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Recurrence relations for marginal and joint moment generating functions of upper k-record values from Gompertz distribution

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Abstract. In this paper the recurrence relations for marginal and joint moment generating functions of upper record values as well as upper k-record values from Gompertz distribution have been obtained.

1. Introduction

A random variable X is said to have the Gompertz distribution if its probability density function is given by

$$f(x) = \beta e^{\alpha x} \exp\left(-\frac{\beta}{\alpha}(e^{\alpha x} - 1)\right), \quad x \geq 0, \alpha > 0, \beta > 0, \tag{1}$$

and the cumulative distribution function is given by

$$F(x) = 1 - \exp\left(-\frac{\beta}{\alpha}(e^{\alpha x} - 1)\right), \ x \geq 0. \tag{2}$$

The Gompertz distribution in (1) was introduced by Gompertz (1825). This distribution is applicable as a model for surviving distributions which has an increasing hazard rate for the life of the creatures and systems. Prentice and El-Shaarawi (1973) have used this model in their studies, Elandt-Johnson and Johnson (1980) have shown that this distribution is widely used in actuarial works.

It is easy to see from (1) and (2) that for the Gompertz distribution

$$f(x) = [\beta + \alpha\{-\ln(1 - F(x))\}][1 - F(x)]. \tag{3}$$

Suppose $\{X_n, n \geq 1\}$ is an infinite sequence of independent, identically distributed (i.i.d.) random variables with common cumulative distribution function (c.d.f.) $F(x)$ and probability density function (p.d.f.) $f(x)$. Let us assume that F is continuous so that ties are not possible. Let $Y_n = \max\{X_1, X_2, \ldots, X_n\}$, $n = 1, 2, \ldots$ We say X_j is an upper record value of this sequence if $Y_j > Y_{j-1}$, $j \geq 2$. The indices at which the upper record values occur are given by the upper record times $\{U(n), n \geq 1\}$, where

$$U(n) = \min\{j \mid j > U(n-1), X_j > X_{U(n-1)}, n > 1\} \text{ with } U(1) = 1.$$

Then $X_{U(n)}$ and $U(n)$ are the sequences of upper record values and upper record times, respectively.

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Similarly, for a fixed $k \geq 1$, we define the sequence $\{U(n:k), n \geq 1\}$ of upper k-record times of $\{X_n, n \geq 1\}$ as follows

$$U(1:k) = 1$$
$$U(n+1:k) = \min\{j > U(n:k) : X_{j:j+k-1} > X_{U(n:k):U(n:k)+k-1}\},$$

where $X_{j:n}$ denotes the jth order statistic of a sample $(X_1, X_2, \ldots, X_n)$, (cf. Kamps (1995a,b)). Then $X_{U(n:k)}$ and $U(n:k)$ are called the sequences of upper k-record values and upper k-record times, respectively.

For $k = 1$ and $n = 1, 2, \ldots$, we write $U(n:1) = U(n)$ and $X_{U(n:1)} = X_{U(n)}$.

Chandler (1952) introduced record values and record value times. Properties of record values of i.i.d. random variables have been extensively studied in the literature. Various developments on records and related topics have been reviewed by a number of authors including Glick (1978), Nevzorov (1987), Resnick (1987), Nagaraja (1988), Ahsanullah (1988, 1995), Arnold and Balakrishnan (1989), Arnold et al. (1992, 1998) and Ahsanullah and Nevzorov (2001).

In this paper, we establish some recurrence relations for marginal and joint moment generating functions of upper k-record values from the Gompertz distribution. The corresponding results for upper record values ($k = 1$) have also been deduced as special cases. Similar work has been done for the power function and Gumbel distributions by Raqab and Ahsanullah (2000) and Ahsanullah and Raqab (1999), respectively.

For convenience, let us denote the marginal moment generating function of $X_{U(n:k)}$ by $M_{U(n:k)}(t)$ and its rth derivative with respect to t by $M^{(r)}_{U(n:k)}(t)$. Similarly, let $M_{U(m,n:k)}(t_1, t_2)$ and $M^{(r,s)}_{U(m,n:k)}(t_1, t_2)$ denote the joint moment generating function of $X_{U(m:k)}$ and $X_{U(n:k)}$ and its (r, s)-th partial derivative with respect to t_1 and t_2, respectively. Also, for $k = 1$, we write $M_{U(n:1)}(t) \equiv M_{U(n)}(t)$, $M^{(r)}_{U(n:1)}(t) \equiv M^{(r)}_{U(n)}(t)$, $M_{U(m,n:1)}(t_1, t_2) \equiv M_{U(m,n)}(t_1, t_2)$ and $M^{(r,s)}_{U(m,n:1)}(t_1, t_2) \equiv M^{(r,s)}_{U(m,n)}(t_1, t_2)$.

2. Relations for marginal moment generating functions of upper *k*-record values

The p.d.f. of upper k-record values $X_{U(n:k)}$, $n \geq 1$, as given by Dziubdziela and Kopocinski (1976) is as follows (for $k \geq 1$):

$$f_{U(n:k)}(x) = \frac{k^n}{(n-1)!}\{-\ln(1-F(x))\}^{n-1}[1-F(x)]^{k-1}f(x), \quad -\infty < x < \infty. \tag{4}$$

Theorem 2.1. *For $n \geq 2$, $r = 0, 1, 2, \ldots,$ and $\alpha, \beta > 0$,*

$$M^{(r+1)}_{U(n+1:k)}(t) = M^{(r+1)}_{U(n:k)}(t) + \frac{1}{n\alpha}\Big[tM^{(r+1)}_{U(n:k)}(t) + (r+1)M^{(r)}_{U(n:k)}(t) - \beta k\Big\{M^{(r+1)}_{U(n:k)}(t) - M^{(r+1)}_{U(n-1:k)}(t)\Big\}\Big], \tag{5}$$

where $M^{(r)}_{U(n:k)}(t) = \dfrac{d^r M_{U(n:k)}(t)}{dt^r}$ and $M^{(0)}_{U(n:k)} = M_{U(n:k)}(t)$, provided the marginal moment generating function exists.

