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Relations for single and joint moment generating functions of lower generalized order statistics from generalized exponential distribution

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Abstract. Distributional and inferential results are derived for ARMA models under the multivariate exponential power (EP) distribution, which includes the Gaussian and Laplace as special cases. Marginal distributions are obtained in the important Laplace case, both under a multivariate Laplace distributed process, as well as a process driven by univariate Laplace noise. Asymptotic results are established for the maximum likelihood estimators under a full EP likelihood, and under a conditional likelihood resulting from a driving univariate EP noise distribution. Whenever tractability permits, results are fully worked out with respect to the MA(1) model. The methodology is illustrated on a fund of real estate returns.

1. Introduction

The concept of generalized order statistics (*gos*) was introduced by Kamps (1995). A variety of order models of random variables is contained in this concept. Pawlas and Syznal (2001) introduced the concept of lower generalized order statistics (*lgos*) to enable a common approach to descending order random variables like reversed order statistics and lower record values. In this article we will consider the *lgos* , which is as follows:

Let $n \in N$, $k \geq 1$, $m \in \Re$, be the parameters such that

$$\gamma_r = k + (n-r)(m+1) \geq 0, \quad \text{for all} \;\; 1 \leq r \leq n.$$

The random variables $X^*(1,n,m,k)\ldots X^*(n,n,m,k)$ are said to be *lgos* from an absolutely continuous distribution function (*df*) $F\,()$ with the probability density function (*pdf*) $f\,()$, if their joint *pdf* is of the form

$$k\Big(\prod_{r=1}^{n-1}\gamma_r\Big)\Big(\prod_{j=1}^{n-1}[F(x_j)]^m f(x_j)\Big)[F(x_n)]^{k-1} f(x_n) \tag{1}$$

for $\quad F^{-1}(1) > x_1 \geq x_2 \geq \ldots \geq x_n \leq F^{-1}(0).$

Note that for $m=0$, $k=1$, we obtain the joint *pdf* for the order statistics and when $m=-1$, we get the joint *pdf* of the $k-$th lower record values.

Keywords. Lower generalized order statistics, order statistics, lower record values, generalized exponential distribution, single and joint moment generating function, recurrence relations and characterization.
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In view of (1), the marginal *pdf* of $r-$th *lgos* is given by

$$f_{X^*(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}\,[F(x)]^{\gamma_r - 1}\, f(x)\, g_m^{r-1}(F(x)). \tag{2}$$

and the joint *pdf* of $r-$th and $s-$th *lgos* , $r < s$, is

$$f_{X^*(r,n,m,k),X^*(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!\;(s-r-1)!}\,[F(x)]^m f(x) g_m^{r-1}(F(x))$$

$$\times [h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s - 1} f(y), \quad x > y, \tag{3}$$

where

$$C_{r-1} \;=\; \prod_{i=1}^{r} \gamma_i, \quad h_m(x) \;=\; \begin{cases} -\dfrac{1}{m+1}\, x^{m+1}, & m \neq -1 \\ -\log x\ , & m = -1 \end{cases}$$

and

$$g_m(x) \;=\; h_m(x) - h_m(1), \quad x \in [0,1).$$

The work of Burkschat *et al.* (2003) may also be seen for dual (lower) generalized order statistics.

Ahsanullah and Raqab (1999), Raqab and Ahsanullah (2000, 2003) have established recurrence relations for moment generating functions of record values from Pareto and Gumble, power function and extreme value distributions.

Recurrence relations for marginal and joint moment generating functions of *gos* from power function distribution are derived by Saran and Pandey (2003). Al-Hussaini *et al.* (2005, 2007) have established recurrence relations for conditional and joint moment generating functions of *gos* based on mixed population, respectively. Khan *et al.* (2010) have established explicit expressions and some recurrence relations for moment generating function of *gos* from Gompertz distribution.

In the present study, we have established exact expressions and some recurrence relations for single and joint moment generating functions of *lgos* from generalized exponential distribution. Results for order statistics and lower record values are deduced as special cases and a theorem for characterizing this distribution is stated and proved.

A random variable X is said to have generalized exponential distribution (Gupta and Kundu, 1999), if its *pdf* is of the form

$$f(x) = \alpha(1 - e^{-x})^{\alpha - 1} e^{-x}, \quad x > 0, \alpha > 0 \tag{4}$$

and the corresponding *df* is

$$F(x) = (1 - e^{-x})^{\alpha}, \quad x > 0, \alpha > 0. \tag{5}$$

Here α is the shape parameter. The location and scale parameters can be added to this model. Without loss of generality, the location and scale parameters are taken to the zero and unity, respectively. When $\alpha = 1$, this distribution corresponds to standard exponential distribution.

