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Kiefer's law of the iterated logarithm for r^{th} upper extreme of a random number of random variables

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Abstract. Let $(X_n)_{n\geq 1}$ be a sequence of independent and identically distributed random variables, defined over a common probability space $(\Omega, \mathcal{F}, P)$ with a continuous distribution function F. Let $M_{r,n}$ denote the r^{th} upper extreme among $(X_1, X_2, \ldots, X_n), n \geq 1$. For a large class of distributions, we obtain Kiefer's form of law of the iterated logarithm for (M_{r,τ_k}), properly normalized, where (τ_k) is a sequence of integer valued random variables.

1. Introduction

Let $(X_n)_{n\geq 1}$ be a sequence of independent and identically distributed (i.i.d.) random variables (r.v.s.) defined over a common probability space $(\Omega, \mathcal{F}, P)$ and let the common distribution function (d.f.) F be continuous. Denote the right extremity of F by $\omega(F)$ and note that $\omega(F) = \infty$ if $F(x) < 1$, for all x real.

Let (τ_k) denote an increasing sequence of integer valued r.v.s. defined on the same probability space. Assume that there exists a non-decreasing subsequence (n_k) of positive integers (natural numbers) such that $\frac{\tau_k}{n_k} \to 1$ almost surely $(a.s.)$ as $k \to \infty$.

For any $n \geq 1$, let $X_{1,n} \leq X_{2,n} \leq X_{3,n} \leq \ldots \leq X_{n,n}$ denote the order statistics of $X_1, X_2, \ldots, X_n$. Then, $X_{n-r+1,n}$ stands for r^{th} upper order statistic or the r^{th} upper extreme. In this paper, we denote $X_{n-r+1,n}$ by $M_{r,n}$, so that $M_{1,n}$ stands for the partial maxima.

Under the setup of de Haan and Hordijk (1972), Hüsler (1985) obtained the law of iterated logarithm (L.I.L.) for $(M_{1,n})$ over subsequence (n_k) of integers, which is either atmost geometrically increasing or atleast geometrically increasing. He showed that if

$$\liminf_{k\to\infty} \frac{n_{k-1}}{n_k} > 0$$

(atmost geometrically increasing), then the limit superior of (M_{1,n_k}), properly normalized, is the same as for $(M_{1,n})$ and if

$$\liminf_{k\to\infty} \frac{n_{k-1}}{n_k} = 0$$

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(atleast geometrically increasing), then the limit superior of (M_{1,n_k}), properly normalized, will be a function of γ', where

$$\gamma' = \inf\{y > 0 : \sum_k (\log n_k)^{-y} < \infty\}.$$

As noted by de Haan and Hordijk (1972), some distributions belonging to the domain of attraction of Gumbel law satisfy the conditions imposed in their paper. However, Vasudeva and Savitha (1992) observed that distribution functions ($d.f.s$) belonging to the domain of attraction of Fréchet law or Weibull law donot come under the setup of de Haan and Hordijk (1972). Assuming that the common d.f. F belongs to the domain of attraction of Fréchet law, they obtained the L.I.L. for (M_{1,τ_k}) over random subsequences (τ_k) such that $\frac{\tau_k}{n_k} \to 1$ $a.s.$, with (n_k) either atmost geometrically increasing or atleast geometrically increasing.

In the case of partial sums, Gut (1986) has established L.I.L. for subsequences, under a similar setup, and the same has been extended to random subsequences by Torrang (1987).

When

$$\lim_{k\to\infty} \frac{n_{k+1}}{n_k} = \infty$$

(n_k is rapidly increasing), Gut and Schwabe (1996) obtained the classical L.I.L. of Hartman and Wintner (1941) over (n_k) by replacing $\log\log n_k$ by $\log k$.

Vasudeva and Divanji (2006), obtained a L.I.L. for (M_{1,n_k}) assuming that the common d.f. F belongs to the domain of attraction of Fréchet law, and (n_k) is rapidly increasing, with $\log k$ in the place of $\log\log n_k$.

When the common d.f. is Uniform $(0, 1)$, Kiefer (1971) has proved that

$$\liminf_{n\to\infty} \frac{\log(n(1 - M_{r,n}))}{\log\log n} = \frac{-1}{r} \; a.s. \, ,$$

which gives an a.s. lower bound for $\log(1 - M_{r,n})$.