Proof. For $n \geq 2$, $r = 0, 1, 2, \ldots$, we have from (4)

$$M_{U(n:k)}(t) = \frac{k^n}{(n-1)!}\int_{-\infty}^{\infty} e^{tx}[-\ln(1-F(x))]^{n-1}[1-F(x)]^{k-1}f(x)dx.$$

On using (3), we have

$$\begin{aligned} M_{U(n:k)}(t) &= \frac{k^n\beta}{(n-1)!}\int_0^{\infty} e^{tx}[-\ln(1-F(x))]^{n-1}[1-F(x)]^k dx + \frac{k^n\alpha}{(n-1)!}\int_0^{\infty} e^{tx}[-\ln(1-F(x))]^{n}[1-F(x)]^k dx \\ &= k\beta I_{n-1,k} + n\alpha I_{n,k}, \end{aligned} \tag{6}$$

where

$$I_{n,k} = \frac{k^n}{n!} \int_0^\infty e^{tx}[-\ln(1-F(x))]^n [1-F(x)]^k dx.$$

Integrating by parts treating e^{tx} for integration and rest of the integrand for differentiation, we get

$$I_{n,k} = \frac{k^{n+1}}{n!\,t} \int_0^\infty e^{tx}[-\ln(1-F(x))]^n [1-F(x)]^{k-1} f(x)dx$$
$$- \frac{k^n}{(n-1)!\,t} \int_0^\infty e^{tx}[-\ln(1-F(x))]^{n-1} [1-F(x)]^{k-1} f(x)dx. \tag{7}$$

Similarly,

$$I_{n-1,k} = \frac{k^n}{(n-1)!\,t} \int_0^\infty e^{tx}[-\ln(1-F(x))]^{n-1} [1-F(x)]^{k-1} f(x)dx$$
$$- \frac{k^{n-1}}{(n-2)!\,t} \int_0^\infty e^{tx}[-\ln(1-F(x))]^{n-2} [1-F(x)]^{k-1} f(x)dx\,. \tag{8}$$

By using (7) and (8) in (6), we get

$$tM_{U(n:k)}(t) = k\beta\Big[M_{U(n:k)}(t) - M_{U(n-1:k)}(t)\Big] + n\alpha\Big[M_{U(n+1:k)}(t) - M_{U(n:k)}(t)\Big]. \tag{9}$$

Differentiating (9), $(r+1)$ times with respect to t, we get

$$tM^{(r+1)}_{U(n:k)}(t) + (r+1)M^{(r)}_{U(n:k)}(t) = k\beta\Big[M^{(r+1)}_{U(n:k)}(t) - M^{(r+1)}_{U(n-1:k)}(t)\Big] + n\alpha\Big[M^{(r+1)}_{U(n+1:k)}(t) - M^{(r+1)}_{U(n:k)}(t)\Big],$$

which, when rewritten, yields (5). □

Remark 2.2. The recurrence relation in Theorem 4 can be used in a simple recursive process to obtain all the marginal moment generating functions of all upper k-record values. By putting $t=0$ in (5), we deduce the recurrence relation for single moments of upper k-record values from Gompertz distribution as given below:

$$\alpha^{(r+1)}_{n+1:k} = \alpha^{(r+1)}_{n:k} + \frac{1}{n\alpha}\Big[(r+1)\alpha^{(r)}_{n:k} - \beta k\Big\{\alpha^{(r+1)}_{n:k} - \alpha^{(r+1)}_{n-1:k}\Big\}\Big], \quad n \geq 2, \tag{10}$$

where $\alpha^{(r)}_{n:k} = E\Big(X^r_{U(n:k)}\Big)$ denotes the rth moment of nth upper k-record value. Also, for $k=1$, we write $\alpha^{(r)}_{n:1} \equiv \alpha^{(r)}_n$. It may be noted that the relation (10) for $k=1$ verifies a result obtained by Khan and Zia (2009) for single moments of upper record values from Gompertz distribution.

Now, applying the relation (5) recursively in itself, one can easily establish some simple recurrence relations as given in the following two corollaries:

Corollary 2.3. *For $n \geq 2$, $1 \leq m \leq n$ and $r = 0,1,2,\ldots,$*

$$M^{(r+1)}_{U(n+1:k)}(t) = M^{(r+1)}_{U(m:k)}(t) + \sum_{i=m}^{n} \frac{1}{i\alpha}\Big[tM^{(r+1)}_{U(i:k)}(t) + (r+1)M^{(r)}_{U(i:k)}(t) - \beta k\Big\{M^{(r+1)}_{U(i:k)}(t) - M^{(r+1)}_{U(i-1:k)}(t)\Big\}\Big]. \tag{11}$$

Corollary 2.4. *For $n \geq 2$, $r = 0,1,2,\ldots,$*

$$M^{(r+1)}_{U(n+1:k)}(t) = \frac{(r+1)^{(r+1)}}{(n)^{(r+1)}\alpha^{r+1}} M^{(0)}_{U(n-r:k)}(t) + \sum_{i=0}^{r} \frac{(r+1)^{(i)}}{(n)^{(i+1)}\alpha^{i+1}}$$
$$\cdot \Big[\{(n-i)\alpha + t\}M^{(r+1-i)}_{U(n-i:k)}(t) - \beta k\Big\{M^{(r+1-i)}_{U(n-i:k)}(t) - M^{(r+1-i)}_{U(n-1-i:k)}(t)\Big\}\Big]. \tag{12}$$

Remark 2.5. By putting $t = 0$ in (11) and (12), we get the simple relations for the single moments of upper k-record values as follows.

