ProbStat Forum, Volume 07, April 2014, Pages 26–38
ISSN 0974-3235

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Limit Laws for Maxima of Functions of Independent Non-identically Distributed Random Variables

Sreehari Maddipatla

Visiting Professor, C R Rao AIMSCS, Gachibowli, Hyderabad, India.

Abstract. Suppose distribution functions $F_1, F_2, \cdots, F_k$ are in the max domains of attraction of max stable laws. Let F be a function of the dfs $F_1, F_2, \cdots, F_k$. We consider some distribution functions F and investigate conditions on $F_1, F_2, \cdots, F_k$ under which F belongs to max domain of attraction of max stable laws. Sums, mixtures and maxima of independent random variables are covered by the results proved.

AMS 2000 Subject Classifications 60G70, 60E05.

1. Introduction

Consider a collection of distribution functions (dfs) $\{F_1, F_2, \ldots, F_k\}$. Suppose each F_j is in the max domain of attraction (MDA) of a max stable law. Let F be a function of $F_1, F_2, \ldots, F_k$. We consider functions F that are dfs and find conditions on $F_j,\ j = 1, 2, \ldots, k$ so that F belongs to MDA of a max stable law. Some work of this nature has already been done; specifically when F is a convolution and when F is a mixture. This is reviewed here and we refer to Sreehari and Ravi (2010) and Sreehari *et al.* (2011) for details. The main aim of this paper is to consider the case when F is a product of dfs F_j. Resnick (1971 *a, b*) has earlier studied this problem. We briefly review Resnick's work and investigate the problem in power normalization set up.

To make the paper self contained we present definitions and some important results that we need concerning ℓ-max stable laws and p-max stable laws. It is known that if $M_n = \max(X_1, \ldots, X_n)$, where X_is are independent identically distributed (iid) random variables (rvs) with df F, and if

$$\lim_{n\to\infty} P\left(\frac{M_n - b_n}{a_n} \le x\right) = \lim_{n\to\infty} F^n(a_n.x + b_n) = G(x), x \in \mathcal{C}(G),$$

where $a_n > 0, b_n \in \mathbf{R}$ are norming constants, G is a non-degenerate df and $\mathcal{C}(G)$ is the set of all continuity points of G, then G has to be one of the three types of the well known extreme value distributions, namely, the Fréchet law: $\Phi_\alpha(x) = \begin{cases} 0, & x < 0, \\ \exp(-x^{-\alpha}), & 0 \le x; \end{cases}$

Keywords. ℓ-max stable laws, p-max stable laws, General max stable laws, Domain of attraction, Convolution, Mixtures and Products of distribution functions.
Received: 05 February 2014; Revised: 09 March 2014, Accepted: 12 March 2014
Email address: msreehari03@yahoo.co.uk (Sreehari Maddipatla)

the Weibull law: $\Psi_\alpha(x) = \begin{cases} \exp(-|x|^\alpha), & x < 0, \\ 1, & 0 \le x; \end{cases}$

the Gumbel law: $\Lambda(x) = \exp(-\exp(-x)), x \in \mathbf{R}$.

Here $\alpha > 0$ is a parameter.

These laws have been called ℓ-max stable dfs in Mohan and Ravi (1993), to emphasize that these are obtained as limit laws of linearly normalized partial maxima of iid rvs and F belongs to the ℓ-max domain of attraction (ℓ-MDA) of the limit law. Necessary and sufficient conditions for a df F to belong to ℓ-MDA of G, henceforth denoted by $\mathcal{D}_\ell(G)$, for each of the three types of ℓ-max stable dfs are well known and are given below.

Theorem A *Let the right extremity of F be given by $r(F) = \sup\{x : F(x) < 1\}$*

1. $F \in \mathcal{D}_\ell(\Phi_\alpha)$ for some $\alpha > 0$ iff $1 - F$ is regularly varying with exponent $-\alpha$, that is, iff $\lim_{t\to\infty} \frac{1-F(tx)}{1-F(t)} = x^{-\alpha}$, $x > 0$. In this case, one may take $a_n = F^-(1 - \frac{1}{n})$ and $b_n = 0$ so that $\lim_{n\to\infty} F^n(a_n.x + b_n) = \Phi_\alpha(x), x \in R$. Here $F^-(y) = \inf\{x : F(x) \ge y\}, y \in R$.

2. $F \in \mathcal{D}_\ell(\Psi_\alpha)$ for some $\alpha > 0$ iff $r(F) < \infty$ and $1 - F\left(r(F) - \frac{1}{.}\right)$ is regularly varying with exponent $-\alpha$, that is, iff $\lim_{t\to\infty} \frac{1-F\left(r(F)-\frac{1}{tx}\right)}{1-F\left(r(F)-\frac{1}{t}\right)} = x^{-\alpha}$, $x > 0$. In this case, one may take $a_n = r(F) - F^-(1 - \frac{1}{n})$ and $b_n = r(F)$ so that $\lim_{n\to\infty} F^n(a_n.x + b_n) = \Psi_\alpha(x), x \in R$.

3. $F \in \mathcal{D}_\ell(\Lambda)$ iff there exists a positive function f such that $\lim_{t\uparrow r(F)} \frac{1-F(t+f(t).x)}{1-F(t)} = \exp(-x)$, $x \ge 0$, $r(F) \le \infty$. If this condition holds for some f, then $\int_a^{r(F)} (1 - F(s))\, ds < \infty$, $a < r(F)$, and the condition holds with the choice $f(t) = \frac{\int_t^{r(F)}(1-F(s))ds}{(1-F(t))}$. In this case, one may take $a_n = f(b_n)$ and $b_n = F^-\left(1 - \frac{1}{n}\right)$ so that $\lim_{n\to\infty} F^n(a_n.x + b_n) = \Lambda(x), x \in R$. One may also take $a_n = F^-\left(1 - \frac{1}{ne}\right) - b_n$, $b_n = F^-\left(1 - \frac{1}{n}\right)$. Also, $f(.)$ may be taken as the mean residual life time of a random variable X given $X > t$ where X has df F.

