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Distributions of sums of geometrically weighted finite valued discrete random variables

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Abstract. Based on the α-binary representation of a number $t \in [0,1]$, described in the introduction, we define a random variable (r.v.) X which is a geometrically weighted sum of a sequence of independent and identically distributed (i.i.d.) Bernoulli random variables. Distributional properties of X and some of its generalizations are studied.

1. Introduction

It is known that every number t, $0 \leq t \leq 1$ has a binary representation $t = \sum_{i=1}^{\infty} a_i \, (1/2)^i$, where $\{ a_n\}_1^{\infty}$ is sequence of zeros and ones. However $t_k = \sum_{i=1}^{k} a_i \, (1/2)^i$ with $a_k = 1$, can also be represented as $\sum_{i=1}^{\infty} a_i \, (1/2)^i$ with $a_k = 0$ and $a_j = 1$ for $j > k$, and for such a number we use first representation. Under the above convention, $t \in [0,1]$ corresponds to a unique binary sequence $\{ a_n\}_1^{\infty}$ and conversely. If $t = \sum_{i=1}^{\infty} a_i \, (1/2)^i$, then $0 \leq t < 1/2$ if and only if $a_1 = 0$. Such a number t can be obtained as the limit of subintervals identified by selecting appropriate half subintervals based on the sequence $\{a_n\}_1^{\infty}$.

Now instead of considering the half intervals, if we partition the interval [0,1) in the ratio $\alpha : (1-\alpha)$, $0 < \alpha < 1$, in the first stage, we obtain subintervals $[0, \alpha)$, and $[\alpha, 1)$. Further by partitioning each one of these intervals into subintervals in the same ratio $\alpha : (1-\alpha)$, in the second stage, we obtain subintervals $[0, \alpha^2)$, $[\alpha^2, \alpha)$, $[\alpha, \alpha + \alpha(1-\alpha))$, $[\alpha + \alpha(1-\alpha), 1)$. This process can be continued. It can be verified that after the k^{th} stage, the 2^k lower boundaries of subintervals are of the form $\sum_{i=1}^{k} \alpha^{i-s_{i-1}} \, (1-\alpha)^{s_{i-1}} \, a_i$, $k = 1, 2, 3, \cdots$ with $a_i = 0$ *or* 1, where $s_i = \sum_{j=1}^{i} a_j, i = 1, 2, 3, \cdots$ and $s_0 = 0$.

For example, if $k = 3$, the eight lower boundary points are 0, α^3, α^2, $\alpha^2 + \alpha^2(1-\alpha)$, α, $\alpha + \alpha^2(1-\alpha)$, $\alpha + \alpha(1-\alpha)$, $\alpha + \alpha(1-\alpha) + \alpha(1-\alpha)^2$ and correspond to the vectors ($\underline{a} = (a_1, a_2, a_3)$) (0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0), (1,1,1). Every number t, $0 < t < 1$, can be obtained as the limit of subintervals identified by selecting appropriate subintervals, that determines the sequence $\{a_n\}_1^{\infty}$ of zeros and ones. Thus every real number t, $0 \leq t < 1$, has unique representation through $\{a_n\}_1^{\infty}$ as

$$t = \sum_{i=1}^{\infty} \alpha^{i-s_{i-1}} \, (1-\alpha)^{s_{i-1}} \, a_i. \tag{1}$$

We refer the representation (1) as the α-binary representation and said to have finite (infinite) α-binary representation according as $a_j = 1$ for finite (infinite) number of values of j.

Keywords. Binary Representation, Independent and Identically Distribu-ted Random Variables, Geometrically Weighted Bernoulli Random Variables, Distribution Function.

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Let the selection of a subinterval and its lower limit be governed by a sequence $Z_1, Z_2, \cdots, Z_k$ i.i.d. Bernouli(1, p) random variables. At the i^{th} stage we select upper or the lower subinterval according as Z_i is 1 or 0. For example, if $Z_1 = 0, Z_2 = 1, Z_3 = 1$, the intervals selected are $[0, \alpha), [\alpha^2, \alpha), [\alpha^2 + \alpha(\alpha - \alpha^2), \alpha)$. Thus corresponding to the sequence $Z_1 = 0, Z_2 = 1, Z_3 = 1$, the lower limit of the interval is $\alpha^2 + \alpha(\alpha - \alpha^2)$.