For $n \geq 2$, $1 \leq m \leq n$ and $r = 0, 1, 2, \ldots,$

$$\alpha_{n+1:k}^{(r+1)} = \alpha_{m:k}^{(r+1)} + \sum_{i=m}^{n} \frac{1}{i\alpha} \Big[(r+1)\alpha_{i:k}^{(r)} - \beta k\Big\{\alpha_{i:k}^{(r+1)} - \alpha_{i-1:k}^{(r+1)}\Big\}\Big]$$

and for $n \geq 2$, $r = 0, 1, 2, \ldots,$

$$\alpha_{n+1:k}^{(r+1)} = \frac{(r+1)^{(r+1)}}{(n)^{(r+1)}\alpha^{r+1}} \sum_{i=0}^{r} \frac{(r+1)^{(i)}}{(n)^{(i+1)}\alpha^{i+1}} \cdot \Big[\{(n-i)\alpha\}\alpha_{n-i:k}^{(r+1-i)} - \beta k\Big\{\alpha_{n-i:k}^{(r+1-i)} - \alpha_{n-i-1:k}^{(r+1-i)}\Big\}\Big].$$

3. Relations for joint moment generating functions of upper k-record values

The joint density function of $X_{U(m:k)}$ and $X_{U(n:k)}$, $1 \leq m < n$, $n \geq 2$, as discussed by Shy et al. (2010) is given by (for $k \geq 1$):

$$f_{U(m:k),U(n:k)}(x, y) = \frac{k^n}{(m-1)!(n-m-1)!}[-\ln(1-F(x))]^{m-1}[1-F(y)]^{k-1}$$
$$\cdot [-\ln(1-F(y)) + \ln(1-F(x))]^{n-m-1} \frac{f(x)f(y)}{1-F(x)}, \ -\infty < x < y < \infty. \tag{13}$$

Theorem 3.1. *For $m \geq 1$, $r, s = 0, 1, 2, \ldots,$ and $\alpha, \beta > 0$,*

$$M_{U(m,m+1:k)}^{(r+1,s)}(t_1, t_2) = \frac{1}{m\alpha + t_1}\Big[m\alpha M_{U(m+1:k)}^{(r+s+1)}(t_1 + t_2) + \beta k$$
$$\cdot\Big\{M_{U(m:k)}^{(r+s+1)}(t_1 + t_2) - M_{U(m-1,m:k)}^{(r+1,s)}(t_1, t_2)\Big\} - (r+1)M_{U(m,m+1:k)}^{(r,s)}(t_1, t_2)\Big], \tag{14}$$

and, for $1 \leq m \leq n-2$, $r, s = 0, 1, 2, \ldots,$ and $\alpha, \beta > 0$,

$$M_{U(m,n:k)}^{(r+1,s)}(t_1, t_2) = \frac{1}{m\alpha + t_1}\Big[m\alpha M_{U(m+1,n:k)}^{(r+1,s)}(t_1, t_2)$$
$$+\beta k\Big\{M_{U(m,n-1:k)}^{(r+1,s)}(t_1, t_2) - M_{U(m-1,n-1:k)}^{(r+1,s)}(t_1, t_2)\Big\} - (r+1)M_{U(m,n:k)}^{(r,s)}(t_1, t_2)\Big], \tag{15}$$

provided the joint moment generating function exists.

Proof. Using (13), we can write

$$M_{U(m,n:k)}(t_1, t_2) = \frac{k^n}{(m-1)!(n-m-1)!}\int_{-\infty}^{\infty} e^{t_2 y}[1-F(y)]^{k-1} f(y)I(y)dy, \tag{16}$$

where

$$I(y) = \int_{-\infty}^{y} e^{t_1 x}[-\ln(1-F(x))]^{m-1}[-\ln(1-F(y)) + \ln(1-F(x))]^{n-m-1} \cdot \frac{f(x)}{[1-F(x)]}dx.$$

On using (3), we have

$$I(y) = \beta\int_{0}^{y} e^{t_1 x}[-\ln(1-F(x))]^{m-1}[-\ln(1-F(y)) + \ln(1-F(x))]^{n-m-1}dx$$
$$+ \alpha\int_{0}^{y} e^{t_1 x}[-\ln(1-F(x))]^{m}[-\ln(1-F(y)) + \ln(1-F(x))]^{n-m-1}dx$$
$$= \beta Q_{m,n} + \alpha Q_{m+1,n+1}, \tag{17}$$

where

$$Q_{m,n} = \int_0^y e^{t_1 x}[-\ln(1-F(x))]^{m-1}[-\ln(1-F(y))+\ln(1-F(x))]^{n-m-1}dx.$$

Integrating $Q_{m,n}$, by parts treating $e^{t_1 x}$ for integration and rest of the integrand for differentiation, we get

$$Q_{m,n} = \frac{(n-m-1)}{t_1}\int_0^y e^{t_1 x}[-\ln(1-F(x))]^{m-1}\cdot[-\ln(1-F(y))+\ln(1-F(x))]^{n-m-2}\frac{f(x)}{[1-F(x)]}dx$$
$$-\frac{(m-1)}{t_1}\int_0^y e^{t_1 x}[-\ln(1-F(x))]^{m-2}\cdot[-\ln(1-F(y))+\ln(1-F(x))]^{n-m-1}\frac{f(x)}{[1-F(x)]}dx. \quad (18)$$

Similarly,

$$Q_{m+1,n+1} = \frac{(n-m-1)}{t_1}\int_0^y e^{t_1 x}[-\ln(1-F(x))]^{m}\cdot[-\ln(1-F(y))+\ln(1-F(x))]^{n-m-2}\frac{f(x)}{[1-F(x)]}dx$$
$$-\frac{m}{t_1}\int_0^y e^{t_1 x}[-\ln(1-F(x))]^{m-1}\cdot[-\ln(1-F(y))+\ln(1-F(x))]^{n-m-1}\frac{f(x)}{[1-F(x)]}dx. \quad (19)$$