Following Pancheva (1985), Mohan and Ravi (1993) studied the limit laws of power normalized partial maxima of iid rvs and their max domains. It is known, (see, for example, Mohan and Ravi, 1993) that if, for $x \in \mathcal{C}(H)$,

$$\lim_{n\to\infty} P\left(\left(\frac{|M_n|}{\alpha_n}\right)^{1/\beta_n} sign(M_n) \le x\right) = \lim_{n\to\infty} F^n\left(\alpha_n |x|^{\beta_n} sign(x)\right) = H(x),$$

where $\alpha_n > 0, \beta_n > 0$ are norming constants, and H is a non-degenerate df, then H has to be a p-type of one of the six types of p-max stable laws, namely,

$$H_{1,\alpha}(x) = \begin{cases} 0, & x < 1, \\ \exp(-(\log x)^{-\alpha}), & 1 \le x; \end{cases}$$

$$H_{2,\alpha}(x) = \begin{cases} 0, & x < 0, \\ \exp(-|\log x|^{\alpha}), & 0 \le x < 1, \\ 1, & 1 \le x; \end{cases}$$

$$H_{3,\alpha}(x) = \begin{cases} 0, & x < -1, \\ \exp(-|\log|x||^{-\alpha}), & -1 \le x < 0, \\ 1, & 0 \le x; \end{cases}$$

$$H_{4,\alpha}(x) = \begin{cases} \exp(-\log|x|)^{\alpha}, & x < -1, \\ 1, & -1 \le x; \end{cases}$$

$\Phi(x) = \Phi_1(x), x \in \mathbf{R}$; and $\Psi(x) = \Psi_1(x), x \in \mathbf{R}$.

These have been called p-max stable dfs in Mohan and Ravi (1993), to emphasize that these are obtained as limit laws of power normalized partial maxima of iid rvs and F belongs to the p-MDA of the limit law. Necessary and sufficient conditions for a df F to belong to p-MDA of H, henceforth denoted as $\mathcal{D}_p(H)$ for each of the six p-max stable dfs are given in Mohan and Ravi (1993). Max domains under linear and power normalizations have been compared in Mohan and Ravi (1993).(See Theorem B below). It is shown that if a df F belongs to some max domain under linear normalization then it necessarily belongs to the p-MDA of a p-max stable law and that the converse is not true. So p-max stable laws attract more dfs than ℓ-max stable laws.

Theorem B *Let F be a df. Then*

(a) (i) $F \in \mathcal{D}_\ell(\Phi_\alpha) \Rightarrow F \in \mathcal{D}_p(\Phi)$, (ii) $F \in \mathcal{D}_\ell(\Lambda), r(F) = \infty \Rightarrow F \in \mathcal{D}_p(\Phi)$;

(b) $F \in \mathcal{D}_\ell(\Lambda), 0 < r(F) < \infty \Leftrightarrow F \in \mathcal{D}_p(\Phi), r(F) < \infty$;

(c) $F \in \mathcal{D}_\ell(\Lambda), r(F) < 0 \Leftrightarrow F \in \mathcal{D}_p(\Psi), r(F) < 0$;

(d) (i) $F \in \mathcal{D}_\ell(\Lambda), r(F) = 0 \Rightarrow F \in \mathcal{D}_p(\Psi)$, (ii) $F \in \mathcal{D}_\ell(\Psi_\alpha), r(F) = 0 \Rightarrow F \in \mathcal{D}_p(\Psi)$

(e) $F \in \mathcal{D}_\ell(\Psi_\alpha), r(F) > 0 \Leftrightarrow F \in \mathcal{D}_p(H_{2,\alpha})$

(f) $F \in \mathcal{D}_\ell(\Psi_\alpha), r(F) < 0 \Leftrightarrow F \in \mathcal{D}_p(H_{4,\alpha})$.

The theory concerning max stable laws can be presented in greater generality than above.

Definition 1: A df H is said to be max stable if for every positive integer n there exists a strictly monotone continuous transformation $f_n(x)$ such that

$$H^n\left(f_n(x)\right) = H(x) \ \ \forall x \in R.$$

Pancheva (1985) proved that the class of max stable laws is given by the two parameter family $H(x) = \exp\left(-\exp\left(-c.h(x)\right)\right)$, where $0 < c \in R$ and h is a strictly increasing invertible continuous function in $S(H)$, the support of H. A df H is max stable iff H^r is max stable for all $r > 0$. This class of distributions, called general max stable laws, naturally contains both the well known extreme value distributions (Fréchet, Weibull and Gumbel types) as well as the six p-max stable laws derived by Pancheva (1985). The transformation $f_n(.)$ is given by $h^{-1}\left(h(.) + \log n\right)$. Pancheva proved that the class of possible limit distributions of normalized maxima of iid rvs coincides with the class of general max stable laws.

Definition 2: A df F is said to belong to the MDA of general max stable law H if there exists a sequence of strictly monotone continuous transformations $\{g_n(.)\}$ such that

$$F^n\left(g_n(x)\right) \xrightarrow{w} H(x),$$

where $g_n(.)$ is such that $g_\lambda(x) = \lim_{n\to\infty} g_{m_n}^{-1}\left(g_n(x)\right)$, with $m_n < n$, $\frac{m_n}{n} \to \lambda$, and $g_n(.)$ considered as a function of λ is solvable, i.e., $g_\lambda(x) = t$ has a unique solution $\lambda = \bar{g}(x,t)$. Here g_λ has to be continuous and strictly increasing in x. Sreehari (2009) proved a necessary and sufficient condition for a given df F to belong to the MDA of H.

Section 2 contains a few lemmas needed for the analysis. In Sections 3 and 4 we briefly discuss the problems concerning convolutions and mixtures respectively. In section 5 we consider products of dfs.

2. Preliminary Lemmas

Here we give a few lemmas which will be used later.

Lemma 2.1 For independent rvs $X, Y,$ and $t > 0$,

$$P(X+Y>t) \geq P(X>t(1+\epsilon))P(|Y|<t\epsilon) + P(Y>t(1+\epsilon))P(|X|<t\epsilon),\ \epsilon>0,$$

and

$$P(X+Y\geq t) \leq P(X>t(1-\epsilon)) + P(Y>t(1-\epsilon)) + P(X>t\epsilon)P(Y>t\epsilon),\ 0<\epsilon<\tfrac{1}{2}.$$

Lemma 2.2 For independent nonnegative rvs X, Y and $t > 0$,

$$P(X+Y\leq t) \leq P(X\leq t(1+\epsilon))P(Y\leq t(1+\epsilon)),\ \epsilon>0.$$

Lemma 2.3 For independent nonnegative rvs X, Y and $t > 0$,

$$P(X+Y\leq t) \geq P(X\leq t(1-\epsilon))P(Y\leq t(1-\epsilon)) - P(t\epsilon<X<t(1-\epsilon), t\epsilon<Y<t(1-\epsilon)),\ 0<\epsilon<\tfrac{1}{2}.$$

Lemma 2.4 For independent rvs $X \sim F_1$ and $Y \sim F_2$, $r(F_1 * F_2) = r(F_1) + r(F_2)$, and $r(F_1F_2) = \max\{r(F_1), r(F_2)\}$.