Let X_k be the lower limit of the k^{th} sub-interval selected on the bases of random variables $Z_1, Z_2, \cdots, Z_k$ and X be the random point obtained from the sequence $\{Z_i\}_{i=1}^{\infty}$. Thus we have

$$X_k(Z_1, Z_2, \cdots, Z_k; \alpha, p) = \sum_{i=1}^{k} \alpha^{i-s_{i-1}}(1-\alpha)^{s_{i-1}} Z_i, \quad k = 1, 2, \cdots \tag{2}$$

and

$$X(Z_1, Z_2, \cdots; \alpha, p) = \sum_{i=1}^{\infty} \alpha^{i-s_{i-1}}(1-\alpha)^{s_{i-1}} Z_i, \tag{3}$$

where $s_i = \sum_{j=1}^{i} Z_j$ and $\{Z_k\}_1^{\infty}$ is a sequence of i.i.d $B(1, p)$ r.v's. These random variables X_k and X may be referred as discrete and continuous α-Bernoulli random variables respectively. Further it is to be noted that X_k can be represented as

$$X_k(Z_1, Z_2, \cdots, Z_k; \alpha, p) = \alpha Z_1 + \alpha^{Z_1}(1-\alpha)^{Z_1} X_{k-1}(Z_2, Z_3, \cdots, Z_k)$$

For notational simplicity in the following we write $X_k(Z_1, Z_2, \cdots, Z_k; \alpha, p)$ and $X(Z_1, Z_2, \cdots, Z_k; \alpha, p)$ as X_k and X respectively.

We note that for (α, p) fixed, $\{X_k\}_1^{\infty}$ is a pointwise non decreasing sequence of r.v's converging to X. Kunte and Rattihalli (1992) have shown that when $(\alpha, p) = (1/2,\ 1/2)$, X has U(0,1) distribution. Bhati et al. (2011) have obtained the distribution function (d.f.) of X when $\alpha = 1/2$. In Section 2, we obtain (4) $F_k(\cdot)$, d.f. of X_k and $F(\cdot)$, d.f. of X. (5) $\Phi_k(\cdot)$ characteristic function (c.f.) of X_k, (6) recursive properties of F_k, F, Φ_k, and Φ c.f. of X_k and X, (7) an upper bound for $F_k(t) - F(t)$ and remark on stochastic ordering is also given. In Section 3, we consider the vector versions of p and α, $\underline{p} = (p_0, p_1, \ldots, p_{m-1})$ and $\underline{\alpha} = \left(\frac{1}{m}, \frac{1}{m}, \ldots, \frac{1}{m}\right)$. The graphs of d.f. of $X_{10}, F_{10}(\cdot)$, for different α, p are given in the Appendix.

2. The distributional properties of $\alpha-$Bernoulli random variables

2.1. The distribution function of X_k

On A_k, the set of all k-dimensional vectors of zeroes and ones define the ordering $(0, 0, \cdots, 0) \prec (0, 0, \cdots, 0, 1) \prec (0, 0, \cdots, 1, 0) \prec (0, 0, \cdots, 1, 1) \prec \cdots \prec (1, 1, \cdots, 1, 0) \prec (1, 1, 1, \cdots, 1, 1)$ and the corresponding values of t be denoted by $t^{(0)} = 0 < t^{(1)} < \cdots < t^{(2^k-1)}$. For example $t^{(0)} = 0, t^{(2^k-1)} = 1-(1-\alpha)^k$, equivalently for $\underline{a}, \underline{b} \in A_k$, $\underline{a} \prec \underline{b}$ iff $t_a < t_b$. The r.v. X_k takes the values $t^{(0)}, t^{(1)}, \cdot, t^{(2^k-1)}$ and hence the d.f.'s' of X_k is given by

$$F_k(t) = P\left(X_k \le t\right) = \sum_{j=0}^{r} P\left(X_k = t^{(j)}\right), \quad t^{(r)} \le t < t^{(r+1)} \tag{4}$$

Theorem 2.1. *The d.f. of X is given by*

$$F(t) = \begin{cases} 0, & t \le 0 \\ \sum_{i=1}^{\infty} q^{i-s_{i-1}} p^{s_{i-1}} a_i\,, & 0 \le t \le 1 \\ 1, & t \ge 1. \end{cases} \tag{5}$$

where for $0 < t < 1$, t has the α-binary representation $\sum_{i=1}^{\infty} \alpha^{i-s_{i-1}}\left(1-\alpha\right)^{s_{i-1}} a_i$.

