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Application of Power Series in Nonlinear Integral Equation in Relation to Branching Processes

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Abstract. Non-linear integral equations are important in many areas of stochastic processes viz, renewal equations, age dependent branching process etc. Analytical solution of nonlinear integral equation of general type is obtained and its convergence is also shown. In spite of a standard theory of linear integral equations, such compact and analytical form of solution of nonlinear integral equation is not available. Secondly, such power series solution is applied to obtain extinction probability in Bellman-Harris process. A special case is also investigated.

1. Introduction

In many situations of stochastic processes we are to deal with integral equations and some of them are non-linear, even some problems of stochastic processes can be formulated in terms of integral equations. Some of the natural phenomena can be modeled as a branching process (Bhattacharya etal 1990), such as survival of family names, verbal flow of information, electron multipliers etc. and integral equations come out in many cases of stochastic process viz renewal equation, age-dependent branching process, etc. For standard reference one can look at any standard book on stochastic process e.g Bailey (1964). Also the theory of linear integral equations is standard (Lovitt, 1950; Lalesco, 1912). But there is lack of compact form of solution of non-linear integral equations (Davis, 1962).

In Section 2 non-linear integral equation where function under integral as square is attempted for solution of the equation by power series method and its uniform convergence is shown in Section 3 under assumption on the kernel. In Section 4 compact form of solution is obtained. The power series method is strong as evident from (Seal, 2013(b)), where it was used for integral equations of non-Riemann type and stochastic differential equation (Seal, 2013(c)).

For this some simple mathematical results as in the following are used from differential equations:

(a) The power series and its derivative have the same radius of convergence.

(b) Any $F(x, y, y', y'', \cdots) = 0$ has a solution in power series when F is sufficiently smooth.

The second part of this paper is the application of the above development to branching process. Following biological terminology, let us consider a situation where each organism of its generation produces a random number of offsprings to form the next generation. If the probability distribution of the number of offsprings produced by an organism is given, one becomes interested in many characteristics like the distribution of the size of the population in different generations and the probability of extinction.

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Simple mathematical model for branching processes was formulated by Galton and Watson (1874). Even there are lot of works on inference of branching process e.g Guttorp (1991). Some generalizations are in Cohn et al. (2003). Even Seal (2013(a)) considered estimation of allele from data over generations and also estimation of it's extinction probability. But we shall concentrate on Bellman-Harris process (1948) which is age dependant branching process. Let us suppose that an ancestor at time $t = 0$ initiates the process and at the end of its life time it produces a random number of offsprings having a distribution and the process continues as long as objects are present. Here assumption is that the offsprings act independent of each other. Also the lifetime of objects are i.i.d. random variables with distribution function G. Under the above assumption, let $\{X(t),\ t \geq 0\}$ be the number of objects available at time t. Then the stochastic process $\{X(t),\ t \geq 0\}$ is called an age-dependent branching process.

In Section 5, BH process is considered. The power series method for solving integral equation is investigated. Applying this technique, expression for extinction probability at any time t is obtained. This has been done in Section 5. Also an example with two offsprings have been worked out using this. This is important for present day scenario, because in our society we do not prefer more that two children.

2. Solution of some non-linear integral equations by power series method

Let us consider the following integral equation

$$\phi(x) = F(x) + \lambda \int_0^x k(x,\xi)\phi^2(\xi)d\xi. \tag{1}$$

Let us substitute in (1), the unknown function $\phi(\xi)$ under the integral sign and expand the integral. Then we have

$$\begin{aligned}
\phi(x) &= F(x) + \lambda \int_0^x k(x,\xi) \left\{ F(\xi) + \lambda \int_0^\xi k(\xi,\xi_1)\phi^2(\xi_1)d\xi_1 \right\}^2 d\xi \\
&= F(x) + \lambda \int_0^x k(x,\xi) \left\{ F^2(\xi) + 2\lambda F(\xi) \int_0^\xi k(\xi,\xi_1)\phi^2(\xi_1)d\xi_1 + \lambda^2 \left(\int_0^\xi k(\xi,\xi_1)\phi^2(\xi_1)d\xi_1 \right)^2 \right\} d\xi \\
&= F(x) + \lambda \int_0^x k(x,\xi)F^2(\xi)d\xi + 2\lambda^2 \int_0^x F(\xi) \int_0^\xi k(\xi,\xi_1)\phi^2(\xi_1)d\xi_1 d\xi \\
&\qquad + \lambda^3 \int_0^x k(x,\xi) \left(\int_0^\xi k(\xi,\xi_1)\phi^2(\xi_1)d\xi_1 \right)^2 d\xi. \qquad (2)
\end{aligned}$$