Lemma 2.5 (Lemma 4.4.2 in Samoridnitsky and Taqqu, 1994) Suppose X is a rv with a regularly varying tail, i.e., there is a real number $\theta > 0$ such that for all $x > 0$, $\lim_{t\to\infty} \frac{P(X>tx)}{P(X>t)} = x^{-\theta}$. Suppose also that the tail of X dominates the tail of a positive rv Y in the sense that $\lim_{t\to\infty} \frac{P(Y>t)}{P(X>t)} = 0$. Then $\lim_{t\to\infty} \frac{P(X+Y>t)}{P(X>t)} = \lim_{t\to\infty} \frac{P(X-Y>t)}{P(X>t)} = 1$.

3. Results for convolutions

In this section we mainly discuss answers to the following questions.

For independent rvs X and Y,

1. if $X \sim F_1 \in \mathcal{D}_\ell(H_1), Y \sim F_2 \in \mathcal{D}_\ell(H_2)$ for some H_1 and H_2, under what conditions the convolution $X+Y \sim F_1 * F_2 \in \mathcal{D}_\ell(H)$ and what dfs H are possible?

2. if $X+Y \sim F_1 * F_2 \in \mathcal{D}_\ell(H)$ for some df H, then what can we say about the max domains to which the individual dfs F_1 and F_2 may belong?

Consider a service center in which the service has two phases which work one after another for any customer. Suppose the information is available on the maximum service times of the two phases on each day for a fixed period. In order to study the maximum time a customer is likely to spend in the center for service, one needs the information about the behaviour of the total (unobserved) time for each customer. Conversely, one may have information on the maximum time any customer might have spent on a day in the service center but to study the efficiency of each of the phases, one needs information on maximum service time a phase takes for a customer, which may not be available. Thus it is of interest to know the behaviour of $X+Y(\sim F_1 * F_2)$ given information about $X(\sim F_1)$ and $Y(\sim F_2)$ as also the behaviour of X and Y given the information about $X+Y$.

Results similar to those above have been obtained for stable laws in Sreehari (1970), Tucker (1968), etc. There has been some work of this type in some special cases. It is known (see problem 25 in Chapter 2 of Galambos, 1978) that if F_1 is the Gamma distribution with parameters n and α (and hence $F_1 \in \mathcal{D}_l(\Lambda)$) and the df F_2 is absolutely continuous with $r(F_2) = \infty$ and $\int_{-\infty}^{\infty} \exp(\alpha y) y^{n-1} dF_2(y) < \infty$, then $F_2 \in \mathcal{D}_l(\Lambda)$, and $F_1 * F_2 \in \mathcal{D}_l(\Lambda)$. Also, if F_1 is the standard lognormal distribution and F_2 is the standard normal distribution then $F_i \in \mathcal{D}_l(\Lambda)$, $i = 1, 2$, and $F_1 * F_2 \in \mathcal{D}_l(\Lambda)$.

Theorem 3.1 (Sreehari *et al.*, 2011) *For independent rvs X and Y, if* $X \sim F_1 \in D_\ell(\Phi_\alpha), Y \sim F_2 \in D_\ell(\Phi_\beta), 0 < \alpha \leq \beta$ *, then* $X + Y \sim F_1 * F_2 \in D_\ell(\Phi_\alpha)$*.*

Remark. In view of Lemma 2.5, if $X \in \mathcal{D}_\ell(\Phi_\alpha)$ for some $\alpha > 0$, and the tail of X dominates the tail of a positive rv Y, then $X + Y \in \mathcal{D}_\ell(\Phi_\alpha)$. Note that Y need not belong to any ℓ-max domain or Y may belong to $\mathcal{D}_\ell(\Psi_\alpha)$ or $\mathcal{D}_\ell(\Lambda)$. Note that if X belongs to $\mathcal{D}_\ell(\Phi_\alpha)$ and the right extremity of the df of Y is finite, then the tail of X dominates that of Y and hence by Lemma 2.5, $X + Y$ belongs to $\mathcal{D}_\ell(\Phi_\alpha)$ irrespective of Y belonging to any other max domain.

To address the second question above we give an example to show that $F_1 * F_2 \in \mathcal{D}_\ell(\Phi_\alpha)$ but both F_1, F_2 do not belong to $\mathcal{D}_\ell(\Phi_\alpha)$. We then give a sufficient condition for the assertion to hold. Consider independent rvs X and Y with

$$X \sim F_1(x) = \begin{cases} 0 & \text{if} \quad x \leq 1, \\ 1 - \frac{1}{x^{\sqrt{2}}}\left(1 + \frac{1}{12}\sin(\log x)\right) & \text{if} \quad 1 < x, \end{cases}$$

$$Y \sim F_2(x) = \begin{cases} 0 & \text{if} \quad x \leq 1, \\ 1 - \frac{1}{x^{\sqrt{2}}}\left(1 - \frac{1}{12}\sin(\log x)\right) & \text{if} \quad 1 < x, \end{cases}$$

Note that $F_i \notin \mathcal{D}_\ell(\Phi_\alpha)$, since $\overline{F_i}$ is not regularly varying, $i = 1, 2$, However, $G = F_1 * F_2 \in D_{\sqrt{2}}(\Phi_\alpha)$. (For details we refer to Sreehari *et al.*, 2011.)

We now give sufficient conditions for a positive answer to question 2 above.

Theorem 3.2 (Sreehari *et al.*, 2011) *For independent rvs* $X \sim F_1$ *and* $Y \sim F_2$ *with* $r(F_1) = r(F_2) = \infty$, *suppose that*

$$\lim_{x\to\infty} \frac{1 - F_2(x)}{1 - F_1(x)} = A, \ 0 < A < \infty,$$

and

$$\limsup_{x\to\infty} \frac{1 - F_1(x\theta)}{1 - F_1(x)} < \infty, \ \text{for all} \ 0 < \theta < 1.$$

Then,

$$X + Y \sim F_1 * F_2 \in \mathcal{D}_\ell(\Phi_\alpha) \text{ implies } F_i \in \mathcal{D}_\ell(\Phi_\alpha), \ i = 1, 2.$$

Proof is based on Lemma 2.1.