By using (18) and (19) in (17), and then (17) in (16), we get

$$t_1 M_{U(m,n:k)}(t_1,t_2) = \beta k\Big[M_{U(m,n-1:k)}(t_1,t_2) - M_{U(m-1,n-1:k)}(t_1,t_2)\Big] + m\alpha\Big[M_{U(m+1,n:k)}(t_1,t_2) - M_{U(m,n:k)}(t_1,t_2)\Big].$$

Differentiating the above relation, $(r+1)$ times with respect to t_1 and then s times with respect to t_2, we get

$$t_1 M^{(r+1,s)}_{U(m,n:k)}(t_1,t_2) + (r+1)M^{(r,s)}_{U(m,n:k)}(t_1,t_2) = \beta k\Big[M^{(r+1,s)}_{U(m,n-1:k)}(t_1,t_2) - M^{(r+1,s)}_{U(m-1,n-1:k)}(t_1,t_2)\Big]$$
$$+ m\alpha\Big[M^{(r+1,s)}_{U(m+1,n:k)}(t_1,t_2) - M^{(r+1,s)}_{U(m,n:k)}(t_1,t_2)\Big]. \quad (20)$$

Upon rewriting (20), we will get the relation in (15).

Proceeding in a similar manner for the case $n = m+1$, the recurrence relation given in (14) can easily be established. □

Theorem 3.2. *For $m \geq 1$, $r, s = 0, 1, 2, \ldots,$ and $\alpha, \beta > 0$,*

$$\alpha M^{(r,s+1)}_{U(m,m+2:k)}(t_1,t_2) = [t_2 - k\beta + \alpha]M^{(r,s+1)}_{U(m,m+1:k)}(t_1,t_2) + (s+1)M^{(r,s)}_{U(m,m+1:k)}(t_1,t_2)$$
$$+ k\beta M^{(r+s+1)}_{U(m:k)}(t_1+t_2) + m\alpha\Big[M^{(r+s+1)}_{U(m+1:k)}(t_1+t_2) - M^{(r,s+1)}_{U(m+1,m+2:k)}(t_1,t_2)\Big], \quad (21)$$

and, for $1 \leq m \leq n-2$, $r,\ s = 0, 1, 2, \ldots,$ and $\alpha, \beta > 0$,

$$\alpha(n-m)M^{(r,s+1)}_{U(m,n+1:k)}(t_1,t_2) = [t_2 - k\beta + \alpha(n-m)]M^{(r,s+1)}_{U(m,n:k)}(t_1,t_2) + (s+1)M^{(r,s)}_{U(m,n:k)}(t_1,t_2)$$
$$+ k\beta M^{(r,s+1)}_{U(m,n-1:k)}(t_1,t_2) + m\alpha\Big[M^{(r,s+1)}_{U(m+1,n:k)}(t_1,t_2) - M^{(r,s+1)}_{U(m+1,n+1:k)}(t_1,t_2)\Big], \quad (22)$$

provided the joint moment generating function exists.

Proof. By using (13), we have

$$M_{U(m,n:k)}(t_1,t_2) = \frac{k^n}{(m-1)!(n-m-1)!}\int_{-\infty}^{\infty}\int_x^{\infty} e^{t_1 x + t_2 y}[-\ln(1-F(x))]^{m-1}$$
$$\cdot[-\ln(1-F(y)) + ln(1-F(x))]^{n-m-1}\cdot[1-F(y)]^{k-1}\frac{f(x)f(y)}{1-F(x)}dydx.$$

Now, on using (3), we have

$$M_{U(m,n:k)}(t_1,t_2) = \frac{k^n}{(m-1)!(n-m-1)!}\int_0^\infty\int_x^\infty e^{t_1x+t_2y}[-\ln(1-F(x))]^{m-1}$$
$$\cdot[-\ln(1-F(y))+\ln(1-F(x))]^{n-m-1}[\beta[1-F(y)]+\alpha[1-F(y)]\{-\ln(1-F(y))\}]$$
$$\cdot[1-F(y)]^{k-1}\frac{f(x)}{1-F(x)}dydx$$
$$= \frac{1}{(m-1)!(n-m-1)!}\int_0^\infty e^{t_1x}[-\ln(1-F(x))]^{m-1}I(x)\frac{f(x)}{[1-F(x)]}dx, \tag{23}$$

where

$$I(x) = \beta\int_x^\infty e^{t_2y}[-\ln(1-F(y))+\ln(1-F(x))]^{n-m-1}[1-F(y)]^k dy$$
$$+\alpha\int_x^\infty e^{t_2y}[-\ln(1-F(y))+\ln(1-F(x))]^{n-m-1}[1-F(y)]^k[-\ln(1-F(y))]dy$$
$$= [\beta+\alpha\{-\ln(1-F(x))\}]\int_x^\infty e^{t_2y}[-\ln(1-F(y))+\ln(1-F(x))]^{n-m-1}[1-F(y)]^k dy$$
$$+\alpha\int_x^\infty e^{t_2y}\cdot[-\ln(1-F(y))+\ln(1-F(x))]^{n-m}[1-F(y)]^k dy$$
$$= [\beta+\alpha\{-\ln(1-F(x))\}]A_{n-m-1}+\alpha A_{n-m}, \tag{24}$$

where

$$A_{n-m} = \int_x^\infty e^{t_2y}[-\ln(1-F(y))+\ln(1-F(x))]^{n-m}[1-F(y)]^k dy.$$

Integrating A_{n-m} by parts treating e^{t_2y} for integration and rest of the integrand for differentiation, we get

$$A_{n-m} = \frac{k}{t_2}\int_x^\infty e^{t_2y}[-\ln(1-F(y))+\ln(1-F(x))]^{n-m}[1-F(y)]^{k-1}f(y)dy$$
$$-\frac{(n-m)}{t_2}\int_x^\infty e^{t_2y}[-\ln(1-F(y))+\ln(1-F(x))]^{n-m-1}\cdot[1-F(y)]^{k-1}f(y)dy. \tag{25}$$