Proof. Let t have an infinite α-binary representation, i.e $t = \sum_{i=1}^{\infty} \alpha^{i-s_{i-1}}\ (1-\alpha)^{s_{i-1}} a_i$, and $a_j = 1$ for infinite number of values of j. Let a_{k_r} be the r^{th} non-zero element in the sequence $\{a_i\}$, $r = 1, 2, \cdots$. Let t_r be the number having the binary representation $(a_1, a_2, \ldots, a_{k_r}, 0, 0, \ldots)$ and $t_0 = 0$. It is to be noted that t_r has a α-finite binary termination. If t has an infinite α-binary representation then the sequence $\{t_r\}$ increases to t. If t has a finite α-binary termination then $t = t_r$ for some finite r. Note that

$$\begin{aligned} P(X = t_r) &= P(Z_i = a_i; i = 1, 2, \cdots, k_r - 1, Z_{k_r} = 1,\ Z_{k_r+j} = 0, j = 1, 2, \cdots) \\ &\leq P(Z_{k_r} = 1,\ Z_{k_r+j} = 0,\ j = 1, 2, \cdots = 0). \end{aligned}$$

Hence

$$P(X = t_r) = 0 \tag{6}$$

Thus F does not have a jump at t_r and $F(t_r) \uparrow F(t)$. Let t have finite α-binary representation. We note that

$$P(0 \leq X < t_1) = P(Z_1 = 0, \cdots, Z_{k_1} = 0) = q^{k_1},$$

$$P(t_1 \leq X < t_2) = P(Z_1 = 0, \cdots, Z_{k_1-1} = 0, Z_{k_1} = 1, Z_{k_1+1} = 0, \cdots, Z_{k_2} = 0) = q^{k_2-1}p,$$

$$\begin{aligned} P(t_2 \leq X < t_3) &= P(Z_1 = \cdots = Z_{k_1-1} = 0, Z_{k_1} = 1, Z_{k_1+1} = \cdots = Z_{k_2-1} = 0, Z_{k_2} = 1, Z_{k_2+1} = \cdots = Z_{k_3} = 0) \\ &= q^{k_3-2}p^2. \end{aligned}$$

Thus in general for $r = 1, 2, 3, \cdots$, we have

$$P(t_{r-1} \leq X < t_r) = q^{k_r-(r-1)}p^{r-1}.$$

Hence,

$$P(X < t) = \sum_{r=1}^{\infty} P(t_{r-1} \leq X < t_r) = \sum_{r=1}^{\infty} q^{k_r-(r-1)}p^{r-1}.$$

If we let $k_r = j$, then $r - 1 = \sum_{i=1}^{j-1} a_i = s_{j-1}$ and we will have

$$P(X < t) = \sum_{j=1}^{\infty} a_j (q^{j-s_{j-1}} p^{s_{j-1}}).$$

In fact F does not have jump at $t (0 < t < 1)$. Hence,

$$P(X \leq t) = P(X < t) = \sum_{j=1}^{\infty} q^{j-s_{j-1}} p^{s_{j-1}} a_j.$$

However, if t is an finite α-binary termination then $t = \sum_{i=1}^{r} \alpha^{i-s_{i-1}} (1-\alpha)^{s_{i-1}} a_i$ for some finite number r and

$$\begin{aligned} P(X \leq t) &= P(X < t_r) + P(X = t_r) \\ &= P(X < t_r) \qquad \text{from (6)} \\ &= \sum_{j=1}^{\infty} a_j \left(q^{j-s_{j-1}} p^{s_{j-1}}\right) \qquad \text{since } a_j = 0 \text{ for } j = r+1, r+2, \cdots. \end{aligned}$$

□

Particular Case. If $p = \alpha = 1/2$, then

$$P(X < t) = \sum_{j=1}^{\infty} a_j (1/2)^j = t,\; t \in (0, 1).$$

Hence, $X \sim U(\,0,\,1)$.

