

A note on exact moments of order statistics from exponentiated log-logistic distribution

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Abstract. In this paper we establish explicit expressions for single and product moments of order statistics from exponentiated log-logistic distribution. These expressions are used to calculate the mean and variances.

1. Introduction

Let X_1, X_2, \dots, X_n be a random sample of size n from a continuous population having probability density function (pdf) $f(x)$ and distribution function (df) $F(x)$. Let $X_{1:n} \leq X_{2:n} \leq \dots \leq X_{n:n}$ be the corresponding order statistics. The pdf of $X_{r:n}$, the r th order statistic is given by David and Nagaraja (2003)

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!} [F(x)]^{r-1} [1-F(x)]^{n-r} f(x), \quad -\infty < x < \infty, \quad (1)$$

and joint pdf of $X_{r:n}, X_{s:n}, 1 \leq r < s \leq n$ is given as

$$f_{r,s:n}(x,y) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} [F(x)]^{r-1} [F(y)-F(x)]^{s-r-1} [1-F(y)]^{n-s} f(x)f(y), \quad (2)$$

$-\infty < x < y < \infty$.

A random variable X is said to have exponentiated log-logistic distribution (Rosaiah *et al.*, 2006) if its pdf is given by

$$f(x) = \alpha \theta x^{-(\alpha+1)} \left[\frac{x^\alpha}{1+x^\alpha} \right]^{\theta+1}, \quad \theta, \alpha > 0, \quad 0 < x < \infty, \quad (3)$$

and the corresponding df of X is

$$F(x) = \left[\frac{x^\alpha}{1+x^\alpha} \right]^\theta, \quad \theta, \alpha > 0, \quad 0 < x < \infty. \quad (4)$$

For more details on this distribution and its application one may refer to Rosaiah *et al.* (2006, 2007).

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Moments of order statistics for some specific distributions are investigated by several authors in the literature. Malik (1966) derived the expression for exact moments of order statistics from Pareto distribution. Malik (1967) obtained the explicit expression for moments of order statistics of power function distribution. Khan and Khan (1987) obtained the moments of order statistics from Burr distribution. Khan and Ali (1995) derived the ratio and inverse moments of order statistic from Burr distribution. Further, Ali and Khan (1996) established the ratio and inverse moments of order statistic from Weibull and exponential distribution. In this paper we have obtained simple expressions for the exact moments of order statistic from exponentiated log-logistic distribution. Also means and variances are tabulated.

2. Single moments

Lemma 2.1. For exponentiated log-logistic distribution as given in (3) and any non negative finite integers a and b , we have that

$$I_j(a, 0) = \theta B\left(\frac{j}{\alpha} + \theta(a + 1), 1 - \frac{j}{\alpha}\right),$$

where

$$I_j(a, b) = \int_0^\infty x^j [F(x)]^a [1 - F(x)]^b f(x) dx. \tag{5}$$

Proof. From (5), we have

$$I_j(a, 0) = \int_0^\infty x^j [F(x)]^a f(x) dx = \alpha \theta \int_0^\infty x^{j-\alpha-1} \left[\frac{x^\alpha}{1+x^\alpha}\right]^{\theta(a+1)+1} dx. \tag{6}$$

Set $\left(\frac{x^\alpha}{1+x^\alpha}\right)^\theta = t^\theta$ in (6), after simplification we get required result. \square

Lemma 2.2. For exponentiated log-logistic distribution as given in (3) and any non negative finite integers a and b , we have that

$$I_j(a, b) = \sum_{k=0}^b (-1)^k \binom{b}{k} I_j(a+k, 0) = \theta \sum_{k=0}^b (-1)^k \binom{b}{k} B\left(\frac{j}{\alpha} + \theta(a+k+1), 1 - \frac{j}{\alpha}\right).$$

Proof. It can be proved by expanding $[1 - F(x)]^b$ binomially in (5) and using Lemma 2.1. \square

Theorem 2.3. For exponentiated log-logistic distribution as given in (3) and $1 \leq r \leq n$, $\alpha, \theta > 0$, we have that

$$E(X_{r:n}^j) = C_{r:n} \sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} I_j(r+k-1, 0) = \theta C_{r:n} \sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} B\left(\frac{j}{\alpha} + \theta(r+k), 1 - \frac{j}{\alpha}\right), \tag{7}$$

where

$$C_{r:n} = \frac{n!}{(r-1)!(n-r)!}.$$

Proof. From (1), we have

$$E(X_{r:n}^j) = \frac{n!}{(r-1)!(n-r)!} \int_0^\infty x^j [F(x)]^{r-1} [1 - F(x)]^{n-r} f(x) dx. \tag{8}$$