The above law can be equivalently given as

$$\liminf_{n\to\infty} (n(1 - M_{r,n}))^{\frac{1}{\log\log n}} = e^{-\frac{1}{r}} \; a.s. \, . \tag{1}$$

This form of the law of the iterated logarithm with $(\log\log n)^{-1}$ in the power, was established by Chover (1967) for partial sum S_n of symmetric stable random variables.

Hall (1979), has extended Kiefer's law to a large class of d.f.s. His setup includes all d.f.s F with $-\log \overline{F}$ regularly varying where $\overline{F} = 1 - F$.

Vasudeva and Moridani (2011) have studied Kiefer's L.I.L. for vector of upper order statistics and have extended the results for distributions with $-\log \overline{F}$ regularly varying; for those with $\overline{F}$ regularly varying and for d.f.s with finite right extremity. They called the class of all d.f.s with $-\log \overline{F}$ regularly varying as C_1; d.f.s with $\overline{F}$ regularly varying as class C_2 and d.f.s with right extremity finite and belonging to the domain of attraction of a Weibull law as class C_3.

In this paper, we establish Kiefer's L.I.L. for (M_{r,τ_k}), when the d.f. F belongs to the classes C_1, C_2 and C_3.

In section 2 we give the statements of the results and make some interesting observations. In the next section, some Lemmas are proved when the d.f. F is Uniform $(0, 1)$. In the last section, all the results stated in section 2 are established by a transformation technique.

From Hüsler (1985), Vasudeva and Savitha (1992) and Vasudeva and Divanji (2006), we note that L.I.L. results for (M_{r,τ_k}) with $\frac{\tau_k}{n_k} \to 1$ *a.s.* will be meaningful under further assumptions that (n_k) is atmost geometrically fast or (n_k) is atleast geometrically fast/rapidly increasing.

Throughout the paper, we will be obtaining the L.I.L. results under the following setup (assumptions).

$(A.)$ Given the sequence (τ_k) of integer valued r.v.s., there always exists a subsequence (n_k) of natural numbers such that $\frac{\tau_k}{n_k} \to 1$ *a.s.* as $k \to \infty$.

$(B_1.)$ The sequence (n_k) in (A) satisfies
$$\liminf_{k\to\infty} \frac{n_{k+1}}{n_k} > 1.$$

$(B_2.)$ The sequence (n_k) in (A) satisfies
$$\limsup_{k\to\infty} \frac{n_{k+1}}{n_k} < \infty.$$

$(B_3.)$ The sequence (n_k) in (A) satisfies
$$\lim_{k\to\infty} \frac{n_{k+1}}{n_k} = \infty.$$

By Gut (2009), page 172, one may note that the a.s. convergence condition in (A) is necessary in establishing L.I.L. (a.s. results) for random number of r.v.s. .

In the sequel, i.o., a.s. and s.v. respectively mean infinitely often, almost surely and slowly varying. c, ϵ, k and n, with or without a suffix, denote positive constants with k and n confined to be integers.

2. Main results

In this section, we present L.I.L. for (M_{r,τ_k}), properly normalized, for d.f.s F which belong to three major classes in extreme value theory, denoted for convenience by C_1, C_2, and C_3. The class C_1 is that of all d.f.s F with $-\log \overline{F}(x) = x^{\gamma} L(x)$, as $x \to \infty$, where $\gamma > 0$ is some constant and $L(x)$ is a slowly varying function. This class contains distributions with Weibullian right tail (which include Exponential, Gumbel, Normal etc.). By Pakes (2000) we note that distributions with Weibullian tail belong to the domain of attraction of Gumbel law, when $0 < \gamma < 1$ $(F \in DA.(H_3))$. Also, we note that when F is Normal $(\gamma = 2)$ or Exponential $(\gamma = 1)$, $\{M_{1,n}\}$ properly normalized converges to a Gumbel r.v. (see Galambus (1978)). The class C_2 is that of d.f.s F with $\overline{F}(x) = x^{-\gamma} L(x)$, as $x \to \infty$, where $\gamma > 0$ is a constant and $L(x)$ is a slowly varying function. It is well known that C_2 is the class of all d.f.s which belong to the domain of attraction of Fréchet law, denoted by $F \in DA.(H_{1,\gamma})$ (see, Galambos (1978)). C_3 is the class of all d.f.s F (with finite right extremity) belonging to the domain of attraction of a Weibull law, denoted by $F \in DA.(H_{2,\gamma})$. Throughout this section, let B_n be a solution of the equation, $n(1 - F(x)) = 1$.