Remarks

1. If $F_1 * F_2 \in \mathcal{D}_\ell(\Phi_\alpha)$ then $r(F_1 * F_2) = \infty$. So, either only $r(F_1) = \infty$ or only $r(F_2) = \infty$ or both $r(F_1) = \infty$ and $r(F_2) = \infty$. If only $r(F_1) = \infty$ then $A = 0$ in Theorem 3.2. In this case $F_1 \in \mathcal{D}_\ell(\Phi_\alpha)$ and there is no need to assume the second condition in Theorem 3.2. Also, $F_2 \notin \mathcal{D}_\ell(\Phi_\delta)$ for any $\delta > 0$. However, F_2 may belong to $\mathcal{D}_\ell(\Psi_\delta)$ for some $\delta > 0$ or F_2 may belong to $\mathcal{D}_\ell(\Lambda)$, but satisfying the condition concerning A. But, if in addition, $F_2 \in \mathcal{D}_\ell(G)$ for some G, then G may be Φ_β with $\beta > \alpha$, in view of Theorem 3.1.
2. If $r(F_1) < \infty$ but $r(F_2) = \infty$ then $F_2 \in \mathcal{D}_\ell(\Phi_\alpha)$.

Next we consider the ℓ-MDA of Weibull laws. If $r(F_1) > 0, r(F_2) > 0$, then note that $F_1 * F_2 = G \in \mathcal{D}_l(\Psi_\alpha)$ iff $1 - G^*(x) = 1 - G\left(r(F_1) + r(F_2) - \frac{1}{x}\right)$ is regularly varying at ∞ with exponent $(-\alpha)$.

Theorem 3.3 (Sreehari *et al.*, 2011) *If* $r(F_1) > 0, r(F_2) > 0$, *then note that* $F_1 * F_2 = G \in \mathcal{D}_l(\Psi_\alpha)$ *iff* $1 - G^*(x) = 1 - G\left(r(F_1) + r(F_2) - \frac{1}{x}\right)$ *is regularly varying at* ∞ *with exponent* $(-\alpha)$.

The proof is based on Lemmas 2.2, 2.3 and Theorem A in introduction.

Our final result of this section is for the case $\mathcal{D}_\ell(\Lambda)$. The result in Galambos (1978) mentioned earlier pertains to $r(F_2) = \infty$. The following result is for the case when $r(F_2) < \infty$.

Theorem 3.4 (Sreehari *et al.*, 2011) *Let* X *be a Gamma rv with probability density function (pdf)* $f_1(x) = f_1(x; p, \theta) = \frac{\theta^p}{\Gamma(p)} e^{-\theta x} x^{p-1}, x > 0$, *where* $p > 0$, *an integer, and* $\theta > 0$ *are parameters. Let* Y *be a rv with support in* $(0, a]$, $a < \infty$, *and with pdf* f_2, *and let* X *and* Y *be independent. Suppose that* $Z = X + Y$. *Then the df* G *of* Z *belongs to the* ℓ*-MDA of the Gumbel law* Λ.

Remarks

1. In the above theorem, if
   (a) Y has uniform df over $(0, 1)$, then Y belongs to the ℓ- MDA of the Weibull law, and $Z = X + Y$ belongs to the ℓ- MDA of the Gumbel law.

   (b) Y has df F_2 with $1 - F_2(x) = \exp\left(-\frac{x}{1-x}\right), 0 < x < 1$, one can verify that $F_2 \in \mathcal{D}_\ell(\Lambda)$ using the von-Mises sufficient conditions, and $Z = X + Y$ belongs to the ℓ- MDA of the Gumbel law.
2. From the previous two remarks, we conclude that if X and Y are independent rvs with the df of $X + Y$ belonging to the ℓ-MDA of Λ, then it is not true that the dfs of both X and Y should belong to the ℓ- MDA of Λ.

The above results also hold for convolutions of dfs $F_1, F_2, \ldots, F_k$, k a fixed positive integer. Similar results for p-max stable laws are discussed by Sreehari *et al.*, (2011).

4. Results for mixtures

Suppose that $F_1, F_2, \ldots, F_k$ are dfs. Set $F = p_1 F_1 + \cdots + p_k F_k$, where $p_i > 0$, $p_1 + \cdots + p_k = 1$. Then F is a df and we denote the left extremity of a df F by $l(F) = \inf\{x : F(x) > 0\}$. In this section we discuss the following questions.

1. If F_j is in the MDA of a general max stable law for each j, $1 \leq j \leq k$, is F in the MDA of some general max stable law H, and if yes, what is the structure of H?

2. If the mixture F is in the MDA of a general max stable law H, what can be said about F_j, $1 \leq j \leq k$?

These problems are of interest in reliability and statistical analysis concerning mixed populations. Kale and Sebastian (1995) discussed the limit behaviour of the maximum of sample observations from the mixture distribution $G = \alpha F_1 + (1 - \alpha) F_2$, where F_1 is in the ℓ- MDA of an extreme value distribution of Gumbel type or Frećhet type and the support of F_2 is $(-\delta, \delta)$. They were investigating non-normal symmetric distributions with kurtosis 3. AL-Hussaini and El-Adll (2004) also investigated the problems cited above and their results are somewhat ambiguous and partly wrong. We give an affirmative answer to the first question above under some assumptions while the second question has a negative answer. We give some interesting examples in this connection.

Denote support of a df F by $S(F)$.