Similarly,

$$A_{n-m-1} = \frac{k}{t_2}\int_x^\infty e^{t_2y}[-\ln(1-F(y))+\ln(1-F(x))]^{n-m-1}[1-F(y)]^{k-1}f(y)dy$$
$$-\frac{(n-m-1)}{t_2}\int_x^\infty e^{t_2y}[-\ln(1-F(y))+\ln(1-F(x))]^{n-m-2}\cdot[1-F(y)]^{k-1}f(y)dy. \tag{26}$$

By using (25) and (26) in (24), and then (24) in (23), we get

$$t_2M_{U(m,n:k)}(t_2,t_2) = \alpha(n-m)[M_{U(m,n+1:k)}(t_1,t_2)-M_{U(m,n:k)}(t_1,t_2)]$$
$$+m\alpha[M_{U(m+1,n+1:k)}(t_1,t_2)-M_{U(m+1,n:k)}(t_1,t_2)]$$
$$+\beta k[M_{U(m,n:k)}(t_1,t_2)-M_{U(m,n-1:k)}(t_1,t_2)].$$

Differentiating the above expression, $(s+1)$ times with respect to t_2 and then r times with respect to t_1, we get

$$t_2M^{(r,s+1)}_{U(m,n:k)}(t_1,t_2)+(s+1)M^{(r,s)}_{U(m,n:k)}(t_1,t_2) = \alpha(n-m)\Big[M^{(r,s+1)}_{U(m,n+1:k)}(t_1,t_2)-M^{(r,s+1)}_{U(m,n:k)}(t_1,t_2)\Big]$$
$$+m\alpha\Big[M^{(r,s+1)}_{U(m+1,n+1:k)}(t_1,t_2)-M^{(r,s+1)}_{U(m+1,n:k)}(t_1,t_2)\Big]+\beta k\Big[M^{(r,s+1)}_{U(m,n:k)}(t_1,t_2)-M^{(r,s+1)}_{U(m,n-1:k)}(t_1,t_2)\Big]. \tag{27}$$

Upon rewriting (27), we will get the relation in (22).

Proceeding in a similar manner for the case $n = m + 1$, the recurrence relation given in (21) can easily be established. □

Remark 3.3. The recurrence relations in Theorems 3.1 and 3.2 can be used in a simple recursive process to obtain all the product moment generating functions of all upper k-record values.

Remark 3.4. By putting $t_1 = t_2 = 0$ in Theorem 3.1, we get the recurrence relations for the product moments of upper k-record values as follows:

For $m \geq 1$, $r, s = 0, 1, 2, \ldots$, and $\alpha, \beta > 0$,

$$\alpha_{m,m+1:k}^{(r+1,s)} = \alpha_{m+1:k}^{(r+s+1)} + \frac{1}{m\alpha}\Big[\beta k\{\alpha_{m:k}^{(r+s+1)} - \alpha_{m-1,m:k}^{(r+1,s)}\} - (r+1)\alpha_{m,m+1:k}^{(r,s)}\Big],$$

and for $1 \leq m \leq n-2$, $r, s = 0, 1, 2, \ldots$, and $\alpha, \beta > 0$,

$$\alpha_{m,n:k}^{(r+1,s)} = \alpha_{m+1,n:k}^{(r+1,s)} + \frac{1}{m\alpha}\Big[\beta k\{\alpha_{m,n-1:k}^{(r+1,s)} - \alpha_{m-1,n-1:k}^{(r+1,s)}\} - (r+1)\alpha_{m,n:k}^{(r,s)}\Big], \tag{28}$$

where $\alpha_{m,n:k}^{(r,s)} = E(X_{U(m:k)}^r X_{U(n:k)}^s)$ denotes the (r,s)-th product moment of the mth and nth upper k-record values.

Also, for $k = 1$, we write $\alpha_{m,n:1}^{(r,s)} \equiv \alpha_{m,n}^{(r,s)}$.

It may be noted that the relation (28), for $k = 1$, verifies the result obtained by Khan and Zia (2009) for product moments of upper record values from Gompertz distribution.

Remark 3.5. By putting $t_1 = t_2 = 0$ in Theorem 3.2, we get some other recurrence relations for the product moments of upper k-record values as follows:
For $m \geq 1$, $r, s = 0, 1, 2, \ldots$, and $\alpha, \beta > 0$,

$$\alpha_{m,m+2:k}^{(r,s+1)} = \left[1 - \frac{k\beta}{\alpha}\right]\alpha_{m,m+1:k}^{(r,s+1)} + \frac{1}{\alpha}\Big[(s+1)\alpha_{m,m+1:k}^{(r,s)} + k\beta\alpha_{m:k}^{(r+s+1)} + m\alpha\Big\{\alpha_{m+1:k}^{(r+s+1)} - \alpha_{m+1,m+2:k}^{(r,s+1)}\Big\}\Big],$$

and for $1 \leq m \leq n-2$, $r, s = 0, 1, 2, \ldots$, and $\alpha, \beta > 0$,

$$\alpha_{m,n+1:k}^{(r,s+1)} = \left[1 - \frac{k\beta}{\alpha(n-m)}\right]\alpha_{m,n:k}^{(r,s+1)} + \frac{1}{\alpha(n-m)}\Big[(s+1)\alpha_{m,n:k}^{(r,s)} + k\beta\alpha_{m,n-1:k}^{(r,s+1)} + m\alpha\Big\{\alpha_{m+1,n:k}^{(r,s+1)} - \alpha_{m+1,n+1:k}^{(r,s+1)}\Big\}\Big].$$

Remark 3.6. By putting $t_2 = 0$, $t_1 = t$ and $m = n$ in (15) or, equivalently, putting $t_2 = t$, $t_1 = 0$ and $m = n$ in (22), one can easily deduce and verify the recurrence relation for marginal moment generating functions of upper k-record values, as given in (5).