It is observed in Gupta and Kundu (1999) that the generalized exponential distribution can be used quite effectively in analyzing many lifetime data, particularly in place of gamma and Weibull distributions.

For more details on this distribution and its applications one may refer to Gupta and Kundu (2001a, b).

2. Relations for single moment generating function

Note that for generalized exponential distribution defined in (4)

$$\alpha F(x) = (e^x - 1) f(x). \tag{6}$$

The relation in (6) will be used to derive some recurrence relations for the moment generating functions of *lgos* from generalized exponential distribution.

Let us denote the single moment generating functions of $X^*(r,n,m,k)$ by $M_{X^*(r,n,m,k)}(t)$ and its $j-$th derivative by $M^{(j)}_{X^*(r,n,m,k)}(t)$.

We shall first establish the explicit expression for $M_{X^*(r,n,m,k)}(t)$. Using (2), we have when $m \neq -1$

$$M_{X^*(r,n,m,k)}(t) = \frac{C_{r-1}}{(r-1)!} \int_0^\infty e^{tx} [F(x)]^{\gamma_r - 1} f(x)\, g_m^{r-1}(F(x)) dx$$

$$= \frac{C_{r-1}}{(r-1)!} J(\gamma_r{-}1, r-1), \tag{7}$$

where

$$J(a,b) = \int_0^\infty e^{tx} [F(x)]^a f(x)\, g_m^b(F(x)) dx. \tag{8}$$

On expanding $g_m^b(F(x)) = \left[\frac{1}{m+1}\{1 - (F(x))^{m+1}\}\right]^b$ binomially in (8), we get when $m \neq -1$

$$J(a,b) = A \int_0^\infty e^{tx} [F(x)]^{a+u(m+1)} f(x) dx, \tag{9}$$

where

$$A = \frac{1}{(m+1)^b} \sum_{u=0}^{b} (-1)^u \binom{b}{u}.$$

Making the substitution $z = [F(x)]^{1/\alpha}$ in (9), we find that

$$J(a,b) = \alpha A \int_0^1 (1-z)^{-t} z^{\alpha[a+u(m+1)+1]-1} dz.$$

On using the Maclaurine series expansion

$$(1-z)^{-t} = \sum_{p=0}^{\infty} \frac{(t)_p\ z^p}{p\ !},$$

where

$$(t)_p \quad = \quad \begin{cases} t(t+1)\dots(t+p-1), & p = 1,2,\dots \\ 1, & p = 0 \end{cases}$$

and integrating the resulting expression, we get

$$J(a,b) = A \sum_{p=0}^{\infty} \frac{(t)_p}{p![a + u(m+1) + 1 + p/\alpha]}. \tag{10}$$

Now on substituting for $J(\gamma_r - 1, r-1)$ from (10) in (7) and simplifying, we obtain when $m \neq -1$

$$M_{X^*(r,n,m,k)}(t) = \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{p=0}^{\infty}\sum_{u=0}^{r-1}(-1)^u \binom{r-1}{u} \frac{(t)_p}{p!\ [\gamma_{r-u} + p/\alpha]}. \tag{11}$$

Applying D'Alembert's ratio test for convergence, it can easily be seen that $M_{X^*(r,n,m,k)}(t)$ exists $\forall\ t,\ -\infty < t < \infty$ and is analytic in t.

And when $m = -1$ that

$$M_{X^*(r,n,m,k)}(t) = \frac{0}{0} \qquad as \ \sum_{u=0}^{r-1}(-1)^u \binom{r-1}{u} = 0\ .$$

Since (11) is of the form $\frac{0}{0}$ at $m = -1$, therefore, we have

$$M_{X^*(r,n,m,k)}(t) = A^* \sum_{u=0}^{r-1}(-1)^u \binom{r-1}{u} \frac{[k + (n-r+u)(m+1) + p/\alpha]^{-1}}{(m+1)^{r-1}}, \tag{12}$$

where

$$A^* = \frac{C_{r-1}}{(r-1)!} \sum_{p=0}^{\infty} \frac{(t)_p}{p!}.$$

Differentiating numerator and denominator of (12) $(r-1)$ times with respect to m , we get

$$\begin{aligned} M_{X^*(r,n,m,k)}(t) &= A^* \sum_{u=0}^{r-1}(-1)^{u+(r-1)} \binom{r-1}{u} \frac{(n-r+u)^{r-1}}{[k + (n-r+u)(m+1) + p/\alpha]^r} \\ &= A^* \sum_{u=0}^{r-1}(-1)^u \binom{r-1}{u} \frac{(r-n-u)^{r-1}}{[k + (n-r+u)(m+1) + p/\alpha]^r}. \end{aligned}$$