Theorem 4.1 (Sreehari and Ravi, 2010) Let $F_1, \ldots, F_k$ be dfs such that

$$F_j^n\left(g_n(x)\right) \xrightarrow{w} H_j(x),\ 1 \le j \le k,$$

where $g_n(x)$ is a strictly monotone continuous function for each n. Let $p_i > 0,\ 1 \le i \le k$, and $\sum_{i=1}^{k} p_i = 1$. Set $F(x) = p_1.F_1(x) + \cdots + p_k.F_k(x)$. Let $S(H_i) \cap S(H_j) \neq \phi$ for $1 \le i < j \le k$. Then as $n \to \infty$,

$$F^n(g_n(x)) \xrightarrow{w} H(x) = \begin{cases} 0 & \text{if} \quad x \le \max_{1\le i\le k} l(H_i) \\ \prod_{i=1}^{k} H_i^{p_i}(x) & \text{if} \quad x > \max_{1\le i\le k} l(H_i). \end{cases}$$

Remarks

1. The above result is essentially the sufficiency part of Theorem 1 in AL-Hussaini and El-Adll (2004). They also claimed the converse of the above result to be true. The following examples demonstrate that the converse of Theorem 3.2 is false in linear normalization setup and nonlinear normalization setup.
2. In case $r(H_i) < r(H_j)$ for some pair (i, j) then the corresponding $H_i^{p_i}$ will become unity in the product term in H.

Next we give two examples that demonstrate that the converse to theorem 4.1 is false in the ℓ-max and non-linear normalization setup.

Example 1. Let

$$F_1(x) = \begin{cases} 0 & \text{if} \quad x < 1, \\ 1 - x^{-\alpha}.\left(1 + \frac{1}{c}\sin(\log x)\right) & \text{if} \quad 1 \le x, \end{cases}$$

and

$$F_2(x) = \begin{cases} 0 & \text{if} \quad x < 1, \\ 1 - x^{-\alpha}.\left(1 - \frac{1}{c}\sin(\log x)\right) & \text{if} \quad 1 \le x, \end{cases},$$

where $c > 1 + \frac{1}{\alpha}$. Let

$$F(x) = \frac{1}{2}.F_1(x) + \frac{1}{2}.F_2(x) = \begin{cases} 0 & \text{if} \quad x < 1, \\ 1 - x^{-\alpha} & \text{if} \quad 1 \le x. \end{cases}$$

Then F_1 and F_2 do not belong to the max domain of attraction of any max stable law under linear norming but

$$F^n(n^{\frac{1}{\alpha}}.x) \xrightarrow{w} \Phi_\alpha(x) = \begin{cases} 0 & \text{if} \quad x < 0, \\ \exp-(x^{-\alpha}) & \text{if} \quad 0 \le x. \end{cases}$$

Example 2. Suppose $p_1 = p_2 = \frac{1}{2}$. Let

$$F_j(x) = \begin{cases} 0 & \text{if} \quad x < 0, \\ 1 - \frac{1}{(1+x)^j} & \text{if} \quad 0 \le x,\ j = 1, 2. \end{cases}$$

Then $F(x) = \frac{F_1(x)+F_2(x)}{2}$. Set $g_n(x) = \frac{nx^2}{1+x}, x > 0$. Then $F^n\left(g_n(x)\right) \xrightarrow{w} \prod_{j=1}^{2} H_j^{\frac{1}{2}}(x)$, where the max stable law H_j is given by

$$H_j(x) = \begin{cases} 0 & \text{if} \quad x < 0, \\ \exp\left(-x^{-j}\right) & \text{if} \quad 0 \le x. \end{cases}$$

However, $F_j^n\left(g_n(x)\right)$ does not converge weakly to $H_j(x),\ j = 1, 2$.

Remarks

1. In the above example, $H(x) = (H_1(x).H_2(x))^{\frac{1}{2}}$ is a general max stable law with the transformation

$$f_n(x) = \frac{nx^2 + \sqrt{n^2x^4 + 4nx^2(1+x)}}{2(1+x)}$$

in definition 1.

2. Theorem 4.1 goes through even when $F(x) = \sum_{i=1}^{\infty} p_i.F_i(x)$, $p_i > 0$, $\sum_{i=1}^{\infty} p_i = 1$ under the additional condition on H_is that $\sum_{i=1}^{\infty}(1 - H_i^{p_i}(x)) < \infty$ for $x > \sup_{1\leq i<\infty} l(H_i)$. (See, for example, Lemma 6.1, Karlin and Taylor, 1975).

5. Results for products of dfs

Consider the product $G = F_1F_2$ of dfs F_1 and F_2. Then G is a df and $1-G = (1-F_1)+(1-F_2)-(1-F_1)(1-F_2)$. Since $(1-F_1(x))(1-F_2(x)) = o(\min\{1-F_1(x), 1-F_2(x)\})$ the behavior of $1-F_1(x)F_2(x)$ depends only on the behavior of $1-F_1(x)$ and $1-F_2(x)$. If

$$\lim_{x\to\infty} \frac{1-F_2(x)}{1-F_1(x)} = A \tag{1}$$

exists and is finite, $1-G$ is regularly varying at ∞ with exponent (-α) iff $1-F_1$ is regularly varying at ∞ with exponent(-α). If the limit in (1) is ∞ then $1-G$ is regularly varying at ∞ with exponent (-α) iff $1-F_2$ is regularly varying at ∞ with exponent(-α). In particular if $F_1 = F_2 = F$, $1-G$ is regularly varying at ∞ with exponent (-α) iff $1-F_1$ is regularly varying at ∞ with exponent (-α). If $1-F_1$ is regularly varying at ∞ with exponent (-α) and $1-F_2$ is regularly varying at ∞ with exponent (-β) , $0 < \alpha < \beta$ the limit in (1) is 0. We then have

Theorem 5.1 *For independent rvs X and Y, if $X \sim F_1 \in \mathcal{D}_\ell(\Phi_\alpha)$ and $Y \sim F_2 \in \mathcal{D}_\ell(\Phi_\beta)$, $0 < \alpha < \beta$, then $\max(X,Y) \sim G = F_1F_2 \in \mathcal{D}_\ell(\Phi_\alpha)$. Further, for a fixed positive integer k and iid rvs $X_1, X_2, \ldots, X_k$ with df F, $\max(X_1, X_2, \cdots, X_k) \sim F^k \in \mathcal{D}_\ell(\Phi_\alpha)$ iff $F \in \mathcal{D}_\ell(\Phi_\alpha)$.*

Remarks

1. In general if $\alpha = \beta$

$$\lim_{x\to\infty} \frac{1-F_2(x)}{1-F_1(x)}$$

may not exist as one can construct functions $1-F_1$ and $1-F_2$ both regularly varying at ∞ but the above ratio oscillating (see, Sreehari, 1973-74).

2. Theorem 5.1 and similar results for ℓ-max setup were discussed in Resnick (1971 *a, b*). The proofs depend on the tail equivalence property and Khinchine's convergence of types theorem (see lemma 1, P. 246, Feller, 1966).