Now, applying the relations (15) and (22) recursively in itself, respectively, one can easily establish some simple recurrence relations as given in the following corollary:

Corollary 3.7. *For $1 \leq m \leq n-2$, $r, s = 0, 1, 2, \ldots$, and $\alpha, \beta > 0$,*

$$M_{U(m,n:k)}^{(r+1,s)}(t_1, t_2) = \left[\frac{-1}{m\alpha + t_1}\right]^{r+1} (r+1)^{(r+1)} M_{U(m,n:k)}^{(0,s)}(t_1, t_2) + \sum_{i=0}^{r} (-1)^i \left[\frac{1}{m\alpha + t_1}\right]^{i+1} (r+1)^{(i)}$$
$$\cdot \Big[m\alpha M_{U(m+1,n:k)}^{(r+1-i,s)}(t_1, t_2) + \beta k\Big\{M_{U(m,n-1:k)}^{(r+1-i,s)}(t_1, t_2) - M_{U(m-1,n-1:k)}^{(r+1-i,s)}(t_1, t_2)\Big\}\Big], \tag{29}$$

and

$$M^{(r,s+1)}_{U(m,n+1:k)}(t_1,t_2) = \left[\frac{1}{\alpha(n-m)}\right]^{s+1} (s+1)^{(s+1)} M^{(r,0)}_{U(m,n-s:k)}(t_1,t_2)$$
$$+ \left[\frac{t_2-\beta k}{\alpha(n-m)}+1\right]\sum_{i=0}^{s}\left[\frac{1}{\alpha(n-m)}\right]^{i}(s+1)^{(i)} M^{(r,s+1-i)}_{U(m,n-i:k)}(t_1,t_2)$$
$$+ \sum_{i=0}^{s}\left[\frac{1}{\alpha(n-m)}\right]^{i+1}(s+1)^{(i)}\Big[\beta k M^{(r,s+1-i)}_{U(m,n-1-i:k)}(t_1,t_2)$$
$$+ m\alpha\Big\{M^{(r,s+1-i)}_{U(m+1,n-i:k)}(t_1,t_2) - M^{(r,s+1-i)}_{U(m-1,n+1-i:k)}(t_1,t_2)\Big\}\Big]. \quad (30)$$

Remark 3.8. By putting $t_1 = t_2 = 0$ in (29) and (30), we get the recurrence relations for the product moments of upper k-record values as follows:

$$\alpha^{(r+1,s)}_{m,n:k} = \left[\frac{-1}{m\alpha}\right]^{r+1}(r+1)^{(r+1)}\alpha^{(s)}_{n:k} + \sum_{i=0}^{r}(-1)^i$$
$$\cdot\left[\frac{1}{m\alpha}\right]^{i+1}(r+1)^{(i)}\Big[m\alpha\cdot\alpha^{(r+1-i,s)}_{m+1,n:k} + \beta k\Big\{\alpha^{(r+1-i,s)}_{m,n-1:k} - \alpha^{(r+1-i,s)}_{m-1,n-1:k}\Big\}\Big],$$

and

$$\alpha^{(r,s+1)}_{m,n+1:k} = \left[\frac{1}{\alpha(n-m)}\right]^{s+1}(s+1)^{(s+1)}\alpha^{(r)}_{m:k} + \left[1-\frac{\beta k}{\alpha(n-m)}\right]\sum_{i=0}^{s}\left[\frac{1}{\alpha(n-m)}\right]^{i}(s+1)^{(i)}\alpha^{(r,s+1-i)}_{m,n-i:k}$$
$$+\sum_{i=0}^{s}\left[\frac{1}{\alpha(n-m)}\right]^{i+1}(s+1)^{(i)}\Big[\beta k\alpha^{(r,s+1-i)}_{m,n-1-i:k} + m\alpha\Big\{\alpha^{(r,s+1-i)}_{m+1,n-i:k} - \alpha^{(r,s+1-i)}_{m+1,n+1-i:k}\Big\}\Big].$$

4. Relations for upper record values

Putting $k = 1$ in the results obtained in Sections 2 and 3, one can easily deduce the corresponding results for upper record values from the Gompertz distribution which are given below.

As defined earlier in Section 1, the moment generating function of $X_{U(n)}$ is denoted by $M_{U(n)}(t)$, the rth derivative of $M_{U(n)}(t)$ with respect to t by $M^{(r)}_{U(n)}(t)$, the joint moment generating function of $X_{U(m)}$ and $X_{U(n)}$ by $M_{U(m,n)}(t_1,t_2)$, and the (r,s)-th partial derivative of $M_{U(m,n)}(t_1,t_2)$ with respect to t_1 and t_2, respectively, by $M^{(r,s)}_{U(m,n)}(t_1,t_2)$. Also $\alpha^{(r)}_n = E(X^r_{U(n)})$ and $\alpha^{(r,s)}_{m,n} = E(X^r_{U(m)}X^s_{U(n)})$.

Theorem 4.1. *For $n \geq 2$, $r = 0,1,2,\ldots$, and $\alpha,\beta > 0$,*

$$M^{(r+1)}_{U(n+1)}(t) = M^{(r+1)}_{U(n)}(t) + \frac{1}{n\alpha}\Big[tM^{(r+1)}_{U(n)}(t) + (r+1)M^{(r)}_{U(n)}(t) - \beta\Big\{M^{(r+1)}_{U(n)}(t) - M^{(r+1)}_{U(n-1)}(t)\Big\}\Big]. \quad (31)$$

Remark 4.2. Setting $t = 0$ in (31), we get

$$\alpha^{(r+1)}_{n+1} = \alpha^{(r+1)}_{n} + \frac{1}{n\alpha}\Big[(r+1)\alpha^{(r)}_n - \beta\Big\{\alpha^{(r+1)}_n - \alpha^{(r+1)}_{n-1}\Big\}\Big],$$

which is in agreement with Khan and Zia (2009).