On applying the L' Hospital rule, we have

$$\lim_{m\to -1} M_{X^*(r,n,m,k)}(t) = \frac{A^*}{(k+p/\alpha)^r} \sum_{u=0}^{r-1}(-1)^u \binom{r-1}{u} (r-n-u)^{r-1}. \tag{13}$$

But for all integers $n \geq 0$ and for all real numbers x , we have Ruiz (1996)

$$\sum_{i=0}^{n}(-1)^i \binom{n}{i} (x-i)^n = n!. \tag{14}$$

Therefore

$$\sum_{u=0}^{b}(-1)^{u+b} \binom{b}{u} u^b = b!. \tag{15}$$

Now on substituting (14) in (13), we find that

$$M_{X^*(r,n,-1,k)}(t) = M_{(Z_r^{(k)})}(t) = k^r \sum_{p=0}^{\infty} \frac{(t)_p}{p!(k+p/\alpha)^r}. \tag{16}$$

Differentiating $M_{X^*(r,n,m,k)}(t)$ and evaluating at $t=0$, we get the mean of the $r-$th *lgos* when $m \neq -1$

$$E[X^*(r,n,m,k)] = \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{p=1}^{\infty}\sum_{u=0}^{r-1}(-1)^u \binom{r-1}{u} \frac{1}{p\ (\gamma_{r-u}+p/\alpha)}$$

and when $m=-1$ that

$$E[X^*(r,n,-1,k)] = E(Z_r^{(k)}) = k^r \sum_{p=1}^{\infty}\frac{1}{p\ (k+p/\alpha)^r}.$$

Special cases

i) Putting $m=0$, $k=1$ in (11), the explicit formula for single moment generating function of order statistics of the generalized exponential distribution can be obtained as

$$M_{X_{n-r+1:n}}(t) = C_{r:n}\sum_{p=0}^{\infty}\sum_{u=0}^{r-1}(-1)^u \binom{r-1}{u} \frac{(t)_p}{p!(n-r+1+u+p/\alpha)}.$$

That is

$$M_{X_{r:n}}(t) = C_{r:n}\sum_{p=0}^{\infty}\sum_{u=0}^{n-r}(-1)^u \binom{n-r}{u} \frac{(t)_p}{p!(r+u+p/\alpha)},$$

where

$$C_{r:n} = \frac{n!}{(r-1)!\ (n-r)!}.$$

ii) Setting $k=1$ in (16), we get the explicit expression for single moment generating function of lower record values from generalized exponential distribution, which verify the result of Raqab (2002).

A recurrence relation for single moment generating function for *lgos* from *df* (5) can be obtained in the following theorem.

Theorem 2.1. *For the distribution given in* (5) *and for* $2 \le r \le n$, $n \ge 2$ *and* $k = 1, 2 \ldots,$

$$\begin{aligned}\Big(1-\frac{t}{\alpha\gamma_r}\Big)M^{(j)}_{X^*(r,n,m,k)}(t) &= M^{(j)}_{X^*(r-1,n,m,k)}(t) + \frac{j}{\alpha\gamma_r}M^{(j-1)}_{X^*(r,n,m,k)}(t)\\ &-\frac{1}{\alpha\gamma_r}\{tM^{(j)}_{X^*(r,n,m,k)}(t+1) + jM^{(j-1)}_{X^*(r,n,m,k)}(t+1)\}.\end{aligned} \tag{17}$$

Proof. From (2), we have

$$M_{X^*(r,n,m,k)}(t) = \frac{C_{r-1}}{(r-1)!}\int_{-\infty}^{\infty} e^{tx}[F(x)]^{\gamma_r-1}\, f(x)\, g_m^{r-1}(F(x))dx. \tag{18}$$