Again substituting $\phi^2(\xi_1)$ under second integral and making some simplifications, above expression (2) becomes

$$\begin{aligned}
\phi(x) &= F(x) + \lambda \int_0^x k(x,\xi)F^2(\xi)d\xi + 2\lambda^2 \int_0^x F(\xi) \left\{ \int_0^\xi k(\xi,\xi_1)F^2(\xi_1)d\xi_1 \right. \\
&\quad \left. + 2\lambda \int_0^x F(\xi_1)k(\xi,\xi_1) \int_0^{\xi_1} k(\xi_1,\xi_2)\phi^2(\xi_2)d\xi_2 d\xi_1 + \lambda^2 \int_0^\xi k(\xi,\xi_1) \left(\int_0^{\xi_1} k(\xi_1,\xi_2)\phi^2(\xi_2)d\xi_2 \right)^2 d\xi_1 \right\} d\xi \\
&\qquad + \lambda^3 \int_0^x k(x,\xi) \left(\int_0^\xi k(\xi,\xi_1)\phi^2(\xi_1)d\xi_1 \right)^2 d\xi
\end{aligned}$$

$$= F(x) + \lambda\int_0^x k(x,\xi)F^2(\xi)d\xi + 2\lambda^2\int_0^x F(\xi)\int_0^\xi k(\xi,\xi_1)F^2(\xi_1)d\xi_1 d\xi$$
$$+\lambda^3\left[4\int_0^x F(\xi)\int_0^\xi F(\xi_1)k(\xi,\xi_1)\int_0^{\xi_1} k(\xi_1,\xi_2)\phi^2(\xi_2)d\xi_2 d\xi_1 d\xi + \int_0^x k(x,\xi)\left\{\int_0^\xi k(\xi,\xi_1)\phi^2(\xi_1)d\xi_1\right\}^2 d\xi\right]$$
$$+2\lambda^4\int_0^x F(\xi)\int_0^\xi k(\xi,\xi_1)\left\{\int_0^{\xi_1} k(\xi_1,\xi_2)\phi^2(\xi_2)d\xi_2\right\}^2 d\xi_1 d\xi$$

$$= F(x) + \lambda\int_0^x k(x,\xi)F^2(\xi)d\xi + 2\lambda^2\int_0^x F(\xi)\int_0^\xi k(\xi,\xi_1)F^2(\xi_1)d\xi_1 d\xi$$
$$+\lambda^3\left[4\int_0^x F(x)\int_0^\xi F(\xi_1)\int_0^{\xi_1} k(\xi_1,\xi_2)\phi^2(\xi_2)d\xi_2 d\xi_1 d\xi\right.$$
$$\left. + \int_0^x k(x,\xi)\left\{\int_0^\xi k(\xi,\xi_1)\phi^2(\xi_1)d\xi_1\right\}^2 d\xi\right] + \lambda^4\{\ \} + \text{ higher order terms of } \lambda. \quad (3)$$

Thus the form of the solution becomes

$$\phi(x) = F(x) + \lambda\int_0^x k(x,\xi)F^2(\xi)d\xi + 2\lambda^2\int_0^x F(\xi)\int_0^\xi k(\xi,\xi_1)F^2(\xi_1)d\xi_1 d\xi + \cdots$$

So we have proof of the following theorem.

Theorem 2.1. *The form of the solution to* (1) *is given by*

$$\phi(x) = \phi_0(x) + \lambda\phi_1(x) + \lambda^2\phi_2(x) + \cdots$$

Proof. From the derivation of $\phi(x)$ as in (3). □

As we know the form of the solution of the integral equation in (1), we shall find out the functions $\phi_0(x)$, $\phi_1(x)$, $\cdots$ etc in terms of the known functions.