2.1. Law of the iterated logarithm when $F \in C_1$

. Define,

$$U(x) = -\log(1 - F(x)), x > 0$$

and denote its inverse function by V. When $U(x) = x^{\gamma} L(x)$, by Hall (1979), note that for all functions $a(.)$ with $0 \neq a(x) \to 0$, as $x \to \infty$,

$$\frac{V\left(x(1 + a(x))\right) - V(x)}{a(x)V(x)} \to \gamma^{-1} \; as \; x \to \infty \tag{2}$$

which implies that V is continuous for all large x, and regularly varying with exponent γ^{-1}. We have, the following theorem

Theorem 2.1. 1. *If (τ_k) and (n_k) satisfy (A) and (B_1) then,*

$$\limsup_{k\to\infty} \frac{\log n_k}{(\log\log n_k)}\left(\frac{M_{r,\tau_k}}{V(\log n_k)} - 1\right) = \frac{c}{r\gamma} \quad a.s. \, ,$$

where

$$c = \inf\{d : \sum_k \frac{1}{(\log n_k)^d} < \infty\}.$$

2. *If (τ_k) and (n_k) satisfy (A) and (B_2) then,*

$$\limsup_{k\to\infty} \frac{\log n_k}{\log\log n_k}\left(\frac{M_{r,\tau_k}}{V(\log n_k)} - 1\right) = \frac{1}{r\gamma} \; a.s. \, .$$

3. *If (τ_k) and (n_k) satisfy (A) and (B_3) then,*

$$\limsup_{k\to\infty} \frac{\log n_k}{\log k}\left(\frac{M_{r,\tau_k}}{V(\log n_k)} - 1\right) = \frac{1}{r\gamma} \quad a.s. \, .$$

Theorem 2.2. *Whenever (τ_k), and (n_k) satisfy (A) and $n_k \to \infty$ as $k \to \infty$, then*

$$\liminf_{k\to\infty} \frac{\log n_k}{\log\log n_k}\left(\frac{M_{r,\tau_k}}{V(\log n_k)} - 1\right) = 0 \; a.s. \, .$$

Now, we consider standard probability distributions and study the L.I.L., when $n_k = 2^k$, $n_k = 2^{k^2}$ and $n_k = k^k$.

Example 2.3. *Let $\{X_n\}$ be a sequence of i.i.d. unit exponential r.v.s. . Then, one can see that $V(x) = x$ and $\gamma = 1$. By Theorem 2.1, we have*

1. *for $n_k = 2^{k^2}$,*

$$\limsup_{k\to\infty}\left(\frac{M_{r,\tau_k} - k^2\log 2}{\log k}\right) = \frac{1}{r} \; a.s. \, .$$

2. *for $n_k = 2^k$,*

$$\limsup_{k\to\infty}\left(\frac{M_{r,\tau_k} - k\log 2}{\log k}\right) = \frac{1}{r} \; a.s. \, .$$

3. *for $n_k = k^k$,*

$$\limsup_{k\to\infty}\left(\frac{M_{r,\tau_k} - k\log k}{\log k}\right) = \frac{1}{r} \; a.s. \, .$$

Example 2.4. *When $\{X_n\}$ is i.i.d. with common d.f. $F(x) = e^{-e^{-x}}, -\infty < x < \infty$, note that, $1 - F(x) \sim e^{-x}$, as $x \to \infty$. Consequently, the L.I.L. coincides with that obtained in the case of unit exponential distribution.*

Example 2.5. *Let $\{X_n\}$ be i.i.d. standard normal. Then, one gets $\gamma = \sqrt{2}$ and $V(x) = \sqrt{2x - \log x - 2\log\sqrt{2\pi} - \log 2} \simeq \sqrt{2x}$ for large x. We have,*