Integrating by parts treating $[F(x)]^{\gamma_r-1}f(x)$ for integration and rest of the integrand for differentiation, we get

$$M_{X^*(r,n,m,k)}(t) = M_{X^*(r-1,n,m,k)}(t) - \frac{tC_{r-1}}{(r-1)!\gamma_r}\int_{-\infty}^{\infty} e^{tx}[F(x)]^{\gamma_r} g_m^{r-1}(F(x))dx,$$

the constant of integration vanishes since the integral considered in (18) is a definite integral. On using (6), we obtain

$$M_{X^*(r,n,m,k)}(t) = M_{X^*(r-1,n,m,k)}(t)$$
$$-\frac{tC_{r-1}}{(r-1)!\gamma_r}\int_0^\infty e^{tx}[F(x)]^{\gamma_r-1}\left\{\frac{(e^x-1)}{\alpha}f(x)\right\}g_m^{r-1}(F(x))dx$$
$$= M_{X^*(r-1,n,m,k)}(t) - \frac{t}{\alpha\gamma_r}\{M_{X^*(r,n,m,k)}(t+1) - M_{X^*(r,n,m,k)}(t)\}. \tag{19}$$

Differentiating both the sides of (19) j times with respect to t , we get

$$M^{(j)}_{X^*(r,n,m,k)}(t) = M^{(j)}_{X^*(r-1,n,m,k)}(t) - \frac{t}{\alpha\gamma_r}M^{(j)}_{X^*(r,n,m,k)}(t+1)$$
$$-\frac{j}{\alpha\gamma_r}M^{(j-1)}_{X^*(r,n,m,k)}(t+1) + \frac{t}{\alpha\gamma_r}M^{(j)}_{X^*(r,n,m,k)}(t) + \frac{t}{\alpha\gamma_r}M^{(j-1)}_{X^*(r,n,m,k)}(t).$$

The recurrence relation in equation (17) is derived simply by rewriting the above equation.

By differentiating both sides of equation (17) with respect to t and then setting $t = 0$, we obtain the recurrence relations for moments of *lgos* from generalized exponential distribution in the form

$$E[X^{*j}(r,n,m,k)] = E[X^{*j}(r-1,n,m,k)]$$
$$+\frac{j}{\alpha\gamma_r}\{E[X^{*j-1}(r,n,m,k)] - E[\phi(X^*(r,n,m,k))]\},$$

where
$\phi(x) = x^{j-1}e^x$.

Remark 2.2. *Putting $m = 0$, $k = 1$ in (17), we obtain the recurrence relation for single moment generating function of order statistics for generalized exponential distribution in the form*

$$\Big(1-\frac{t}{\alpha(n-r+1)}\Big)M^{(j)}_{X_{n-r+1:n}}(t) = M^{(j)}_{X_{n-r+2:n}}(t) + \frac{j}{\alpha(n-r+1)}M^{(j-1)}_{X_{n-r+1:n}}(t)$$
$$-\frac{1}{\alpha(n-r+1)}\{tM^{(j)}_{X_{n-r+1:n}}(t+1) + jM^{(j-1)}_{X_{n-r+1:n}}(t+1)\}.$$

Replacing $(n-r+1)$ by $(r-1)$, we have

$$M^{(j)}_{X_{r:n}}(t) = \Big(1-\frac{t}{\alpha(r-1)}\Big)M^{(j)}_{X_{r-1:n}}(t) - \frac{j}{\alpha(r-1)}M^{(j-1)}_{X_{r-1:n}}(t)$$
$$+\frac{1}{\alpha(r-1)}\{tM^{(j)}_{X_{r-1:n}}(t+1) + jM^{(j-1)}_{X_{r-1:n}}(t+1)\}.$$

For $r = r+1$, the result was obtained by Raqab (2004).

Remark 2.3. *Setting $m = -1$ and $k \geq 1$ in Theorem 2.1, we get a recurrence relation for single moment generating function of lower k record values for generalized exponential distribution in the form*

$$\Big(1-\frac{t}{\alpha k}\Big)M^{(j)}_{Z_r^{(k)}}(t) = M^{(j)}_{Z_{r-1}^{(k)}}(t) + \frac{j}{\alpha k}M^{(j-1)}_{Z_r^{(k)}}(t) - \frac{1}{\alpha k}\{tM^{(j)}_{Z_r^{(k)}}(t+1) + jM^{(j-1)}_{Z_{r-1}^{(k)}}(t+1)\}.$$

3. Relations for joint moment generating function

Before coming to main results we shall prove the following Lemmas.