Now putting this $\phi(x)$ in equation (1) we get

$$\begin{aligned}\sum_{i=0}^\infty \lambda^i\phi_i(x) &= F(x) + \lambda\int_0^x k(x,\xi)\left\{\phi_0(\xi) + \lambda\phi_1(\xi) + \lambda^2\phi_2(\xi) + \cdots\right\}^2 d\xi \\ &= F(x) + \lambda\int_0^x k(x,\xi)\left[\left\{\phi_0^2(\xi) + \lambda^2\phi_1^2(\xi) + \lambda^4\phi_2^2(\xi) + \cdots\right\} + \sum_{i\neq j}\lambda^{i+j}\phi_i(\xi)\phi_j(\xi)\right] d\xi \\ &= F(x) + \lambda\int_0^x k(x,\xi)\left[\sum_{i=0}^\infty \lambda^{2i}\phi_i^2(\xi)\right] d\xi + \sum_{i\neq j}\int_0^x k(x,\xi)\phi_i(\xi)\phi_j(\xi)\lambda^{i+j+1}d\xi.\end{aligned}$$

Now comparing the coefficients of different powers of λ from both sides and equating them we have

$$\phi_0(x) = F(x) \quad \text{and} \quad \phi_1(x) = \int_0^x k(x,\xi)\phi_0^2(\xi)d\xi.$$

Thus recursively we have the following theorem.

Theorem 2.2. *The $\phi_k(x)$ functions of Theorem 2.1 are given by*

$$\phi_k(x) = \int_0^x k(x,\xi)\left[\sum_{\substack{i+j=k-1\\ i\neq j}} \phi_i(\xi)\phi_j(\xi)\right] d\xi + \delta_k \int_0^x k(x,\xi)\phi^2_{\frac{k-1}{2}}(\xi)d\xi,$$

where

$$\delta_k = \begin{cases} 0, & \text{if } k \text{ is even} \\ 1, & \text{if } k \text{ is odd} \end{cases}$$

Now for any λ if the given equation has a solution, the solution function must be of the form given above. In the next section we shall see that the above series is uniformly convergent in many situations.

3. Uniform convergence of the solution

Under the assumptions we have $|F(x)| \leq A$ and $|k(x,\xi)| \leq B$, i.e. bounded kernel we have

$$|\phi(x)| \leq a_0 + a_1\lambda x + a_2\lambda^2x^2 + a_3\lambda^3x^3 + \cdots$$

where

$$\int_0^x \left| k(x,\xi)\left[\sum_{\substack{i+j=k-1\\ i\neq j}} \phi_i(\xi)\phi_j(\xi)\right] + \delta_k k(x,\xi)\phi^2_{\frac{k-1}{2}}(\xi)\right| d\xi \leq a_k x^k, \quad \forall\ k = 0,1,2,\cdots \tag{4}$$

i.e., we have

$$|\phi_k(x)| \leq a_k x^k,\ \forall\ k$$

It is to be noted that step (4) is valid because

$$|\phi_0(x)| \leq |F(x)| \leq A = a_0 \text{ (say)},$$

$$|\phi_1(x)| \leq A^2Bx = a_1x,$$

$$|\phi_2(x)| \leq \int_0^x |2\phi_0(.)\phi_1(.)|xd\xi \leq 2a_0a_1\frac{x^2}{2} = a_2x^2,$$

$$|\phi_k(x)| \leq \sum_{i+j=k-1} a_ia_j\frac{x^k}{k} = a_kx^k,$$

and we have relationship as

$$ka_k = a_0a_{k-1} + a_1a_{k-2} + \cdots + a_{k-1}a_0. \tag{5}$$

Now let us define the following power series

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots = \sum_{n=0}^{\infty} a_nx^n,$$

where a_i's are such that above power series is convergent in some non-degenerate interval of x. Then we have

$$f^2(x) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_ka_{n-k}\right)x^n = \sum_{n=0}^{\infty}(n+1)a_nx^n = f'(x).$$

As we know that $f(x)$ and $f'(x)$ have same radius of convergence, $f(x)$ must satisfy the following differential equation

$$y' = y^2. \tag{6}$$

Also by result (b) in Section 1, we know that the above differential equation has a solution in terms of power series, where a_i's satisfy relationship (5).