1. *for $n_k = 2^{k^2}$,*

$$\limsup_{k\to\infty}\left(\frac{k\,(M_{r,\tau_k} - V(\log n_k))}{\log k}\right) = \frac{1}{r\sqrt{\log 2}} \; a.s. \, .$$

2. *for $n_k = 2^k$,*

$$\limsup_{k\to\infty}\left(\frac{\sqrt{k}\,(M_{r,\tau_k} - V(\log n_k))}{\log k}\right) = \frac{1}{r\sqrt{\log 2}} \; a.s. \, .$$

3. *for $n_k = k^k$,*

$$\limsup_{k\to\infty}\left(\frac{\sqrt{k}\,(M_{r,\tau_k} - V(\log n_k))}{\sqrt{\log k}}\right) = \frac{1}{r} \; a.s. \, .$$

2.2. Law of the iterated logarithm when $F \in C_2$

. Define,

$$U^*(x) = 1 - F(x), x > 0$$

and note that $U^*(x) = x^{-\gamma} L(x)$, where $\gamma > 0$, is some constant and L is a slowly varying function. Let V^* be the inverse of U^*. Observe that,

$$V^*(y) = y^{-\frac{1}{\gamma}}\, l\left(\frac{1}{y}\right), 0 < y \leq 1, \tag{3}$$

where l is slowly varying. Note that $U^*(.)$ and $V^*(.)$ are decreasing functions and that B_n is a solution of the equation $nU^*(x) = 1$. By (3), one can see that $B_n = n^{\frac{1}{\gamma}} l(n)$.

Theorem 2.6. 1. *If (τ_k) and (n_k) satisfy (A) and (B_1) then,*

$$\limsup_{k\to\infty} \left(\frac{M_{r,\tau_k}}{B_{n_k}}\right)^{\frac{1}{\log\log n_k}} = e^{\frac{c}{r\gamma}} \ a.s. ,$$

where

$$c = \inf\{d : \sum_k \frac{1}{(\log n_k)^d} < \infty\}.$$

2. *If (τ_k) and (n_k) satisfy (A) and (B_2) then,*

$$\limsup_{k\to\infty} \left(\frac{M_{r,\tau_k}}{B_{n_k}}\right)^{\frac{1}{\log\log n_k}} = e^{\frac{1}{r\gamma}} \ a.s.$$

3. *If (τ_k) and (n_k) satisfy (A) and (B_3) then,*

$$\limsup_{k\to\infty} \left(\frac{M_{r,\tau_k}}{B_{n_k}}\right)^{\frac{1}{\log k}} = e^{\frac{1}{r\gamma}} \quad a.s. .$$

Theorem 2.7. *Whenever (τ_k), and (n_k) satisfy (A) and $n_k \to \infty$ as $k \to \infty$,*

$$\liminf_{k\to\infty} \left(\frac{M_{r,\tau_k}}{B_{n_k}}\right)^{\frac{1}{\log\log n_k}} = 1 \ a.s. .$$

Example 2.8. *Let $\{X_n\}$ be i.i.d. Pareto with $F(x) = 1 - \frac{1}{x^\gamma}$ if $x > 0, = 0$ otherwise, $\gamma > 0$. Observe that, $B_n \simeq n^{\frac{1}{\gamma}}$. By Theorem 2.6, we have*

1. *for $n_k = 2^{k^2}$,*

$$\limsup_{k\to\infty} \left(\frac{M_{r,\tau_k}}{n_k^{\frac{1}{\gamma}}}\right)^{\frac{1}{\log k}} = e^{\frac{1}{r\gamma}} \ a.s. .$$

2. *for $n_k = 2^k$,*

$$\limsup_{k\to\infty} \left(\frac{M_{r,\tau_k}}{n_k^{\frac{1}{\gamma}}}\right)^{\frac{1}{\log k}} = e^{\frac{1}{r\gamma}} \ a.s. .$$

3. *for $n_k = k^k$,*

$$\limsup_{k\to\infty} \left(\frac{M_{r,\tau_k}}{n_k^{\frac{1}{\gamma}}}\right)^{\frac{1}{\log k}} = e^{\frac{1}{r\gamma}} \ a.s. .$$