On application of Lemma 2.2 in (8), we get required result. \square

Identity 2.1. For $1 \leq r \leq n - 1$, we have that

$$C_{r:n} \sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{1}{r+k} = 1. \tag{9}$$

Proof. At $j = 0$ in (7), we have

$$1 = C_{r:n} \sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{1}{r+k}$$

and hence the result given in (9). \square

3. Product moments

Lemma 3.1. For exponentiated log-logistic distribution as given in (3) and any non negative finite integers a and c , we have that

$$I_{i,j}(a, 0, c) = \frac{\theta^2}{\frac{j}{\alpha} + \theta(c+1)} B\left(\frac{i+j}{\alpha} + \theta(a+c+2), 1 - \frac{i}{\alpha}\right) \times {}_3F_2 \left[\begin{matrix} \frac{j}{\alpha} + \theta(c+1), & \frac{j}{\alpha}, & \frac{i+j}{\alpha} + \theta(a+c+2) \\ \frac{j}{\alpha} + \theta(c+1) + 1, & \frac{j}{\alpha} + \theta(a+c+2) + 1 & \end{matrix} ; 1 \right],$$

where

$$I_{i,j}(a, b, c) = \int_0^\infty \int_0^x x^i y^j [F(x)]^a [F(y) - F(x)]^b [F(y)]^c f(x) f(y) dy dx \tag{10}$$

and

$${}_pF_q \left[\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{matrix} , 1 \right] = \sum_{r=0}^\infty \left[\prod_{j=1}^p \frac{\Gamma(a_j + r)}{a_j} \right] \left[\prod_{j=1}^q \frac{\Gamma(b_j)}{b_j + r} \right].$$

For $p = q + 1$ and $\sum_{j=1}^q b_j - \sum_{j=1}^p a_j > 0$, see Mathai and Saxena (1973).

Proof. We have from (10)

$$I_{i,j}(a, 0, c) = \int_0^\infty \int_0^x x^i y^j [F(x)]^a [F(y)]^c f(x) f(y) dy dx. \tag{11}$$

In view of (3), (4) and (11), we have

$$I_{i,j}(a, 0, c) = (\alpha\theta)^2 \int_0^\infty x^{i-\alpha-1} \left[\frac{x^\alpha}{1+x^\alpha} \right]^{\theta(a+1)+1} \left[\int_0^x y^{j-\alpha-1} \left[\frac{y^\alpha}{1+y^\alpha} \right]^{\theta(c+1)+1} dy \right] dx. \tag{12}$$

By setting $\left(\frac{y^\alpha}{1+y^\alpha} \right)^\theta = t^\theta$ in (12), we get after simplification

$$I_{i,j}(a, 0, c) = \alpha\theta^2 \int_0^\infty x^{i-\alpha-1} \left[\frac{x^\alpha}{1+x^\alpha} \right]^{\theta(a+1)+1} B_{\frac{x^\alpha}{1+x^\alpha}} \left(\frac{j}{\alpha} + \theta(c+1), 1 - \frac{j}{\alpha} \right) dx \tag{13}$$

where

$$B_x(p, q) = \int_0^x t^{p-1} (1-t)^{q-1} dt$$

is the incomplete beta function.

Also note that Mathai and Saxena (1973) proved that

$$B_x(p, q) = \frac{x^p}{p} {}_2F_1(p, 1 - q, p + 1; x)$$

and

$$\int_0^1 t^{a-1}(1-t)^{b-1} {}_2F_1(c, d; e; t) dt = B(a, b) {}_3F_2(c, d, a; e, a + b; 1).$$

Substituting these value in (13), we get the required result. \square

Lemma 3.2. For exponentiated log-logistic distribution as given in (3) and any non negative finite integers a, b and c , we have that

$$\begin{aligned} I_{i,j}(a, b, c) &= \sum_{k=0}^b (-1)^k \binom{b}{k} I_{i,j}(a + k, 0, b + c - k) \\ &= \frac{\theta^2}{\frac{j}{\alpha} + \theta(c + 1)} B\left(\frac{i + j}{\alpha} + \theta(a + c + 2), 1 - \frac{i}{\alpha}\right) \\ &\quad \times {}_3F_2 \left[\begin{matrix} \frac{j}{\alpha} + \theta(c + 1), & \frac{j}{\alpha}, & \frac{i + j}{\alpha} + \theta(a + c + 2) \\ \frac{j}{\alpha} + \theta(c + 1) + 1, & \frac{j}{\alpha} + \theta(a + c + 2) + 1 & \end{matrix} ; 1 \right]. \end{aligned}$$