3. Interestingly Resnick arrived at this problem while investigating the limit distributions of $\{X_n\}$, when the rvs are defined on a finite Markov chain. Products of dfs come up in a natural way in survival analysis. For example, consider a system with two non-identical components with life-time dfs F_1 and F_2 working simultaneously. Suppose the system works as long as at least one of the components works and working or non-working status of a component has no impact on the life-time of the other component. Further suppose administrative policy for the system is to immediately replace both the components when the system becomes non-functional. Then the duration between two successive non-functional situations of the system

follows the df F_1F_2.

4. Resnick (1971 *b*) discussed, with suitable examples, the possibilities (a) of $F_1 \in \mathcal{D}_\ell(\Lambda)$ and $F_2 \in \mathcal{D}_\ell(\Lambda)$ but $F_1F_2 \notin \mathcal{D}_\ell(\Lambda)$, and (b) of $F_1F_2 \in \mathcal{D}_\ell(\Lambda)$ but neither $F_1 \in \mathcal{D}_\ell(\Lambda)$ nor $F_2 \in \mathcal{D}_\ell(\Lambda)$.

The main result of Resnick is

Theorem 5.2 *Let F_1, F_2 be dfs and let φ be an extreme value distribution. Suppose $F_1 \in \mathcal{D}_\ell(\varphi)$ and that $F_1^n(a_nx+b_n) \stackrel{w}{\to} \varphi(x)$ for normalizing constants $a_n > 0, b_n$ real. Then $F_2^n(a_nx+b_n) \stackrel{w}{\to} \varphi^*(x)$, non-degenerate, iff for some $A > 0, B$ real*

$$\varphi^*(x) = \varphi(Ax + B), r(F_1) = r(F_2) = x_0,$$

$\lim_{x\to x_0-} \frac{1-F_1(x)}{1-F_2(x)}$ *exists, and if*
(a) $\varphi(x) = \Phi_\alpha(x)$, then $B = 0$ and $\lim_{x\to\infty} \frac{1-F_1(x)}{1-F_2(x)} = A^\alpha$;
(b) $\varphi(x) = \Psi_\alpha(x)$, then $B = 0$ and $\lim_{x\to x_0-} \frac{1-F_1(x)}{1-F_2(x)} = A^{-\alpha}$;
(c) $\varphi(x) = \Lambda(x)$, then $A = 1$ and $\lim_{x\to x_0-} \frac{1-F_1(x)}{1-F_2(x)} = e^B$.

Next we discuss similar results in the p-max stable setup. Before we proceed further we note that in the above result of Resnick one is checking if $F_2^n(a_nx + b_n) \stackrel{w}{\to} \varphi^*(x)$ where $\varphi^*(x)$ is of same type as φ and the norming constants are same as those in $F_1^n(a_nx + b_n) \stackrel{w}{\to} \varphi(x)$. Hence it may seem natural to look for such a possibility in the p-max setup.

If $F_1(x) = 0$ for $x < 1$ and $= 1 - \exp(-\log^2 x)$ for $x \geq 1$, then

$$F_1^n\left(e^{\sqrt{\log n}}|x|^{\frac{1}{2\sqrt{\log n}}}\right) \stackrel{w}{\to} \Phi(x),$$

and if $F_2(x) = 0$ for $x < 1$ and $= 1 - \frac{1}{x}$ for $x \geq 1$,

$$F_2^n(nx) \stackrel{w}{\to} \Phi(x).$$

However, $F_2^n\left(e^{\sqrt{\log n}}|x|^{\frac{1}{2\sqrt{\log n}}}\right) \to 0$ while $F_1^n(nx)$ converges weakly to the df which is degenerate at 1.

We recall the Remark 2 after Theorem 5.1. In the context of power normalization we need a convergence of types theorem (See Lemma 5.1 below). We recall that Mohan and Ravi defined two dfs F_1 and F_2 to be same p -type if there exist positive numbers A, B such that $F_1(x) = F_2\left\{A|x|^B sign(x)\right\}$.

Lemma 5.1 *Let U and V be two non-degenerate dfs satisfying $V(x) = U(A|x|^B sign(x)),\ A > 0, B > 0$. If for a sequence $\{F_n\}$ of dfs and constants $a_n > 0,\ \alpha_n > 0,\ b_n > 0,$ and $\beta_n > 0$*

$$F_n\left(a_n|x|^{b_n} sign(x)\right) \to U(x), \quad F_n\left(\alpha_n|x|^{\beta_n} sign(x)\right) \to V(x) \tag{2}$$

at all continuity points of U and V, then

$$\left(\frac{\alpha_n}{a_n}\right)^{\frac{1}{b_n}} \to A > 0, \quad \frac{\beta_n}{b_n} \to B > 0 \tag{3}$$

Conversely, if (3) *holds then each of the two limiting relations in* (2) *holds and implies the other with $V(x) = U(A|x|^B sign(x))$.*

Proof Suppose the two limiting relations in (2) hold with $V(x) = U(A|x|^B sign(x))$. Suppose $S(U) \cap S(V) \cap (0, \infty) \neq \phi$. Let $x', x'' \in C(U)$ such that $0 < x' < x'' < \infty$. Then there exist $0 < y' < y'' < \infty$ such that $V(y') < U(x')$ and $V(y'') > U(x'')$. Then for sufficiently large n

$$0 < \alpha_n \, (y')^{\beta_n} < a_n \, (x')^{b_n} < a_n \, (x'')^{b_n} < \alpha_n \, (y'')^{\beta_n} < \infty,$$

which in turn gives the following relations :

$$(a) \quad \frac{x''}{(y'')^{\frac{\beta_n}{b_n}}} < \left(\frac{\alpha_n}{a_n}\right)^{\frac{1}{b_n}} < \frac{x'}{(y')^{\frac{\beta_n}{b_n}}},$$

and

$$(b) \quad \frac{\beta_n}{b_n} > \frac{\log x'' - \log x'}{\log y'' - \log y'} = \theta_1 > 0,$$

say.

