy_1 dy_2. \tag{59}$$

In view of (50) and (59), we have

$$E\Big(Y^{(a)}_{[r,n,m,k]} Y^{(b)}_{[s,n,m,k]}\Big) =$$
$$= A \int_0^\infty \int_0^\infty y_1^a y_2^b \frac{(\alpha_2)}{(\alpha_2 y_1)^\beta} \frac{(\alpha_2)}{(\alpha_2 y_2)^\beta} F^{1:1;0}_{1:2;1} \left[\begin{matrix} (g); & (\beta); & (d); & (\beta); & \\ & & & ; & \frac{-1}{\alpha_2 y_2}, \frac{-1}{\alpha_2 y_1} \\ (g+1); & (d+1); & & & \end{matrix} \right] dy_1 dy_2.$$
$$= A \int_0^\infty \int_0^\infty y_1^a y_2^b \frac{(\alpha_2)}{(\alpha_2 y_1)^\beta} \frac{(\alpha_2)}{(\alpha_2 y_2)^\beta} \sum_{p=0}^\infty \sum_{l=0}^\infty \frac{(g)_{l+p}}{(g+1)_{l+p}} \frac{(\beta)_l (d)_l (\beta)_p}{(d+1)_l} \frac{(\frac{-1}{\alpha_2 y_1})^p}{p!} \frac{(\frac{-1}{\alpha_2 y_2})^l}{l!} dy_1 dy_2. \tag{60}$$

On applying (15) in (60), we have

$$= A \int_0^\infty y_2^b \frac{(\alpha_2)}{(\alpha_2 y_2)^\beta} \sum_{l=0}^\infty \frac{(g)_l}{(g+1)_l} \frac{(\beta)_l (d)_l}{(d+1)_l} \frac{(\frac{-1}{\alpha_2 y_2})^l}{l!} \Big\{ \int_0^\infty y_1^a \frac{(\alpha_2)}{(\alpha_2 y_1)^\beta} \sum_{p=0}^\infty \frac{(\beta)_p (g+l)_p}{(g+1+l)_p} \frac{(\frac{-1}{\alpha_2 y_1})^p}{p!} dy_1 \Big\} dy_2. \tag{61}$$

Using relation (26) in (6), we have

$$= A \int_0^\infty y_2^b \frac{(\alpha_2)}{(\alpha_2 y_2)^\beta} \sum_{l=0}^\infty \frac{(g)_l}{(g+1)_l} \frac{(\beta)_l (d)_l}{(d+1)_l} \frac{(\frac{-1}{\alpha_2 y_2})^l}{l!}$$
$$\times \int_0^\infty y_1^a \frac{(\alpha_2)}{(\alpha_2 y_1)^\beta} \ {}_2F_1 \left[\begin{matrix} (\beta); & (g+l) & \\ & ; & \frac{-1}{\alpha_2 y_1}{}_1 \\ (g+l+1) & & \end{matrix} \right] dy_1 dy_2. \tag{62}$$

Setting $t_1 = \frac{1}{\alpha_2 y_1}$ in (62), to get

$$= \frac{A}{(\alpha_2)^a} \int_0^\infty y_2^b \frac{(\alpha_2)}{(\alpha_2 y_2)^\beta} \sum_{l=0}^{\infty} \frac{(g)_l}{(g+1)_l} \frac{(\beta)_l (d)_l}{(d+1)_l} \frac{(\frac{-1}{\alpha_2 y_2})^l}{l!}$$
$$\times \int_0^\infty t^{(\beta-a-1)-1} {}_2F_1 \left[\begin{matrix} (\beta); & (g+l) \\ & & ; & -t_1 \\ (g+l+1) & \end{matrix} \right] dt_1 dy_2. \tag{63}$$

On using relation (18) in (63), we have

$$= \frac{A}{(\alpha_2)^a} \frac{(g+l)\Gamma(\beta-a-1)\Gamma(a+1)}{\Gamma(\beta)(g+l+1-\beta+a)} \int_0^\infty y_2^b \frac{(\alpha_2)}{(\alpha_2 y_2)^\beta} \sum_{l=0}^{\infty} \frac{(g)_l}{(g+1)_l} \frac{(\beta)_l (d)_l}{(d+1)_l} \frac{(\frac{-1}{\alpha_2 y_2})^l}{l!} \ dy_2.$$

Now in view of relation (13) and (26), we have

$$= \frac{A}{(\alpha_2)^a} \frac{g}{(g+1-\beta+a)} \frac{\Gamma(\beta-a-1)\Gamma(a+1)}{\Gamma(\beta)} \int_0^\infty y_2^b \frac{(\alpha_2)}{(\alpha_2 y_2)^\beta}$$
$$\times {}_3F_2 \left[\begin{matrix} (d); & (g+1-\beta+a); & (\beta) \\ & & ; & -\frac{1}{\alpha_2 y_2} \\ (d+1); & (g+2-\beta+a) & \end{matrix} \right] dy_2. \tag{64}$$