It is to be noted that $f(0) = a_0$ and $f'(0) = a_1$. In a case to be considered later, for a kernel $k(x, \xi) \leq 1$, it is enough to consider one initial condition $f(0) = a_0$, say. So we see that the power series is convergent with appropriate radius of convergence. In our case we can assume $a_n \leq 1$, thus $ka_k = a_0 a_{k-1} + \cdots + a_{k-1} a_0$. By direct verification we see that some of the first terms are less than 1 and from the above relationship we see that it follows $\forall\ n,\ n \geq 1$. With this development our next result becomes.

Theorem 3.1. *If $F(x) \leq A$ and $k(x, \xi) \leq B$ then for $\lambda \leq 1$ the series in Theorem 2.2 is uniformly convergent.*

Proof. We know that there exists solution to differential equation (6) satisfying the initial condition. Then we must have some a_i's satisfying (5) and those will dominate the term as in (4) and hence by comparison test $\sum_{i=0}^{\infty} \lambda^i \phi_i(x)$ will be uniformly convergent. □

Remark 3.2. For our purpose we may assume $|F(x) \leq 1|$ and $k(x, \xi) \leq 1$. We shall use these in subsequent works and in the context of probability generating function.

4. Compact form of the solution

We have already seen that the previous solution (3) gives

$$\begin{aligned}\phi(x) \quad = \quad & F(x) + \lambda \int_0^x k(x, \xi) F^2(\xi) d\xi + 2\lambda^2 \int_0^x F(\xi) \int_0^\xi k(\xi, \xi_1) F^2(\xi_1) d\xi_1 d\xi \\ & + \lambda^3 \left[4 \int_0^x F(x) \int_0^\xi F(\xi_1) k(\xi, \xi_1) \int_0^{\xi_1} k(\xi_1, \xi_2) \phi^2(\xi_2) d\xi_2 d\xi_1 d\xi + \int_0^x k(x, \xi) \left\{ \int_0^\xi k(\xi, \xi_1) \phi^2(\xi_1) d\xi_1 \right\}^2 d\xi \right].\end{aligned}$$

Now let us look at the term

$$\int_0^x F(\xi) \int_0^\xi k(\xi, \xi_1) F^2(\xi_1) d\xi_1 d\xi.$$

By interchanging the order of integration we get this

$$\int_0^x F(\xi) \int_0^\xi k(\xi, \xi_1) F^2(\xi_1) d\xi_1 d\xi = \int_0^x F^2(\xi_1) d\xi_1 \int_{\xi_1}^x F(\xi) k(\xi, \xi_1) d\xi = \int_0^x k_2(x, \xi_1) F^2(\xi_1) d\xi_1,$$

where $k_2(x, \xi_1) = \int_{\xi_1}^x F(\xi) k(\xi, \xi_1) d\xi$, $k_1(x, \xi) = k(x, \xi)$ and $\phi_0(x) = F(x)$. So $\phi_1(x) = \int_0^x k_1(x, \xi) F^2(\xi) d\xi$ and $\phi_2(x) = \int_0^x k_2(x, \xi) F^2(\xi) d\xi$. Proceeding like this we have

$$\phi(x) = F(x) + \sum_{i=1}^{\infty} \lambda^{i+1} \int_0^x k_i(x, \xi) F^2(\xi) d\xi.$$

Let us put

$$R(x, \xi; \lambda) = k_1(x, \xi) + \lambda k_2(x, \xi) + \lambda^2 k_3(x, \xi) + \cdots = \sum_0^\infty \lambda^i k_{i+1}(x, \xi).$$

So the solution turns out to be next theorem in terms of the kernel $R(x, \xi; \lambda)$.

Theorem 4.1. *The function $\phi(x)$ satisfies*

$$\phi(x) = F(x) + \lambda \int_0^x R(x, \xi; \lambda) F^2(\xi) d\xi.$$

5. General expression for probability of extinction at time t

Let $\{X(t), t \geq 0\}$ be the number of objects alive at time t and if the number of offsprings obtained from a unit has p.g.f $P(s)$ then we see that the probability generating function $F(t,s)$ of $X(t)$ is

$$F(t,s) = [1 - G(t)]s + \int_0^t P[F(t-u,s)]dG(u).$$

Then extinction probability at time t can be obtained as $F(t,0)$, i.e. putting $s = 0$ we have that

$$F(t,0) = \int_0^t P[F(t-u,0)]dG(u).$$

Putting $t - u = v$, we have that

$$F(t,0) = \int_0^t P[F(v,0)]g(t-v)dv,$$

where g is density of G.