Lemma 3.1. *For generalized exponential distribution as given in* (4) *and non-negative integers a, b and c with $m \neq -1$,*

$$J(a,0,c) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{(t_1)_q (t_2)_p}{p!\ q!\ (c+1+p/\alpha)[a+c+2+(p+q)/\alpha]}, \tag{20}$$

where

$$J(a,b,c) = \int_0^{\infty}\int_0^{x} e^{t_1 x + t_2 y}[F(x)]^a f(x) \times [h_m(F(y)) - h_m(F(x))]^b [F(y)]^c f(y) dy dx. \tag{21}$$

Proof. From (21), we have

$$J(a,0,c) = \int_0^{\infty} e^{t_1 x}[F(x)]^a f(x) G(x) dx, \tag{22}$$

where

$$G(x) = \int_0^{x} e^{t_2 y}[F(y)]^c f(y) dy. \tag{23}$$

By setting $z = [F(y)]^{1/\alpha}$ in (23), we get

$$G(x) = \sum_{p=0}^{\infty} \frac{(t_2)_p}{p!\ (c+1+p/\alpha)} [F(x)]^{c+1+p/\alpha}.$$

On substituting the above expression of $G(x)$ in (22), we find that

$$J(a,0,c) = \sum_{p=0}^{\infty} \frac{(t_2)_p}{p!\ (c+1+p/\alpha)} \int_0^{\infty} e^{t_1 x}[F(x)]^{a+c+1+p/\alpha} f(x) dx. \tag{24}$$

Again by setting $w = [F(x)]^{1/\alpha}$ in (24) and simplifying the resulting expression, we derive the relation given in (20).

Lemma 3.2. *For the distribution as given in* (4) *and any non-negative integers a, b and c ,*

$$J(a,b,c) = \frac{1}{(m+1)^b} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{v=0}^{b} (-1)^b \binom{b}{v} \frac{(t_1)_q}{p!\ q![c+v(m+1)+1+p/\alpha]} \times \frac{(t_2)_p}{[a+c+b(m+1)+2+(p+q)/\alpha]}, \quad m \neq -1 \tag{25}$$

$$= b! \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{(t_1)_q (t_2)_p}{p!\ q!(c+1+p/\alpha)^{b+1}[a+c+2+(p+q)/\alpha]}, \quad m = -1, \tag{26}$$

where $J(a,b,c)$ is as given in (21).

Proof. When $m \neq -1$, we have

$$[h_m(F(y)) - h_m(F(x))]^b = \frac{1}{(m+1)^b}[(F(x))^{m+1} - (F(y))^{m+1}]^b$$

$$= \frac{1}{(m+1)^b}\sum_{v=0}^{b}(-1)^v\binom{b}{v}[F(x)]^{(b-v)(m+1)}[F(y)]^{v(m+1)}.$$

Now substituting for $[h_m(F(y)) - h_m(F(x))]^b$ in (21), we get

$$J(a,b,c) = \frac{1}{(m+1)^b}\sum_{v=0}^{b}(-1)^v\binom{b}{v}J(a+(b-v)(m+1),0,c+v(m+1)).$$

Making use of Lemma 3.1, we derive the relation given in (25).

When $m = -1$, $J(a,b,c) = \frac{0}{0}$, as $\sum_{v=0}^{b}(-1)^v\binom{b}{v} = 0$, so after applying L' Hospital rule and (15), (26) can be proved on the lines of (16).

Theorem 3.3. *For the distribution as given in* (5) *and* $1 \le r < s \le n$, $k = 1, 2, ...,$ $m \neq 1$,

$$M_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2) = \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{u=0}^{r-1}(-1)^u\binom{r-1}{u}$$

$$\times J(m+u(m+1), s-r-1, \gamma_s - 1) \tag{27}$$

$$= \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}(-1)^{u+v}$$

$$\times\binom{r-1}{u}\binom{s-r-1}{v}\frac{(t_1)_q(t_2)_p}{p!\ q!(\gamma_{s-v}+p/\alpha)[\gamma_{r-u}+(p+q)/\alpha]}. \tag{28}$$

Proof. From (3), we have

$$M_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2) = \frac{C_{s-1}}{(r-1)!\ (s-r-1)!}\int_0^{\infty}\int_0^{x} e^{t_1x+t_2y}[F(x)]^m f(x)$$

$$\times g_m^{r-1}(F(x))[h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s-1}f(y)dydx. \tag{29}$$

On expanding $g_m^{r-1}(F(x))$ binomially in (29), we get the relation given in (27).

Making use of the Lemma 3.2 in (27), we derive the relation in (28).