Remark 2.9. *When F is Fréchet with parameter γ, the d.f. is given by $F(x) = e^{-\frac{1}{x^\gamma}}, x > 0$. One can see that $B_n \simeq n^{\frac{1}{\gamma}}$. Hence, the L.I.L. results of example 2.8 hold good, when (X_n) is i.i.d. with Fréchet distribution (parameter γ).*

Example 2.10. *Let $\{X_n\}$ be i.i.d. Burr with $F(x) = 1 - \frac{1}{(1+x^p)^\gamma}, x > 0, = 0$ otherwise, $p > 0, \gamma > 0$. Note that, $1 - F(x) \simeq \frac{1}{x^{p\gamma}}$. Note that, $B_n \simeq n^{\frac{1}{p\gamma}}$*

1. *for* $n_k = 2^{k^2}$,
$$\limsup_{k\to\infty}\left(\frac{M_{r,\tau_k}}{n_k^{\frac{1}{\gamma}}}\right)^{\frac{1}{\log k}} = e^{\frac{1}{r\gamma}} \ a.s. .$$
2. *for* $n_k = 2^k$,
$$\limsup_{k\to\infty}\left(\frac{M_{r,\tau_k}}{n_k^{\frac{1}{\gamma}}}\right)^{\frac{1}{\log k}} = e^{\frac{1}{r\gamma}} \ a.s. .$$
3. *for* $n_k = k^k$,
$$\limsup_{k\to\infty}\left(\frac{M_{r,\tau_k}}{n_k^{\frac{1}{\gamma}}}\right)^{\frac{1}{\log k}} = e^{\frac{1}{r\gamma}} \ a.s. .$$

2.3. Law of the iterated logarithm when $F \in C_3$

. Let $\{X_n\}$ be i.i.d. with common d.f. F and let $\omega(F) = \sup\{x : F(x) < 1\}$ be finite. Suppose that F belongs to the domain of attraction of the Weibull law ie., $F\varepsilon DA.(H_{2,\gamma}), \gamma > 0$. Let $M'_{r,n}$ be the r^{th} maxima of $(Z_1, Z_2, \ldots, Z_n)$ where $\{Z_n\}$ are i.i.d. r.v.s. given by
$$Z_n = \frac{1}{\omega(F) - X_n}, n \geq 1. \tag{4}$$
Let F^* denote the d.f. of $Z_n, n \geq 1$. Note that $F^*\varepsilon DA.(H_{1,\gamma})$ (for details, see Galambos (1978)). Define $U^* = 1 - F^*$. Let V^* denote the inverse. Then, we have the following theorem.

Theorem 2.11. 1. *If* (τ_k) *and* (n_k) *satisfy* (A) *and* (B_1) *then,*
$$\liminf_{k\to\infty}\left(B_{n_k}\left(\omega(F) - M_{r,\tau_k}\right)\right)^{\frac{1}{\log\log n_k}} = e^{\frac{-c}{r\gamma}} \ a.s. ,$$
where
$$c = \inf\{d : \sum_k \frac{1}{(\log n_k)^d} < \infty\}.$$

2. *If* (τ_k) *and* (n_k) *satisfy* (A) *and* (B_2) *then,*
$$\liminf_{k\to\infty}\left(B_{n_k}\left(\omega(F) - M_{r,\tau_k}\right)\right)^{\frac{1}{\log\log n_k}} = e^{\frac{-1}{r\gamma}} \ a.s. .$$
3. *If* (τ_k) *and* (n_k) *satisfy* (A) *and* (B_3) *then,*
$$\liminf_{k\to\infty}\left(B_{n_k}\left(\omega(F) - M_{r,\tau_k}\right)\right)^{\frac{1}{\log k}} = e^{\frac{-1}{r\gamma}} \quad a.s. .$$

Theorem 2.12. *Whenever* (τ_k), *and* (n_k) *satisfy* (A) *and* $n_k \to \infty$ *as* $k \to \infty$,
$$\limsup_{k\to\infty}\left(B_{n_k}\left(\omega(F) - M_{r,\tau_k}\right)\right)^{\frac{1}{\log\log n_k}} = 1 \ a.s. .$$