Next starting with $0 < v' < v'' \in C(V)$ we can find $0 < u' < u'' < \infty$ such that for sufficiently large n

$$0 < a_n \, (u')^{b_n} < \alpha_n \, (v')^{\beta_n} < \alpha_n \, (v'')^{\beta_n} < a_n \, (u'')^{b_n} < \infty$$

which gives the inequality

$$(c) \quad \frac{\beta_n}{b_n} < \frac{\log u'' - \log u'}{\log v'' - \log v'} = \theta_2$$

say. From (a), (b) and (c) we observe that $\left\{\frac{\beta_n}{b_n}\right\}$ and $\left\{\left(\frac{\alpha_n}{a_n}\right)^{\frac{1}{b_n}}\right\}$ are bounded sequences satisfying

$$\theta_1 < \frac{\beta_n}{b_n} < \theta_2; \quad \frac{x''}{(y'')^{\theta_2}} < \left(\frac{\alpha_n}{a_n}\right)^{\frac{1}{b_n}} < \frac{x'}{(y')^{\theta_1}}.$$

In case $S(U) \cap S(V) \cap (0, \infty) = \phi$, we may consider continuity points that are negative and proceed as above and arrive at similar conclusion. Let $\{n'\} \subset \{n\}$ be a subsequence for which both the above converge to give

$$\frac{\beta_{n'}}{b_{n'}} \to B^* > 0, \quad \left(\frac{\alpha_{n'}}{a_{n'}}\right)^{\frac{1}{b_{n'}}} \to A^* > 0.$$

Then $G_{n'}(a_{n'}|x|^{b_{n'}} sign(x)) \to U(x)$, while

$$G_{n'}(\alpha'_n|x|^{\beta_{n'}} sign(x)) = G_{n'}\left(a_{n'}\left\{\left(\frac{\alpha_{n'}}{a_{n'}}\right)^{\frac{1}{b_{n'}}} |x|^{\frac{\beta_{n'}}{b_{n'}}}\right\}^{b_{n'}} sign(x)\right) \to U(A^*|x|^{B^*} sign(x)). \quad (4)$$

But by assumption $G_{n'}(\alpha'_n|x|^{\beta_{n'}} sign(x)) \to V(x) = U(A|x|^B sign(x))$ also. Hence $A = A^*$ and $B = B^*$ proving (3). Sufficiency of (3) is easily seen from (4).

Proceeding as in Resnick (1971 a) one can get similar results in the $p-$ max set up. But the proofs are going to be cumbersome. However, we derive results using Resnick's results in $\ell-$ max set up. It will be noted that this method is useful in case of $F_i \in \mathcal{D}_p(\Psi), r(F_i) < \infty$ and $F_i \in \mathcal{D}_p(H_{2,.})$ and $F_i \in \mathcal{D}_p(H_{4,.})$.

Theorem 5.3 *Let X and Y be independent positive rvs. If $X \sim F_1 \in \mathcal{D}_p(H_{1,\alpha})$ and $Y \sim F_2 \in \mathcal{D}_p(H_{1,\beta})$, $0 < \alpha < \beta$, then $G = F_1F_2 \in \mathcal{D}_p(H_{1,\alpha})$.*

Proof

$$X \sim F_1 \in \mathcal{D}_p(H_{1,\alpha}) \Leftrightarrow \log X \sim G_1 \in \mathcal{D}_\ell(\Phi_\alpha)$$

where $G_1(x) = F_1(e^x)$. Similarly

$$Y \sim F_2 \in \mathcal{D}_p(H_{1,\beta}) \Leftrightarrow \log Y \sim G_2 \in \mathcal{D}_\ell(\Phi_\beta)$$

where $G_2(x) = F_2(e^x)$. Since $\log[\max(X,Y)] = \max[\log(X,Y)]$, and by theorem 5.1 $\max(\log X, \log Y) \sim G_1G_2 \in \mathcal{D}_\ell(\Phi_\alpha)$, we have $\log[\max(X,Y)] \sim G_1G_2 \in \mathcal{D}_\ell(\Phi_\alpha)$ or equivalently $\max(X,Y) \sim F_1F_2 \in \mathcal{D}_p(H_{1,\alpha})$.

Remark If $\alpha = \beta$ this result may not hold in view of remark 1 following Theorem 5.1. It will however hold if

$$\lim_{x\to\infty} \frac{1 - F_2(e^x)}{1 - F_1(e^x)}$$

exists and is finite.

We now give the results for dfs in the MDA of $H_{2,\alpha}$.

Theorem 5.4 *Let X and Y be independent positive rvs. If $X \sim F_1 \in \mathcal{D}_p(H_{2,\alpha})$ and $Y \sim F_2 \in \mathcal{D}_p(H_{2,\beta})$, $0 < \alpha < \beta$ and $0 < r(F_1) = r(F_2) = x_0 < \infty$, then $F = F_1F_2 \in \mathcal{D}_p(H_{2,\alpha})$.*

Proof By Theorem B, $F_1 \in \mathcal{D}_\ell(\Psi_\alpha), F_2 \in \mathcal{D}_\ell(\Psi_\beta)$ and $r(F) = x_0$ by Lemma 2.4. Hence by Theorem A, $1 - F_1(x_0 - \frac{1}{x})$ is RV $(-\alpha)$ while $1 - F_2(x_0 - \frac{1}{x})$ is RV $(-\beta)$. Then $1 - F_1F_2(x_0 - \frac{1}{x})$ is RV $(-\alpha)$ and by Theorem A, $F \in \mathcal{D}_\ell(\Psi_\alpha)$. By Theorem B we have the result.

In the case of dfs in the MDA of $H_{3,\alpha}$, the above line of proof will not work because Theorem B is not applicable.

Theorem 5.5 *Let X and Y be independent negative rvs. If $X \sim F_1 \in \mathcal{D}_p(H_{3,\alpha})$ and $Y \sim F_2 \in \mathcal{D}_p(H_{3,\beta})$, $0 < \alpha < \beta$ then $F = F_1F_2 \in \mathcal{D}_p(H_{3,\alpha})$.*

Proof By theorem 2.3 in Mohan and Ravi (1993), $X \sim F_1 \in \mathcal{D}_p(H_{3,\alpha})$ implies $r(F_1) = 0$ and

$$\lim_{t\to\infty} \frac{1 - F_1(-e^{-tx})}{1 - F_1(-e^{-t})} = x^{-\alpha}.$$

Let $Z_1 = -\log(-X),\ \ Z_2 = -\log(-Y)$ and $Z = \max(Z_1, Z_2)$. Then

$$\frac{P(Z_1 > tx)}{P(Z_1 > t)} = \frac{1 - F_1(-e^{-tx})}{1 - F_1(-e^{-t})} \to x^{-\alpha}$$

as $t \to \infty$. Similarly $1 - P(Z_2 > t)$ is RV $(-\beta)$ and hence $1 - P(Z > t)$ is RV $(-\alpha)$. Thus with $F = F_1F_2$ we have, $1 - F(-e^{-t})$ is RV $(-\alpha)$. Further $r(F) = 0$ by Lemma 2.4. Hence by theorem 2.3 in Mohan and Ravi (1993) again it follows that $F \in \mathcal{D}_p(H_{3,\alpha})$.