Remark 3.4. *Putting* $m = 0, k = 1$ *in* (28)*, the explicit formula for joint moment generating functions of order statistics for generalized exponential distribution can be obtained as*

$$M_{X_{n-r+1,n-s+1:n}}(t_1,t_2) = C_{r,s:n}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}(-1)^{u+v}\binom{r-1}{u}\binom{s-r-1}{v}$$

$$\times\frac{(t_1)_q(t_2)_p}{p!q!\ (n-s+1+v+p/\alpha)[n-r+1+u+(p+q)/\alpha]}.$$

That is

$$M_{X_{r,s:n}}(t_1,t_2) = C_{r,s:n}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{u=0}^{n-s}\sum_{v=0}^{s-r-1}(-1)^{u+v}\binom{n-s}{u}\binom{s-r-1}{v}$$

$$\times\frac{(t_1)_q(t_2)_p}{p!q!\ (r+v+p/\alpha)[s+u+(p+q)/\alpha]},$$

where

$$C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}.$$

Remark 3.5. *Setting $m=-1$ in (28), we deduce the explicit expression for joint moment generating function of lower k record values for generalized exponential distribution in view of (27) and (26) in the form*

$$M_{(Z_r^{(k)},Z_s^{(k)})}(t_1,t_2) = k^s\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(t_1)_q(t_2)_p}{p!q!\ (k+p/\alpha)^{s-r}[k+(p+q)/\alpha]^r}$$

and hence for lower records

$$M_{X_{L(r)},X_{L(s)}}(t_1,t_2) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(t_1)_q(t_2)_p}{p!q!\ (1+p/\alpha)^{s-r}[1+(p+q)/\alpha]^r}.$$

Differentiating $M_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2)$ and evaluating at $t_1=t_2=0$, we get the product moments of *lgos* when $m\neq -1$

$$E[X^*(r,n,m,k)X^*(s,n,m,k)] = \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}$$

$$\times(-1)^{u+v}\binom{r-1}{u}\binom{s-r-1}{v}\frac{1}{p\ q\ (\gamma_{s-v}+p/\alpha)[\gamma_{r-u}+(p+q)/\alpha]}$$

and when $m=-1$ that

$$E(Z_r^{(k)}Z_s^{(k)}) = k^s\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{1}{p\ q\ (k+p/\alpha)^{s-r}[k+(p+q)/\alpha]^r}.$$

Making use of (6), we can derive the recurrence relations for joint moment generating function of *lgos* from (5).

Theorem 3.6. *For the distribution given in (5) and for $1\le r<s\le n$, $n\ge 2$ and $k=1,2,\ldots,$*

$$\Big(1-\frac{t_2}{\alpha\gamma_s}\Big)M^{(i,j)}_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2) = M^{(i,j)}_{X^*(r,n,m,k),X^*(s-1,n,m,k)}(t_1,t_2)$$

$$-\frac{1}{\alpha\gamma_s}\{t_2M^{(i,j)}_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2+1)+jM^{(i,j-1)}_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2+1)\}$$

$$+\frac{j}{\alpha\gamma_s}M^{(i,j-1)}_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2). \tag{30}$$

Proof. Using (3), the joint moment generating function of $X^*(r,n,m,k)$ and $X^*(s,n,m,k)$ is given by

$$M_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2) = \frac{C_{s-1}}{(r-1)!\ (s-r-1)!} \int_0^\infty [F(x)]^m f(x) \times g_m^{r-1}(F(x)) J(x) dx, \tag{31}$$

where

$$J(x) = \int_0^x e^{t_1 x + t_2 y} [h_m(F(y)) - h_m(F(x))]^{s-r-1} [F(y)]^{\gamma_s - 1} f(y) dy.$$

Solving the integral in $J(x)$ by parts and substituting the resulting expression in (31), we get

$$M_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2) = M_{X^*(r,n,m,k),X^*(s-1,n,m,k)}(t_1,t_2) - \frac{t_2\ C_{s-1}}{(r-1)!(s-r-1)!\gamma_s} \int_0^\infty \int_0^x e^{t_1 x + t_2 y} [F(x)]^m f(x) g_m^{r-1}(F(x)) \times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [F(y)]^{\gamma_s} dy dx,$$

the constant of integration vanishes since the integral in $J(x)$ is a definite integral. On using the relation (6), we obtain

$$M_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2) = M_{X^*(r,n,m,k),X^*(s-1,n,m,k)}(t_1,t_2) - \frac{t_2}{\alpha\gamma_s} M_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2+1) + \frac{t_2}{\alpha\gamma_s} M_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2). \tag{32}$$