Example 2.13. *Let* $\{X_n\}$ *be i.i.d. Weibull with d.f.* $F(x) = e^{-(-x^{\gamma})}$ *if* $x < 0$, $= 1$ *otherwise,* $\gamma > 0$, *we have* $B_n \simeq n^{\frac{1}{\gamma}}$. *Then,*

1. *for* $n_k = 2^{k^2}$,
$$\liminf_{k\to\infty}\left(n_k^{\frac{1}{\gamma}}\left(-M_{r,\tau_k}\right)\right)^{\frac{1}{\log k}} = e^{\frac{-1}{r\gamma}} \quad a.s. .$$
2. *for* $n_k = 2^k$,
$$\liminf_{k\to\infty}\left(n_k^{\frac{1}{\gamma}}\left(-M_{r,\tau_k}\right)\right)^{\frac{1}{\log k}} = e^{\frac{-1}{r\gamma}} \ a.s. .$$
3. *for* $n_k = k^k$,
$$\liminf_{k\to\infty}\left(n_k^{\frac{1}{\gamma}}\left(-M_{r,\tau_k}\right)\right)^{\frac{1}{\log k}} = e^{\frac{-1}{r\gamma}} \quad a.s. .$$

3. Results for Uniform population

In this section, we extend Kiefer's law in (1) to random number of r.v.s. . Suppose that $\{U_n\}$ is a sequence of i.i.d. Uniform $(0,1)$ r.v.s. defined over the same probability space $(\Omega, \mathcal{F}, P)$ on which $\{X_n\}$ is defined. Let $M^*_{r,n}$ stand for the r^{th} upper extreme among $(U_1, U_2, \ldots, U_n), n \geq 2$, so that M^*_{r,τ_k} is the r^{th} upper extreme among $(U_1, U_2, \ldots, U_{\tau_k})$. We first establish a Lemma giving necessary bounds that are applied in the subsequent Lemmas.

Lemma 3.1. *Given $\theta > 0, \epsilon > 0, \epsilon' > 0$ with $\epsilon < \theta$ and $\epsilon' < 1$, one can find constants $C_1, C_2 > 0$ and integer $n_0 > 0$ such that for all $n \geq n_0$,*

$$P\left(M^*_{r,[(1+\epsilon')n]} > 1 - \frac{1}{n(\log n)^{\frac{\theta+\epsilon}{r}}}\right) \leq \frac{C_1}{(\log n)^{\theta+\epsilon}} \tag{5}$$

and

$$P\left(M^*_{r,[(1-\epsilon')n]} > 1 - \frac{1}{n(\log n)^{\frac{\theta-\epsilon}{r}}}\right) \geq \frac{C_2}{(\log n)^{\theta-\epsilon}}. \tag{6}$$

Proof. To show (5), put $[(1+\epsilon')n] = n'$. Then,

$$P\left(M^*_{r,n'} > 1 - \frac{1}{n(\log n)^{\frac{\theta+\epsilon}{r}}}\right) = 1 - P\left(M^*_{r,n'} \leq 1 - \frac{1}{n(\log n)^{\frac{\theta+\epsilon}{r}}}\right).$$

Let $\beta_n = \dfrac{1}{n(\log n)^{\frac{\theta+\epsilon}{r}}}$. Then,

$$\begin{aligned} P\left(M^*_{r,n'} > 1 - \beta_n\right) =& 1 - \sum_{k=0}^{r-1} \binom{n'}{k} \left(1 - F(1-\beta_n)\right)^k \left(F(1-\beta_n)\right)^{n'-k} \\ =& 1 - \sum_{k=0}^{r-1} \binom{n'}{k} (\beta_n)^k (1-\beta_n)^{n'-k} \end{aligned}$$

Expanding $(1-\beta_n)^{n'-k}$ using Taylor's theorem, one gets

$$\begin{aligned} P\left(M^*_{r,n'} > 1 - \beta_n\right) = 1 - \sum_{k=0}^{r-1}\sum_{l=0}^{r-k} \binom{n'}{k}\binom{n'-k}{l}(-1)^l \beta_n^{k+l} \\ - \sum_{k=0}^{r-1} d_k \binom{n'}{k}\binom{n'-k}{r+1-k}\beta_n^{r+1} \end{aligned}$$

$$= 1 - T_{1,n} - T_{2,n}, \text{ say,} \tag{7}$$

where $d_k = (-1)^{r-k-1}(1-d^*_k)^{r-k-1}$ with $d^*_k \in (0, \beta_n)$.
Consider,

$$T_{1,n} = \sum_{k=0}^{r-1}\sum_{l=0}^{r-k} \binom{n'}{k}\binom{n'-k}{l}(-1)^l \beta_n^{k+l}.$$

Observe that the term of $T_{1,n}$ corresponding to $k = l = 0$ is 1.