Corollary 2.2. *If $p = 1-\alpha$, then $X \sim U\,(0,1)$. By substituting $p = 1-\alpha$ in (5), we have*

$$P(X \le t) = \sum_{i=1}^{\infty} \alpha^{i-s_{i-1}} (1-\alpha)^{s_{i-1}} a_i = t$$

and hence the corollary.

2.2. The characteristic function of X_k

The r.v. X_k defined in (2) takes 2^k distinct values $t_a = \sum_{i=1}^{k} \alpha^{i-s_{i-1}}\ (1-\alpha)^{s_{i-1}} a_i$, where $\underline{a} = (a_1, a_2, \ldots, a_k) \in A_k$, the set of all k-dimensional vectors of zeroes and ones. Let Z_i's be i.i.d. $B(1,p)$ r.v's and $\underline{Z} = (Z_1, Z_2, \ldots, Z_k) \in A_k$, then $P(\underline{Z} = \underline{a}) = p^{s_k}\,(1-p)^{k-s_k}$.

The characteristic function of X_k is given by

$$\begin{aligned}
\Phi_k(u) &= \sum_{\underline{a} \in A_k} p^{\sum\limits_{j=1}^{k} a_j} q^{k-\sum\limits_{j=1}^{k} a_j} e^{iut_a} \\
&= \sum_{\underline{a} \in A_k} p^{\sum\limits_{j=1}^{k} a_j} q^{k-\sum\limits_{j=1}^{k} a_j} e^{iu\sum\limits_{i=1}^{k} \alpha^{i-s_{i-1}}(1-\alpha)^{s_{i-1}} a_i}.
\end{aligned} \tag{7}$$

In particular, if $\alpha = 1/2$ then

$$\Phi_k(u) = \sum_{\underline{a} \in A_k} p^{\sum\limits_{j=1}^{k} a_j} q^{k-\sum\limits_{j=1}^{k} a_j} e^{iu\sum\limits_{i=1}^{k} (\frac{1}{2})^i a_i}.$$

We note that, by conditioning on Z_1 the c.f. of X_k can be also be written as

$$\begin{aligned}
\Phi_k(u) =& E\left(e^{iuX_k}\right) \\
=& p\, E(e^{iu(\alpha + (1-\alpha)X_{k-1})}) + q\, E(e^{iu(\alpha X_{k-1})}) \\
=& p\, e^{iu\alpha} E(e^{iu(1-\alpha)X_{k-1}}) + q\, E(e^{iu\alpha X_{k-1}}) \\
=& p\, e^{iu\alpha} E(e^{iu(1-\alpha)X_{k-1}}) + q\, E(e^{iu\alpha X_{k-1}}) \\
\Phi_k(u) =& p\, e^{iu\alpha} \Phi_{k-1}\left((1-\alpha)\,u\right) + q\, \Phi_{k-1}\left(\alpha u\right)
\end{aligned} \tag{8}$$

One can verify that $\Phi_k(u)$ given by (7) also satisfies the recurrence relation (8). As k tends to infinity, we get

$$\Phi(u) = p\, e^{iu\alpha} \Phi((1-\alpha)u) + q\, \Phi(\alpha u).$$

However for $\alpha = 1/2$ we have $\Phi(u) = \left(p\, e^{iu\alpha} + q\right) \Phi(u/2)$. Repeating this and replacing u by $2u$ each time, we get, for $n = 1, 2, \cdots$.

$$\Phi(u) = \Phi(u/2^n) \prod_{k=1}^{\infty} \left(1 - p + pe^{iu/2^k}\right),$$

which is exactly same relation obtained in (3) Bhati et al. (2011).