Next result deals with the case of $\mathcal{D}_p(H_{4,.})$ and the proof is similar to that of Theorem 5.4 and we omit the details.

Theorem 5.6 *Let X and Y be independent positive rvs. If $X \sim F_1 \in \mathcal{D}_p(H_{4,\alpha})$ and $Y \sim F_2 \in \mathcal{D}_p(H_{4,\beta})$, $0 < \alpha < \beta$ and $r(F_1) = r(F_2)$, then $F = F_1F_2 \in \mathcal{D}_p(H_{4,\alpha})$.*

Ravi (2000) gave an example in which $F_i \in \mathcal{D}_p(\Phi), i = 1, 2$ but $F_1F_2 \notin \mathcal{D}_p(\Phi)$. Recall Φ is the fifth p-max stable law. However, using the A-eqivalence property defined by Resnick (1971 b), we get some positive results in this context.

Associate with the df F of a rv X, its A-function

$$A_F(z) = \frac{1}{z} E(X - z | X > z).$$

Resnick defined A-equivalence classes of dfs and their role in determining the ℓ-max stable laws to which products of dfs belong.

Definition 3: Two dfs F_1 and F_2 are A-equivalent if $r(F_1) = r(F_2) = x_0 \leq \infty$ and $A_{F_1}(z) \sim A_{F_2}(z)$ as $z \uparrow x_0$.

The following result then follows.

Theorem 5.7 *Suppose* $F_i \in \mathcal{D}_p(\Phi), i = 1, 2,\ 0 < r(F_1) = r(F_2) < \infty$*. If* F_1 *and* F_2 *are* A*-equivalent then* $F_1 F_2 \in \mathcal{D}_p(\Phi)$

Proof $F_i \in \mathcal{D}_p(\Phi), i = 1, 2,\ 0 < r(F_1), r(F_2) < \infty \Leftrightarrow F_i \in \mathcal{D}_\ell(\Lambda), i = 1, 2$ by Theorem B. Then $F_1 F_2 \in \mathcal{D}_\ell(\Lambda),\ 0 < r(F_1 F_2) = x_0 < \infty$ by Resnick (1971 *b*) in view of the assumptions that $0 < r(F_1) = r(F_2) < \infty$, and that F_1 and F_2 are A-equivalent. This is equivalent to $F_1 F_2 \in \mathcal{D}_p(\Phi)$ by Theorem B.

On the same lines we have

Theorem 5.8 *Suppose* $F_i \in \mathcal{D}_p(\Psi), i = 1, 2,\ r(F_1) = r(F_2) < 0$*. If* F_1 *and* F_2 *are* A*-equivalent then* $F_1 F_2 \in \mathcal{D}_p(\Psi)$.

Remarks

1. The above results have obvious extensions to the case of products of finite number of dfs.

2. Hebbar (1981) in the linear normalization setup and Ravi (2000) in the power normalization setup studied the following problem, which is an analogue of Gnedenko's hypothesis in the limit theory concerning sums of independent rvs. Suppose $\{X_n\}$ is a sequence of independent rvs with corresponding dfs $\{F_n\}$ such that for each n, the df $F_n \in \{G_1, G_2, \ldots, G_m\}$. Suppose that for each $n, \tau_k(n)$ of $\{F_1, F_2, \ldots, F_n\}$ are equal to $G_k, 1 \leq k \leq m$. Suppose further that each G_k belongs to the MDA of some max stable law H_k. Let $M_n = \max\{X_1, X_2, \ldots, X_n\}$. Will the df F_n^* of M_n, properly normalized, converge weakly to a proper df, which is a function of $H_1, H_2, \ldots, H_m$? Note that $F_n^* = \prod_{k=1}^m G_k^{\tau_k(n)}$. This problem can be analysed in the light of above results for products and mixtures. In particular, we have the following observation comparable to results in Ravi (2000).

Let $\frac{\tau_k(n)}{n} \to a_k,\ 0 < a_k < 1,\ 1 \leq k \leq m$. Let $G_1, G_2, \ldots, G_m$ be dfs such that

$$G_k^n(g_n(x)) \stackrel{w}{\to} H_k(x),\ 1 \leq k \leq m,$$

where $g_n(x)$ is a strictly monotone continuous function for each n. Let $a_i > 0,\ 1 \leq i \leq m$, and $\sum_{i=1}^m a_i = 1$. Set $G(x) = a_1.G_1(x) + \cdots + a_m.G_m(x)$. Let $S(H_i) \cap S(H_k) \neq \phi$ for $1 \leq i < k \leq m.$. Then by Theorem 4.1, $G^n(g_n(x)) \stackrel{w}{\to} H(x)$ given in Theorem 4.1. We also have

$$\prod_{k=1}^m G_k^{na_k}(g_n(x)) \stackrel{w}{\to} \prod_{k=1}^m H_k^{a_k}(x)$$

which in turn gives

$$F_n^*(g_n(x)) = P(M_n \leq (g_n(x)) = \prod_{k=1}^{m} G_k^{\tau_k(n)}(g_n(x)) \stackrel{w}{\to} \prod_{k=1}^{m} H_k^{a_k}(x).$$

Conclusions We reviewed the ℓ-max , p-max stable laws and general max stable laws and their MDAs. We considered functions of dfs such as the convolutions, mixtures and products of dfs which belong to MDAs of some of these max stable laws and investigated if these functions belong to MDAs of max stable laws.

Acknowledgements Part of the work was done while the author was visiting CRRao Advanced Institute for Mathematics, Statistics and Computer Science (AIMSCS), Hyderabad. The author thanks the authorities of AIMSCS for providing the facilities under the DST-CMS Extension Project No. SR/S4/MS:516/07 dated 21-04-2008. The author thanks the referee for carefully reading the original version and pointing out a few typos.

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