Differentiating both sides of (32) i times with respect to t_1 and then j times with respect to t_2, we get

$$M^{(i,j)}_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2) = M^{(i,j)}_{X^*(r,n,m,k),X^*(s-1,n,m,k)}(t_1,t_2) - \frac{t_2}{\alpha\gamma_s} M^{(i,j)}_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2+1) - \frac{j}{\alpha\gamma_s} M^{(i,j-1)}_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2+1) + \frac{t_2}{\alpha\gamma_s} M^{(i,j)}_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2) + \frac{j}{\alpha\gamma_s} M^{(i,j-1)}_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2),$$

which, when rewritten gives the recurrence relation given in (30).
One can also note that Theorem 2.1 can be deduced from Theorem 3.6 by letting t_1 tends to zero.

Remark 3.7. *Putting $m = 0$, $k = 1$ in (30), we obtain the recurrence relations for joint moment generating function of order statistics for generalized exponential distribution in the form*

$$\Big(1 - \frac{t_2}{\alpha(n-s+1)}\Big) M^{(i,j)}_{X_{n-r+1,n-s+1:n}}(t_1,t_2) = M^{(i,j)}_{X_{n-r+1,n-s+2:n}}(t_1,t_2) + \frac{j}{\alpha(n-s+1)} M^{(i,j-1)}_{X_{n-r+1,n-s+1:n}}(t_1,t_2) - \frac{1}{\alpha(n-s+1)} \{t_2\ M^{(i,j)}_{X_{n-r+1,n-s+1:n}}(t_1,t_2+1) + j\ M^{(i,j-1)}_{X_{n-r+1,n-s+1:n}}(t_1,t_2+1)\}.$$

That is

$$M^{(i,j)}_{X_{r,s:n}}(t_1,t_2) = \Big(1 - \frac{t_1}{\alpha(r-1)}\Big) M^{(i,j)}_{X_{r-1,s:n}}(t_1,t_2) - \frac{i}{\alpha(r-1)} M^{(i-1,j)}_{X_{r-1,s:n}}(t_1,t_2) + \frac{1}{\alpha(r-1)} \{t_1\ M^{(i,j)}_{X_{r-1,s:n}}(t_1+1,t_2) + i\ M^{(i-1,j)}_{X_{r-1,s:n}}(t_1+1,t_2)\}$$

as obtained by Raqab (2004) for $r = r+1$.

Remark 3.8. *Substituting $m = -1$ and $k \geq 1$ in Theorem 3.6, we get recurrence relation for joint moment generating function of lower k record values for generalized exponential distribution.*

4. Characterization

Let $X^*(r,n,m,k)$, $r = 1,2,\ldots,n$ be *lgos* from a continuous population with *df* $F(x)$ and *pdf* $f(x)$, then the conditional *pdf* of $X^*(s,n,m,k)$ given $X^*(r,n,m,k) = x$, $1 \le r < s \le n$, in view of (2) and (3), is

$$f_{X^*(s,n,m,k)|X^*(r,n,m,k)}(y|x) = \frac{C_{s-1}}{(s-r-1)!C_{r-1}}[F(x)]^{m-\gamma_r+1}$$
$$\times[h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s-1}f(y). \tag{33}$$

Theorem 4.1. *Let X be a non negative random variable having an absolutely continuous distribution function $F(x)$ with $F(0) = 0$ and $0 < F(x) < 1$ for all $x > 0$, then*

$$E[e^{tX^*(s,n,m,k)}|X^*(l,n,m,k) = x] = \sum_{p=0}^{\infty}\frac{(t)_p(1-e^{-x})^p}{p!}\prod_{j=1}^{s-l}\Big(\frac{\gamma_{l+j}}{\gamma_{l+j}+p/\alpha}\Big),$$
$$l = r,\ r+1 \tag{34}$$

if and only if

$$F(x) = (1-e^{-x})^{\alpha},\quad x > 0,\quad \alpha > 0.$$

Proof. From (33), we have

$$E[e^{tX^*(s,n,m,k)}|X^*(r,n,m,k) = x] = \frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}$$
$$\times\int_0^x e^{ty}\ \Big[1-\Big(\frac{F(y)}{F(x)}\Big)^{(m+1)}\Big]^{s-r-1}\ \Big(\frac{F(y)}{F(x)}\Big)^{\gamma_s-1}\frac{f(y)}{F(x)}dy. \tag{35}$$