2.3. Recursive property

The d.f. $F(t)$ is such that $F(t) = 0$ for $t \leq 0$, $F(t) = 1$ for $t \geq 1$ and for $0 < t < 1$, it satisfies the following

$$F(t) = (1-p)F(t/\alpha) + pF\left((t-\alpha)\,/\,(1-\alpha)\right) \tag{9}$$

By conditioning on Z_1 from (3) we have

$$P(X \leq t) = P(X \leq t|Z_1 = 1) \cdot P(Z_1 = 1) + P(X \leq t|Z_1 = 0) \cdot P(Z_1 = 0). \tag{10}$$

Consider

$$\begin{aligned} P(X \leq t|Z_1 = 1) =& P\big(\alpha + \sum_{i=2}^{\infty} \alpha^{i-s_{i-1}} \,(1-\alpha)^{s_{i-1}}\, Z_i \leq t\big) \\ =& P(\alpha + \alpha \sum_{u=1}^{\infty} \alpha^{u-S_u} \,(1-\alpha)^{s_u}\, Z_{u+1} \leq t) \\ =& P(\alpha + \alpha(1-\alpha) \sum_{u=1}^{\infty} \alpha^{u-1-s'_{u-1}} \,(1-\alpha)^{s'_{u-1}}\, Z'_u \leq t) \\ & (\text{since}\, s_u = \sum_{i=1}^{u} Z_i = 1 + \sum_{i=2}^{u} Z_i = 1 + s'_{u-1}) \\ =& P(\alpha + (1-\alpha)\, Y \leq t), \end{aligned}$$

where $Y = \sum_{u=1}^{\infty} \alpha^{u-s'_{u-1}} \,(1-\alpha)^{s'_{u-1}}\, Z_u$, which is distributed as that of X. Hence,

$$P(\alpha + (1-\alpha)\, Y \leq t) = P(X \leq (t-\alpha)/(1-\alpha)) = F((t-\alpha)/(1-\alpha)). \tag{11}$$

Similarly, by conditioning on $Z_1 = 0$ one can show that

$$P(X < t|Z_1 = 0) = F(t/\alpha) \tag{12}$$

Now (9) follows from (10), (11) and (12).

2.4. Upper bound for $F_k(t) - F(t)$

If $p = 0$ we have $X_k = X = 0$ for all $k = 1, 2, \cdots$ and if $p = 1$ then $X_k = \sum_{i=1}^{k} \alpha(1-\alpha)^{i-1} = 1-(1-\alpha)^k$, $k = 1, 2, \cdots$ and $X = 1$. Hence $F_k(t) - F(t) = 1$ if $t \in [1-(1-\alpha)^k,\, 1)$ and 0 (otherwise).

Let $0 < p < 1$, We note that $\underline{t_k} = (a_1, a_2, \cdots, a_k, \underline{0}) \leq t \leq \overline{t_k} = (a_1, a_2, \cdots, a_k, \underline{1})$, where $\underline{0}\,(\underline{1})$ is the sequence of zeroes (ones) then $\underline{t_k}$ is non-decreasing and increases to t, while $\overline{t_k}$ is non-increasing, decreasing to t as k tends to infinity and $F\left(\underline{t_k}\right) \leq F\left(t\right) \leq F\left(\overline{t_k}\right)$

$$\begin{aligned} F_k(t) - F(t) \leq& F(\overline{t_k}) - F_k(\underline{t_k}) \\ =& \left\{\sum_{i=1}^{k} q^{i-s_{i-1}} p^{s_{i-1}} a_i + \sum_{i=k+1}^{\infty} q^{i-s_{i-1}} p^{s_{i-1}}\right\} - \left\{\sum_{i=1}^{k} q^{i-s_{i-1}} p^{s_{i-1}} a_i\right\} \\ =& \sum_{i=k+1}^{\infty} q^{i-s_{i-1}} p^{s_{i-1}} \\ =& q^{k+1-s_k} p^{s_k} \left(1 + p + p^2 + ...\right) \\ =& q^{k-s_k} p^{s_k} \leq \rho^k \\ \text{where,} \quad & \rho = \max(p, q) \in (0, 1). \end{aligned}$$