Proof. Lemma can be proved by expanding $[F(y) - F(x)]^b$ binomially in (10) and using Lemma 2.1. \square

Theorem 3.3. For exponentiated log-logistic distribution as given in (2) and $\alpha, \theta > 0$ and $1 \leq r < s \leq n - 1$, we have that

$$\begin{aligned} E(X_{r,s:n}^{i,j}) &= C_{r,s:n} \sum_{l=0}^{n-s} \sum_{k=0}^{s-r-1} (-1)^{k+l} \binom{s-r-1}{k} \binom{n-s}{l} I_{i,j}(r + k - 1, 0, s + l - r - 1) \\ &= C_{r,s:n} \sum_{l=0}^{n-s} \sum_{k=0}^{s-r-1} (-1)^{k+l} \binom{s-r-1}{k} \binom{n-s}{l} \frac{\theta^2}{\frac{j}{\alpha} + \theta(s + l - r)} \\ &\quad \times B\left(\frac{i + j}{\alpha} + \theta(s + l + k), 1 - \frac{i}{\alpha}\right) \\ &\quad \times {}_3F_2 \left[\begin{matrix} \frac{j}{\alpha} + \theta(s + l - r), & \frac{j}{\alpha}, & \frac{i + j}{\alpha} + \theta(s + l + k) \\ \frac{j}{\alpha} + \theta(s + l - r) + 1, & \frac{j}{\alpha} + \theta(s + l + k) + 1 & \end{matrix} , 1 \right]. \end{aligned} \tag{14}$$

Proof. From (2), we have

$$E(X_{r,s:n}^{i,j}) = C_{r,s:n} \int_0^\infty \int_0^x [F(x)]^{r-1} [F(y) - F(x)]^{s-r-1} [1 - F(y)]^{n-s} f(x) f(y) dy dx.$$

On application of Lemma 3.2, (14) can be proved. \square

Table 1: Mean of order statistics

n	r	$\theta = 2, \alpha = 2$	$\theta = 3, \alpha = 3$	$\theta = 4, \alpha = 4$	$\theta = 5, \alpha = 5$
1	1	2.3562	1.8809	1.6921	1.5803
2	1	1.2762	1.3468	1.3476	1.3304
	2	3.4361	2.4151	2.0366	1.8303
3	1	1.0124	1.1796	1.2293	1.2404
	2	1.8039	1.6813	1.5842	1.5103
	3	4.2522	2.7819	2.2629	1.9903
4	1	0.8813	1.0896	1.1637	1.1897
	2	1.4058	1.4494	1.4261	1.3925
	3	2.2020	1.9133	1.7422	1.6282
	4	4.9356	3.0714	2.4364	2.1111
5	1	0.7992	1.0308	1.1200	1.1556
	2	1.2094	1.3250	1.3383	1.3258
	3	1.7005	1.6362	1.5578	1.4926
	4	2.5363	2.0981	1.8651	1.7185
	5	5.5353	3.3148	2.5793	2.2092

Here in Table 1, it may be noted that the well known property of order statistics $\sum_{i=1}^n E(X_{i:n}) = nE(X)$ (David and Nagaraja, 2003) is satisfied.

Table 2: Variance of order statistics

n	r	$\theta = 2, \alpha = 3$	$\theta = 2, \alpha = 4$	$\theta = 3, \alpha = 4$	$\theta = 3, \alpha = 5$
1	1	1.4312	0.4285	0.5055	0.2288
2	1	0.2056	0.0997	0.1102	0.0620
	2	2.2005	0.5729	0.6885	0.2874
3	1	0.1073	0.0586	0.0624	0.0373
	2	0.2625	0.1118	0.1289	0.0674
	3	2.8524	0.6885	0.8328	0.3326
4	1	0.0747	0.0434	0.0451	0.0279
	2	0.1224	0.0591	0.0661	0.0369
	3	0.3157	0.1252	0.1469	0.0738
	4	3.4367	0.7876	0.9557	0.3702
5	1	0.0586	0.0355	0.0363	0.0229
	2	0.0789	0.0408	0.0446	0.0258
	3	0.1394	0.0629	0.0720	0.0387
	4	0.3651	0.1379	0.1633	0.0797
	5	3.9749	0.8757	1.0645	0.4028

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