Let $t_2 = \frac{1}{\alpha_2 y_2}$ in (64), we have

$$= \frac{A}{(\alpha_2)^{a+b}} \frac{g}{(g+1-\beta+a)} \frac{\Gamma(\beta-a-1)\Gamma(a+1)}{\Gamma(\beta)} \int_0^\infty t_2^{(\beta-b-1)-1}$$
$$\times {}_3F_2 \left[\begin{matrix} (d); & (g+1-\beta+a); & (\beta) \\ & & ; & -t_2 \\ (d+1); & (g+2-\beta+a) & \end{matrix} \right] dt_2. \tag{65}$$

Using relation (19) in (65), to get

$$= \frac{A}{(\alpha_2)^{a+b}} \frac{\Gamma(\beta-a-1)\Gamma(a+1)}{\Gamma(\beta)} \frac{\Gamma(\beta-b-1)\Gamma(b+1)}{\Gamma(\beta)} \frac{dg}{(d-\beta+b+1)(g-2\beta+a+b+2)}. \tag{66}$$

Now putting the value of A, d, g and β in (66), we have

$$\begin{aligned} E\Big(Y_{[r,n,m,k]}^{(a)} Y_{[s,n,m,k]}^{(b)}\Big) &= \frac{1}{(\alpha_2)^{(a+b)}} \frac{c^2(c+1)^2 C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \frac{\Gamma(c-a+1)\Gamma(a+1)}{\Gamma(c+2)} \\ &\times \frac{\Gamma(c-b+1)\Gamma(b+1)}{\Gamma(c+2)} \sum_{i=0}^{r-1} \sum_{j=0}^{s-r-1} (-1)^{i+j} \binom{r-1}{i} \binom{s-r-1}{j} \\ &\times \frac{1}{[c\{k+(n-s+j)(m+1)\}-b]} \ \frac{1}{[c\{k+(n-r+i)(m+1)\}-a-b]} \\ &= \frac{1}{(\alpha_2)^{(a+b)}} \frac{(c+1)^2 C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \frac{\Gamma(c-a+1)\Gamma(a+1)}{\Gamma(c+2)} \\ &\times \frac{\Gamma(c-b+1)\Gamma(b+1)}{\Gamma(c+2)} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} B\Big(\frac{k}{m+1} + (n-r+i) - \frac{a+b}{c(m+1)} \ , \ 1\Big) \\ &\times \sum_{j=0}^{s-r-1} (-1)^j \binom{s-r-1}{j} B\Big(\frac{k}{m+1} + (n-s+j) - \frac{b}{c(m+1)} \ , \ 1\Big). \end{aligned} \tag{67}$$

In view of relation (22), we get

$$= \frac{1}{(\alpha_2)^{(a+b)}} \frac{(c+1)^2 C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \frac{\Gamma(c-a+1)\Gamma(a+1)}{\Gamma(c+2)} \frac{\Gamma(c-b+1)\Gamma(b+1)}{\Gamma(c+2)}$$
$$\times B\Big(\frac{k}{m+1}+(n-r)-\frac{a+b}{c(m+1)}\ ,\ r\Big) B\Big(\frac{k}{m+1}+(n-s)-\frac{b}{c(m+1)}\ ,\ s-r\Big),$$

which after simplification yields

$$E\Big(Y^{(a)}_{[r,n,m,k]} Y^{(b)}_{[s,n,m,k]}\Big) = \frac{1}{(\alpha_2)^{(a+b)}} \frac{C_{s-1}}{(m+1)^{s-2}} \frac{\Gamma(c-a+1)\Gamma(a+1)}{\Gamma(c+1)} \frac{\Gamma(c-b+1)\Gamma(b+1)}{\Gamma(c+1)}$$
$$\times \frac{\Gamma(\frac{k}{(m+1)}+(n-r)-\frac{a+b}{c(m+1)})}{\Gamma(\frac{k}{(m+1)}+n-\frac{a+b}{c(m+1)})} \frac{\Gamma(\frac{k}{(m+1)}+(n-s)-\frac{b}{c(m+1)})}{\Gamma(\frac{k}{(m+1)}+(n-r)-\frac{b}{c(m+1)})}.$$

$$E\Big(Y^{(a)}_{[r,n,m,k]} Y^{(b)}_{[s,n,m,k]}\Big) = \frac{1}{(\alpha_2)^{(a+b)}} \frac{\Gamma(c-a+1)\Gamma(a+1)}{\Gamma(c+1)} \frac{\Gamma(c-b+1)\Gamma(b+1)}{\Gamma(c+1)}$$
$$\times \frac{1}{\prod_{i=1}^{r}(1-\frac{a+b}{c\gamma_i})\prod_{j=r+1}^{s}(1-\frac{b}{c\gamma_i})}. \tag{68}$$