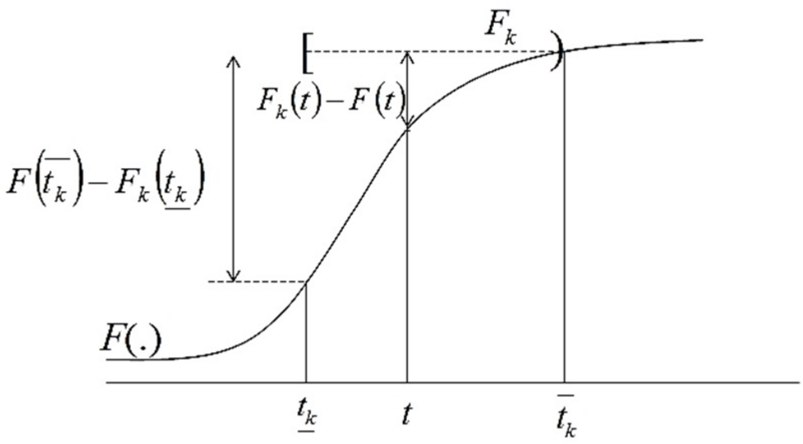

Figure 1: The graphs of $F_k(t)$ and $F(t)$

Theorem 2.3. *Let $u < v$ have α-binary representation in terms of $(a_1, a_2, \cdots, \ a_r, 0, 0, \cdots)$ and $(a_1, a_2, \cdots, a_r, 1, 1, 1, \cdots)$ respectively. Then the conditional distribution of X given $u \leq X \leq v$ is the same as that of $u + \alpha^{r-s_r}(1-\alpha)^{s_r}X$.*

Proof. Observe that the conditioning event implies that $Z_i = a_i, i = 1, 2, \cdots, r$. Further by (3), $X - u$ depends on $Z_i = a_i, i = r+1, r+2, \cdots$, which are independent of $Z_i = a_i$, $i = 1, 2, \cdots, r$. Hence the result. □

Remark 2.4. *From the description of the variables $X_k(\alpha, p) : 0 < \alpha < 1, < p < 1$ given above (2) it follows that $X_k(\alpha, p)$ is stochastically increasing in both α, p. This is also evident from the graphs given in the appendix.*

3. A generalization

In this section, we consider r.v's taking values, $0, 1, 2, \cdots, m-1$ and for $m = 2$ these will be Bernoulli random variables. Let $\{Z_n\}_1^\infty$ be a sequence of i.i.d. r.v's taking values, $0, 1, 2, \cdots, m-1$ with respective probabilities p_0, p_1, $\cdots, p_{m-1}$. Consider the formation of m equal length subintervals of an interval, (in general they may be in certain fixed proportions). We know that every number t, $0 \leq t < 1$ can be represented in terms of the sequence $\{a_n\}_1^\infty$, $a_n = 0, 1, 2, \cdots, m-1$ where

$$t = \sum_{i=1}^{\infty} (1/m)^i a_i. \tag{13}$$

This may be referred as *the m-ary* representation. For $m = 2(10)$ the above representation is referred as binary(decimal) representation of the number t.

Theorem 3.1. *Let $X = \sum_{i=1}^{\infty} (1/m)^i Z_i$. Then the d.f. of X is given by*

$$F(t) = \begin{cases} 0, & if \quad t < 0 \\ \sum_{r=1}^{\infty} p_0^{k_r - r} \left\{ \prod_{i=1}^{r-1} p_{a_{k_i}} \right\} \left\{ \prod_{j=0}^{a_{k_r}-1} p_j \right\}, & if \quad 0 \leq t < 1 \\ 1, & if \quad t \geq 1 \end{cases}$$

where for $0 < t < 1$, t has the m-ary representation (13) and a_{k_r} is the r^{th} non zero element in the sequence $\{a_n\}_{n=1}^{\infty}$.

Proof. Let t_r be the number having the *m-ary* representation $(a_1,\ a_2, \cdots, a_{k_r}\ \ , 0, 0, \cdots)$ with $t_0 = 0$, $t_r = \sum_{i=1}^{r} (1/m)^i a_i$. It is to be noted that $\{t_r\}$ is an non-decreasing sequence increasing to t. We then have

$$\begin{aligned} P(t_1 \le X < t_2) =& P(Z_1 = 0, \cdots, Z_{k_1-1} = 0, Z_{k_1} = a_{k_1}, Z_{k_1+1} = 0, \cdots, Z_{k_2-1} = 0, Z_{k_2} < a_{k_2}) \\ =& p_0^{k_2-2} p_{a_{k_1}} \left(\sum_{j=0}^{a_{k_2}-1} p_j \right) \\ P(t_2 \le X < t_3) =& P(Z_1 = \cdots = Z_{k_1-1} = 0, Z_{k_1} = a_{k_1}, Z_{k_1+1} = \cdots = Z_{k_2-1} = 0, \\ & Z_{k_2} = a_{k_2}, Z_{k_2+1} = \cdots = Z_{k_3-1} = 0, Z_{k_3} = a_{k_3}) \\ =& p_0^{k_3-3} p_{a_{k_1}} p_{a_{k_2}} \left(\sum_{j=0}^{a_{k_3}-1} p_j \right). \end{aligned}$$