Corollary 4.3. *For $n \geq 2$, $1 \leq m \leq n$, $r = 0,1,2,\ldots$, and $\alpha,\beta > 0$,*

$$M^{(r+1)}_{U(n+1)}(t) = M^{(r+1)}_{U(m)}(t) + \sum_{i=m}^{n}\frac{1}{i\alpha}\Big[tM^{(r+1)}_{U(i)}(t) + (r+1)M^{(r)}_{U(i)}(t) - \beta\Big\{M^{(r+1)}_{U(i)}(t) - M^{(r+1)}_{U(i-1)}(t)\Big\}\Big]. \quad (32)$$

Remark 4.4. Putting $t = 0$ in (32), we get

$$\alpha_{n+1}^{(r+1)} = \alpha_m^{(r+1)} + \sum_{i=m}^{n} \frac{1}{i\alpha}\Big[(r+1)\alpha_i^{(r)} - \beta\Big\{\alpha_i^{(r+1)} - \alpha_{i-1}^{(r+1)}\}\Big],$$

Corollary 4.5. *For $n \geq 2$, $r = 0, 1, 2, \ldots,$ and $\alpha, \beta > 0$,*

$$M_{U(n+1)}^{(r+1)}(t) = \frac{(r+1)^{(r+1)}}{(n)^{(r+1)}\alpha^{r+1}} M_{U(n-r)}^{(0)}(t) + \sum_{i=0}^{r} \frac{(r+1)^{(i)}}{(n)^{(i+1)}\alpha^{i+1}}\Big[[(n-i)\alpha+t]M_{U(n-i)}^{(r+1-i)}(t) - \beta\Big\{M_{U(n-i)}^{(r+1-i)}(t) - M_{U(n-1-i)}^{(r+1-i)}(t)\Big\}\Big]. \tag{33}$$

Remark 4.6. By putting $t = 0$ in (33), we get for $n \geq 2$, $r = 0, 1, 2, \ldots,$ and $\alpha, \beta > 0$,

$$\alpha_{n+1}^{(r+1)} = \frac{(r+1)^{(r+1)}}{(n)^{(r+1)}\alpha^{r+1}} + \sum_{i=0}^{r} \frac{(r+1)^{(i)}}{(n)^{(i+1)}\alpha^{i+1}}\Big[\{(n-i)\alpha\}\alpha_{n-i}^{(r+1-i)} - \beta\Big\{\alpha_{n-i}^{(r+1-i)} - \alpha_{n-1-i}^{(r+1-i)}\}\Big].$$

Theorem 4.7. *For $m \geq 1$, $r, s = 0, 1, 2, \ldots,$ and $\alpha, \beta > 0$,*

$$M_{U(m,m+1)}^{(r+1,s)}(t_1, t_2) = \frac{1}{m\alpha + t_1}\Big[m\alpha M_{U(m+1)}^{(r+s+1)}(t_1 + t_2) + \beta\Big\{M_{U(m)}^{(r+s+1)}(t_1 + t_2) - M_{U(m-1,m)}^{(r+1,s)}(t_1, t_2)\Big\} - (r+1)M_{U(m,m+1)}^{(r,s)}(t_1, t_2)\Big],$$

and, for $1 < m \leq n$, $r, s = 0, 1, 2, \ldots,$ and $\alpha, \beta > 0$,

$$M_{U(m,n)}^{(r+1,s)}(t_1, t_2) = \frac{1}{m\alpha + t_1}\Big[m\alpha M_{U(m+1,n)}^{(r+1,s)}(t_1, t_2) + \beta\Big\{M_{U(m,n-1)}^{(r+1,s)}(t_1, t_2) - M_{U(m-1,n-1)}^{(r+1,s)}(t_1, t_2)\Big\} - (r+1)M_{U(m,n)}^{(r,s)}(t_1, t_2)\Big]. \tag{34}$$

Remark 4.8. By setting $t_1 = t_2 = 0$ in (34), we get

$$\alpha_{m,n}^{(r+1,s)} = \alpha_{m+1,n}^{(r+1,s)} + \frac{1}{m\alpha}\Big[\beta\Big\{\alpha_{m,n-1}^{(r+1,s)} - \alpha_{m-1,n-1}^{(r+1,s)}\Big\} - (r+1)\alpha_{m,n}^{(r,s)}\Big],$$

which is in agreement with Khan and Zia (2009).

Remark 4.9. By putting $t_2 = 0$, $t_1 = t$ and $m = n$ in (34), it reduces to the recurrence relation for marginal moment generating functions of upper record values from Gompertz distribution, as given in (31).

Theorem 4.10. *For $m \geq 1$, $r, s = 0, 1, 2, \ldots,$ and $\alpha, \beta > 0$,*

$$\alpha M_{U(m,m+2)}^{(r,s+1)}(t_1, t_2) = [t_2 - \beta + \alpha]M_{U(m,m+1)}^{(r,s+1)}(t_1, t_2) + (s+1)M_{U(m,m+1)}^{(r,s)}(t_1, t_2) + \beta M_{U(m)}^{(r+s+1)}(t_1 + t_2) + m\alpha\Big[M_{U(m+1)}^{(r+s+1)}(t_1 + t_2) - M_{U(m+1,m+2)}^{(r,s+1)}(t_1, t_2)\Big], \tag{35}$$

and for $1 \leq m \leq n-2$, $r, s = 0, 1, 2, \ldots,$ and $\alpha, \beta > 0$,

$$[\alpha(n-m)]M_{U(m,n+1)}^{(r,s+1)}(t_1, t_2) = [t_2 - k\beta + \alpha(n-m)]M_{U(m,n)}^{(r,s+1)}(t_1, t_2) + (s+1)M_{U(m,n)}^{(r,s)}(t_1, t_2) + \beta M_{U(m,n-1)}^{(r,s+1)}(t_1, t_2) + m\alpha\Big[M_{U(m+1,n)}^{(r,s+1)}(t_1, t_2) - M_{U(m+1,n+1)}^{(r,s+1)}(t_1, t_2)\Big]. \tag{36}$$