By setting $u = \frac{F(y)}{F(x)} = \left(\frac{1-e^{-y}}{1-e^{-x}}\right)^{\alpha}$ from (5) in (35), we obtain

$$E[e^{tX^*(s,n,m,k)}|X^*(r,n,m,k) = x] = B\int_0^1[1-(1-e^{-x})u^{1/\alpha}]^{-t}u^{\gamma_s-1}(1-u^{m+1})^{s-r-1}du$$
$$= B\sum_{p=0}^{\infty}\frac{(t)_p(1-e^{-x})^p}{p\ !}\int_0^1 u^{\gamma_s-1+p/\alpha}(1-u^{m+1})^{s-r-1}du, \tag{36}$$

where

$$B = \frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}.$$

Again by setting $v = u^{m+1}$ in (36), we get

$$E[e^{tX^*(s,n,m,k)}|X^*(r,n,m,k) = x] = \frac{B}{m+1}$$
$$\times\sum_{p=0}^{\infty}\frac{(t)_p(1-e^{-x})^p}{p!}\int_0^1 v^{\frac{k+p/\alpha}{m+1}+n-s-1}(1-v)^{s-r-1}dv$$
$$= \frac{B}{m+1}\sum_{p=0}^{\infty}\frac{(t)_p(1-e^{-x})^p\ \Gamma\Big(\frac{k+p/\alpha}{m+1}+n-s\Big)\Gamma(s-r)}{p!\ \Gamma\Big(\frac{k+p/\alpha}{m+1}+n-r\Big)}$$
$$= \frac{C_{s-1}}{C_{r-1}}\sum_{p=0}^{\infty}\frac{(t)_p(1-e^{-x})^p}{p!\ \prod_{j=1}^{s-r}(\gamma_{r+j}+p/\alpha)}$$

and hence the relation given in (34).

To prove sufficient part, we have from (33) and (34)

$$\frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}\int_0^x e^{ty}[(F(x))^{m+1}-(F(y))^{m+1}]^{s-r-1}$$
$$\times [F(y)]^{\gamma_s-1}f(y)dy=[F(x)]^{\gamma_{r+1}}H_r(x), \tag{37}$$

where

$$H_r(x)=\sum_{p=0}^{\infty}\frac{(t)_p(1-e^{-x})^p}{p!}\prod_{j=1}^{s-r}\Big(\frac{\gamma_{r+j}}{\gamma_{r+j}+p/\alpha}\Big).$$

Differentiating (37) both the sides with respect to x , we get

$$\frac{C_{s-1}[F(x)]^m f(x)}{(s-r-2)!C_{r-1}(m+1)^{s-r-2}}\int_0^x e^{ty}[(F(x))^{m+1}-(F(y))^{m+1}]^{s-r-2}$$

$$\times[F(y)]^{\gamma_s-1}f(y)dy=H_r'(x)[F(x)]^{\gamma_{r+1}}+\gamma_{r+1}H_r(x)[F(x)]^{\gamma_{r+1}-1}f(x)$$

or

$$\gamma_{r+1}H_{r+1}(x)[F(x)]^{\gamma_{r+2}+m}f(x)=H_r'(x)[F(x)]^{\gamma_{r+1}}+\gamma_{r+1}H_r(x)[F(x)]^{\gamma_{r+1}-1}f(x),$$

where

$$H_r'(x)=\sum_{p=0}^{\infty}\frac{(t)_p(1-e^{-x})^{p-1}e^{-x}}{p!}\prod_{j=1}^{s-r}\Big(\frac{\gamma_{r+j}}{\gamma_{r+j}+p/\alpha}\Big)$$

and

$$H_{r+1}(x)-H_r(x)=\sum_{p=0}^{\infty}\frac{(t)_p(1-e^{-x})^p\ (p/\alpha)}{p!\ \gamma_{r+1}}\prod_{j=1}^{s-r}\Big(\frac{\gamma_{r+j}}{\gamma_{r+j}+p/\alpha}\Big).$$

Therefore,

$$\frac{f(x)}{F(x)}=\frac{H_r'(x)}{\gamma_{r+1}[H_{r+1}(x)-H_r(x)]}=\frac{\alpha e^{-x}}{1-e^{-x}}$$

which proves that

$$F(x)=(1-e^{-x})^{\alpha},\quad x>0,\alpha>0.$$

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