In general, we have $P(t_{r-1} \le X < t_r) = p_0^{k_r - r} (\prod_{i=1}^{r-1} p_{a_{k_i}})(\sum_{j=0}^{a_{k_r}-1} p_j)$. Thus

$$Pr(X < t) = \sum_{r=1}^{\infty} \Pr(t_{r-1} \le X < t_r) = \sum_{r=1}^{\infty} p_0^{k_r - r} \left(\prod_{i=1}^{r-1} p_{a_{k_i}} \right) \left(\sum_{j=0}^{a_{k_r}-1} p_j \right). \tag{14}$$

However, $\Pr(X = t) = Pr(Z_i = a_i : i = 1, 2, 3, \cdots) = 0$

Hence we have from (14) ,

$$F(t) = \sum_{r=1}^{\infty} p_0^{k_r - r} \left(\prod_{i=1}^{r-1} p_{a_{k_i}} \right) \left(\sum_{j=0}^{a_{k_r}-1} p_j \right).$$

□

Particular Cases.
Case(i): Let $p_0 = p$, $p_j = p^{j+1}$, for $j = 0, 1, \cdots, m-1$, but as $\sum_{j=1}^{m} p^j = 1$, we have $p\left(\frac{1-p^m}{1-p}\right) = 1$. Then $\left(\prod_{i=1}^{r-1} p_{a_{k_i}}\right) = \left(\prod_{i=1}^{r-1} p^{a_{k_i}+1}\right) = p^{\sum_{i=1}^{r-1} a_{k_i} + r - 1}$ and $\left\{\sum_{j=0}^{a_{k_r}-1} p_j\right\} = \left\{\sum_{j=0}^{a_{k_r}-1} p^{j+1}\right\} = p\frac{(1-p^{a_{k_r}})}{(1-p)}$. Hence

$$P(X \le t) = \sum_{r=1}^{\infty} p^{k_r - r} p^{\sum_{i=1}^{r-1} a_{k_i} + r - 1} p \frac{(1 - p^{a_{k_r}})}{(1-p)} = \sum_{r=1}^{\infty} p^{k_r + \sum_{i=1}^{r-1} a_{k_i}} \left(\frac{1 - p^{a_{k_r}}}{1-p} \right).$$

Case(ii): If $p_i = 1/m$, $i = 1, 2, \cdots, m$. then

$$\begin{aligned} P\left(X \le t\right) =& \sum_{r=1}^{\infty} p_0^{k_r - r} \left(\prod_{i=1}^{r-1} p_{a_{k_i}} \right) \left(\sum_{j=0}^{a_{k_r}-1} p_j \right) \\ =& \sum_{r=1}^{\infty} p_0^{k_r - r} \left(\prod_{i=1}^{r-1} (1/m) \right) \left(\sum_{j=0}^{a_{k_r}-1} (1/m) \right) \\ P(X \le t) =& \sum_{r=1}^{\infty} (1/m)^{k_r - r + (r-1)} a_{k_r} (1/m) \\ =& \sum_{r=1}^{\infty} (1/m)^{k_r} a_{k_r} \\ =& \sum_{j=1}^{\infty} (1/m)^j a_j \qquad \text{(since } a_j = 0 \text{ for } j \neq k_r) \\ =& t \end{aligned}$$

Hence $X \sim U(0,1)$
Case(iii) : As a particular case, if we take $m = 2$, $p_0 = 1 - p = q$, $p_1 = p$ then it reduced to $F(t) = \sum_{i=1}^{\infty} q^{i-s_{i-1}} p^{s_{i-1}} a_i$ which is (5) of Bhati et al. (2011). Note that in this case $a_{k_r} = 1$, $r = 1, 2, \cdots$.