Putting

$$E(v) = F(v,0), \tag{7}$$

we have $E(t) = \int_0^t P[E(v)]g(t-v)dv$. Now if the number of offsprings follows the distribution which takes values 0, 1, $\cdots$, k with probabilities p_0, p_1, $\cdots$, p_k, respectively, then $P(s) = \sum_0^k s^i p_i$.

In this case the integral equation (7) changes to

$$E(t) = \int_0^t \sum_0^k p_i \left(E(v)\right)^i g(t-v)dv = \int_0^t g(t-v)\left(\sum_{i=0}^k p_i E(v)^i\right) dv. \tag{8}$$

This is a nonlinear equation of polynomial type. Thus we see that explicit analytic solution is important. So our problem is to find out the solution of $E(t)$. Let us see this in a simpler example.

Example 5.1. Let us give an application when $p_0 = \frac{1}{4}, p_1 = \frac{1}{2}, p_2 = \frac{1}{4}$ which has much practical significance. Here

$$P(s) = \frac{1}{4} + \frac{1}{2}s + \frac{1}{4}s^2 = \frac{1}{4}(1+s)^2,$$

which implies

$$f(t) = \int_0^t g(t-v)\frac{1}{4}(1+E(v))^2 dv \text{ from (7).}$$

By changing $f(t) = 1 + E(t)$, the above changes to

$$f(t) = 1 + \int_0^t g(t-v)\frac{1}{4}f^2(v)dv. \tag{9}$$

Now according to our previous nonlinear integral equation we have

$$F(t) = 1,\ \lambda = \frac{1}{4},\ \text{and } k(t,v) = g(t-v) \quad \Rightarrow \quad \phi_0(t) = 1,$$

$$\phi_1(t) = \int_0^t g(t-v)dv = [-G(t-v)]_0^t = G(t),$$

$$
\begin{aligned}
\phi_2(t) &= \int_0^t 2G'(t-v)(G(v)-G(0))dv \\
&= 2\int_0^t G'(t-v)G(v)dv - 2\int_0^t G'(t-v)G(0)dv \\
&= 2\int_0^t G'(t-v)G(v)dv - 2G(0)\int_0^t G'(t-v)dv \\
&= 2(G' * G)(t) - 2G(0)(G(t)-G(0)) \\
&= 2[(G * G)(t) - G(0)(G(t)-G(0))] \\
&= 2(G' * G)(t) \\
&= 2(G * g)(t),
\end{aligned}
$$

$$
\begin{aligned}
\phi_3(t) &= \int_0^t \phi_1^2(v)g(t-v)dv + \int_0^t 2\phi_0(v)\phi_2(v)g(t-v)dv \\
&= \int_0^t (G(v))^2 g(t-v)dv + 2\int_0^t 2(G * g)(v)g(t-v)dv \\
&= (G^2 * g)(t) + 2^2(G * g * g)(t) \\
&= (G^* * g)(t) + 2^2 G * (g^{*2})(t).
\end{aligned}
$$

Thus successively others functions can be obtained.

So

$$
f(t) = \sum_{k=0}^{\infty}\left(\frac{1}{4}\right)^k \phi_k(t) \qquad \Rightarrow \quad 1 + E(t) = \phi_0(t) + \sum_{k=1}^{\infty}\left(\frac{1}{4}\right)^k \phi_k(t) \quad \text{from (7) and (9)}
$$

$$
\Rightarrow \quad E(t) = \sum_{k=1}^{\infty}\left(\frac{1}{4}\right)^k \phi_k(t).
$$

6. Concluding remarks

In this paper solution and its compact form for second degree type non-linear integral equations are obtained by power series method. This may be used in more combinatorial way to obtain solution of higher degree non-linear integral equations. Then it is used to get expression for extinction probability at any time of an age dependent branching process.

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