Remark 4.11. By setting $t_1 = t_2 = 0$ in (35) and (36), we get the recurrence relations for product moments of upper record values from Gompertz distribution. That is:

For $m \geq 1$, $r, s = 0, 1, 2, \ldots$, and $\alpha, \beta > 0$,

$$\alpha_{m,m+2}^{(r,s+1)} = \left[1 - \frac{\beta}{\alpha}\right] \alpha_{m,m+1}^{(r,s+1)} + \frac{1}{\alpha}\Big[(s+1)\alpha_{m,m+1}^{(r,s)} + \beta\alpha_{m}^{(r+s+1)} + m\alpha\Big\{\alpha_{m+1}^{(r+s+1)} - \alpha_{m+1,m+2}^{(r,s+1)}\Big\}\Big],$$

and for $1 \leq m \leq n-2$, $r, s = 0, 1, 2, \ldots$, and $\alpha, \beta > 0$,

$$\alpha_{m,n+1}^{(r,s+1)} = \left[1 - \frac{\beta}{\alpha(n-m)}\right] \alpha_{m,n}^{(r,s+1)} + \frac{1}{\alpha(n-m)}\Big[(s+1)\alpha_{m,n}^{(r,s)} + \beta\alpha_{m,n-1}^{(r,s+1)} + m\alpha\Big\{\alpha_{m+1,n}^{(r,s+1)} - \alpha_{m+1,n+1}^{(r,s+1)}\Big\}\Big].$$

Remark 4.12. By putting $t_2 = t$, $t_1 = 0$ and $m = n$ in (36), one can easily get and verify the recurrence relation for marginal moment generating functions of upper record values from Gompertz distribution, as given in (31).

Corollary 4.13. *For $1 \leq m \leq n-2$, $r, s = 0, 1, 2, \ldots$, and $\alpha, \beta > 0$,*

$$M_{U(m,n)}^{(r+1,s)}(t_1,t_2) = \left[\frac{-1}{m\alpha + t_1}\right]^{r+1} (r+1)^{(r+1)} M_{U(m,n)}^{(0,s)}(t_1,t_2) + \sum_{i=0}^{r}(-1)^i \left[\frac{1}{m\alpha + t_1}\right]^{i+1} (r+1)^{(i)}$$
$$\cdot \Big[m\alpha M_{U(m+1,n)}^{(r+1-i,s)}(t_1,t_2) + \beta\Big\{M_{U(m,n-1)}^{(r+1-i,s)}(t_1,t_2) - M_{U(m-1,n-1)}^{(r+1-i,s)}(t_1,t_2)\Big\}\Big], \tag{37}$$

and

$$M_{U(m,n+1)}^{(r,s+1)}(t_1,t_2) = \left[\frac{1}{\alpha(n-m)}\right]^{s+1} (s+1)^{(s+1)} M_{U(m,n-s)}^{(r,0)}(t_1,t_2) + \left[\frac{t_2 - \beta}{\alpha(n-m)} + 1\right] \sum_{i=0}^{s} \left[\frac{1}{\alpha(n-m)}\right]^{i}$$
$$\cdot (s+1)^{(i)} M_{U(m,n-i)}^{(r,s+1-i)}(t_1,t_2) + \sum_{i=0}^{s} \left[\frac{1}{\alpha(n-m)}\right]^{i+1} (s+1)^{(i)} \Big[\beta M_{U(m,n-1-i)}^{(r,s+1-i)}(t_1,t_2)$$
$$+ m\alpha\Big\{M_{U(m+1,n-i)}^{(r,s+1-i)}(t_1,t_2) - M_{U(m+1,n+1-i)}^{(r,s+1-i)}(t_1,t_2)\Big\}\Big]. \tag{38}$$

Remark 4.14. By putting $t_1 = t_2 = 0$ in (37) and (38), we have for $1 \leq m \leq n-2$ and $r, s = 0, 1, 2, \ldots$,

$$\alpha_{m,n}^{(r+1,s)} = \left[\frac{-1}{m\alpha}\right]^{r+1} (r+1)^{(r+1)} \alpha_{m,n}^{(0,s)} + \sum_{i=0}^{r}(-1)^i \left[\frac{1}{m\alpha}\right]^{i+1} (r+1)^{(i)} \cdot \Big[m\alpha \cdot \alpha_{m+1,n}^{(r+1-i,s)} + \beta\Big\{\alpha_{m,n-1}^{(r+1-i,s)} - \alpha_{m-1,n-1}^{(r+1-i,s)}\Big\}\Big],$$

and

$$\alpha_{m,n+1}^{(r,s+1)} = \left[\frac{-1}{\alpha(n-m)}\right]^{s+1} (s+1)^{(s+1)} \alpha_{m,n-s}^{(r,0)} + \left[1 - \frac{\beta}{\alpha(n-m)}\right] \sum_{i=0}^{s} \left[\frac{1}{\alpha(n-m)}\right]^{i} (s+1)^{(i)} \alpha_{m,n-i}^{(r,s+1-i)}$$
$$+ \sum_{i=0}^{s} \left[\frac{1}{\alpha(n-m)}\right]^{i+1} (s+1)^{(i)} \Big[\beta\alpha_{m,n-1-i}^{(r,s+1-i)} + m\alpha\Big\{\alpha_{m+1,n-i}^{(r,s+1-i)} - \alpha_{m+1,n+1-i}^{(r,s+1-i)}\Big\}\Big].$$

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