$$\begin{aligned} F(t) &= \sum_{r=1}^{\infty} q^{k_r - r} \left(\prod_{i=1}^{r-1} p \right) \left(\sum_{j=0}^{0} p_j \right). \\ &= \sum_{r=1}^{\infty} q^{k_r - r} p^{r-1} q = \sum_{r=1}^{\infty} q^{k_r - (r-1)} p^{r-1}. \end{aligned}$$

By setting $k_r = i$ then $s_i = r$, $s_{i-1} = r - 1$ and since $a_i = 0$ for $i \neq k_r$,we have $F(t) = \sum_{i=1}^{\infty} q^{i-s_{i-1}} p^{s_{i-1}} a_i$, for $0 < t < 1$.

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Appendix: Graphs of D.F. $F_{10}(t)$ for different values of α and p

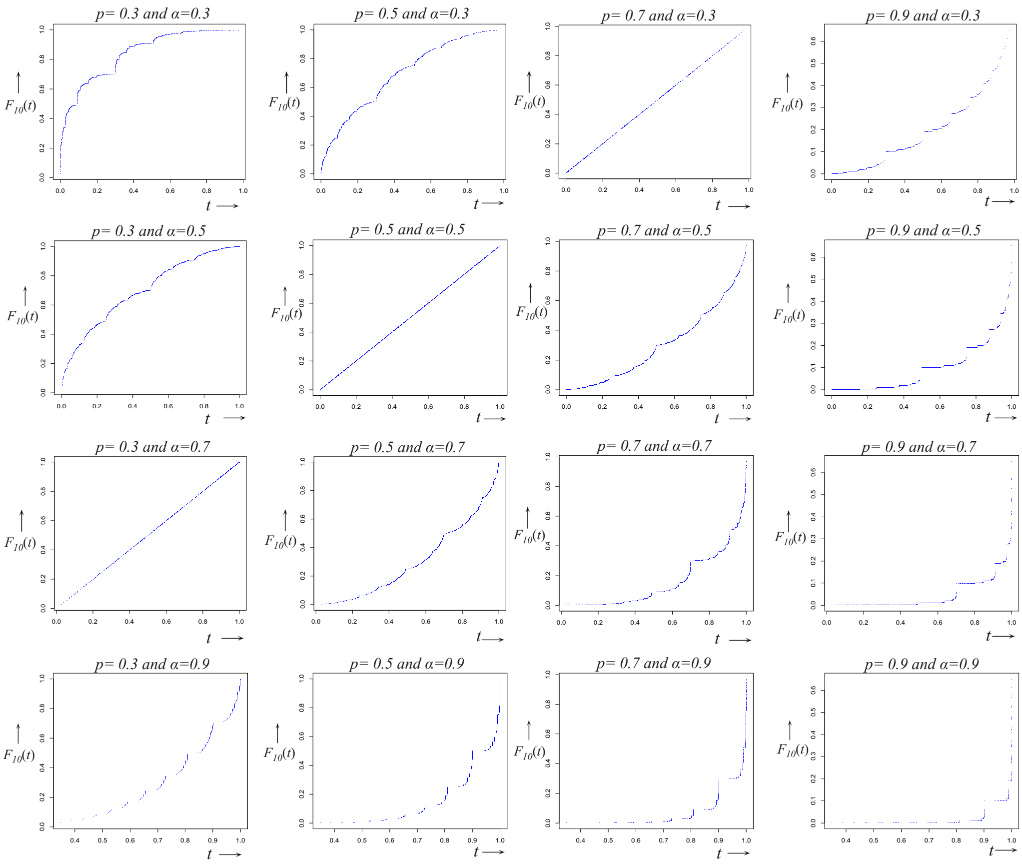

Figure 2: Based on the graphs one can observe (i) for t_0, $(0 < t_0 < 1)$ any fixed value, (a) the function $F_{10}(t_0)$ is decreasing in, for each fixed value of p, (b) $F_{10}(t_0)$ is decreasing in p for each fixed value of α. (ii) for $p = 1 - \alpha$ for $F_{10}(t) = t$ for $t = t_a$.