Remark 5.1. Set $m=0$, $k=1$ in (67), to get product moments of concomitants of order statistics from bivariate Lomax distribution

$$E\Big(Y^{(a)}_{[r:n]} Y^{(b)}_{[s:n]}\Big) = \frac{C_{r,s:n}}{(\alpha_2)^{(a+b)}} \frac{\Gamma(c-a+1)\Gamma(a+1)}{\Gamma(c)} \frac{\Gamma(c-b+1)\Gamma(b+1)}{\Gamma(c)}$$
$$\times \sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j} \frac{1}{[sc-nc-jc-c+b]} \frac{1}{[rc-nc-ic-c+a+b]}.$$

$$E\Big(Y^{(a)}_{[r:n]} Y^{(b)}_{[s:n]}\Big) = \frac{1}{(\alpha_2)^{(a+b)}} \frac{n!}{(n-s)!} \frac{\Gamma(c-a+1)\Gamma(a+1)}{\Gamma(c+1)} \frac{\Gamma(c-b+1)\Gamma(b+1)}{\Gamma(c+1)}$$
$$\times \frac{\Gamma(n-r+1-\frac{a+b}{c})}{\Gamma(n+1-\frac{a+b}{c})} \frac{\Gamma(n-s+1-\frac{b}{c})}{\Gamma(n-r+1-\frac{b}{c})}.$$

Remark 5.2. If $m=-1$, in (68), then we get product moment of concomitants of kth upper record value from bivariate Lomax distribution as

$$E\Big(Y^{(a)}_{[r,n,-1,k]} Y^{(b)}_{[s,n,-1,k]}\Big) = \frac{1}{(\alpha_2)^{(a+b)}} \frac{\Gamma(c-a+1)\Gamma(a+1)}{\Gamma(c+1)} \frac{\Gamma(c-b+1)\Gamma(b+1)}{\Gamma(c+1)} \frac{1}{(1-\frac{a+b}{ck})^r(1-\frac{b}{ck})^{(s-r)}}.$$

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Table 3: Product moments between the concomitants of order statistics:

		$\alpha_2 = 1$			$c = 3$	
n	$s \setminus r$	1	2	3	4	5
1	1	-0.0821				
2	1	0.1231				
	2	-0.0615	-0.1231			
3	1	-0.0633				
	2	0.1266	0.1582			
	3	-0.0633	-0.0791	-0.1582		
4	1	0.5317				
	2	0.3323	0.3798			
	3	0.1329	0.1519	0.1899		
	4	-0.0665	-0.0759	-0.0949	-0.1898	
5	1	0.7669				
	2	0.5577	0.6135			
	3	0.3486	0.3834	0.4382		
	4	0.1394	0.1534	0.1753	0.2191	
	5	-0.0697	-0.0767	-0.0876	-0.1096	-0.2191

Table 4: Product moments between the concomitants of order statistics:

		$\alpha_2 = 2$			$c = 4$	
n	$s \setminus r$	1	2	3	4	5
1	1	-0.0064				
2	1	0.0170				
	2	-0.0057	-0.0085			
3	1	0.0408				
	2	0.0175	0.0204			
	3	-0.0058	-0.0068	-0.0102		
4	1	0.0666				
	2	0.0424	0.0466			
	3	0.0182	0.0199	0.0233		
	4	-0.0061	-0.0067	0.0078	-0.0117	
5	1	0.0942				
	2	0.0691	0.0740			
	3	0.0439	0.0471	0.0518		
	4	0.0188	0.0202	0.0222	0.0259	
	5	-0.0063	-0.0067	-0.0074	-0.0086	-0.0129

Table 5: Product moments between the concomitants of order statistics:

		$\alpha_2 = 3$			$c = 5$	
n	$s \setminus r$	1	2	3	4	5
1	1	-0.0013				
2	1	0.0048				
	2	-0.0012	-0.0016			
3	1	0.0110				
	2	0.0049	0.0055			
	3	-0.0012	-0.0014	-0.0018		
4	1	0.0177				
	2	0.0114	0.0122			
	3	0.0050	0.0054	0.0061		
	4	-0.0013	-0.0014	-0.0015	-0.0020	
5	1	0.0246				
	2	0.0182	0.0192			
	3	0.0117	0.0123	0.0133		
	4	0.0052	0.0055	0.0059	0.0066	
	5	-0.0013	-0.0014	-0.0015	-0.0017	-0.0022

Table 6: Product moments between the concomitants of record statistics:

s	r	$\alpha_2 = 1,\, c = 3$	$\alpha_2 = 2,\, c = 4$	$\alpha_2 = 3,\, c = 5$
1	1	0.3333	0.0313	0.0074
2	1	0.5000	0.0417	0.0093
	2	1.0000	0.0625	0.0123
3	1	0.7500	0.0555	0.0116
	2	1.5000	0.0833	0.0154
	3	3.0000	0.1250	0.0206
4	1	1.1250	0.0741	0.0145
	2	2.2500	0.1111	0.0193
	3	4.5000	0.1667	0.0257
	4	9.0000	0.2500	0.0343
5	1	1.6875	0.0988	0.0181
	2	3.3750	0.1481	0.0241
	3	6.7500	0.2222	0.0322
	4	13.5000	0.3333	0.0429
	5	27.0000	0.5000	0.0572

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