For $k+l=r$, since $0 \le k \le r-1$, $(\beta_n)^r$ will have coefficients

$$\binom{n'}{0}\binom{n'}{r}(-1)^r + \binom{n'}{1}\binom{n'-1}{r-1}(-1)^{r-1} .. + \binom{n'}{r-1}\binom{n'-r+1}{1}(-1)$$

$$= \frac{n'!}{r!(n'-r)!}\left(\frac{(-1)^r r!}{r!} + \frac{(-1)^{r-1}r!}{(r-1)!} + \ldots + \frac{(-1)r!}{(r-1)!} + \frac{r!}{r!} - 1\right) = -\binom{n'}{r}.$$

On similar lines, one can see that the terms of $T_{1,n}$ with $1 \le j+i \le r-1$ will be 0. Consequently,

$$T_{1,n} = 1 - \binom{n'}{r}(\beta_n)^r. \tag{8}$$

Consider,

$$T_{2,n} = \sum_{j=0}^{r-1} d_j \binom{n'}{j}\binom{n'-j}{r+1-j}\beta_n^{r+1}.$$

Then,

$$|T_{2,n}| \ge r(n'-r-1)^{r+1}(\beta_n)^{r+1}.$$

From (8), we have

$$P\left(M^*_{r,n'} > \beta_n\right) = \binom{n'}{r}(\beta_n^r) - T_{2,n}.$$

Note that,

$$\binom{n'}{r}(\beta_n)^r \le \frac{(1+\epsilon)^r}{(\log n)^{\theta+\epsilon}} \text{ and } T_{2,n} = o\left(\frac{1}{(\log n)^{\theta+\epsilon}}\right).$$

Consequently, one can find a $c_1 > 0$ and $n_0 > 0$ such that for all $n \ge n_0$,

$$P\left(M^*_{r,n} > 1 - \frac{1}{n(\log n)^{\frac{\theta+\epsilon}{r}}}\right) \le \frac{c_1}{(\log n)^{\theta+\epsilon}}.$$

One can establish (6) on similar lines. The details are omitted. □

Lemma 3.2. *Let (τ_k) and (n_k) satisfy (A) and (B_1). Then for $r \ge 1$,*

$$\liminf_{k\to\infty}\left(n_k\left(1 - M^*_{r,\tau_k}\right)\right)^{\frac{1}{\log\log n_k}} = e^{-\frac{c}{r}} \quad a.s.\,, \tag{9}$$

where

$$c = \inf\{d : \sum_k \frac{1}{(\log n_k)^d} < \infty\}.$$

Proof. In order to prove (9), it is sufficient if one shows that for $0 < \epsilon < c$,

$$P\left(\left(n_k\left(1 - M^*_{r,\tau_k}\right)\right)^{\frac{1}{\log\log n_k}} < e^{-\frac{(c+\epsilon)}{r}} \; i.o.\right) = 0 \tag{10}$$

and

$$P\left(\left(n_k\left(1 - M^*_{r,\tau_k}\right)\right)^{\frac{1}{\log\log n_k}} < e^{-\frac{(c-\epsilon)}{r}} \; i.o.\right) = 1. \tag{11}$$

Note that,

$$P\left(\left(n_k\left(1 - M^*_{r,\tau_k}\right)\right)^{\frac{1}{\log\log n_k}} < e^{\frac{-(c+\epsilon)}{r}} \; i.o.\right) = P\left(M^*_{r,\tau_k} > 1 - \frac{1}{n_k(\log n_k)^{\frac{(c+\epsilon)}{r}}} \; i.o.\right).$$

Let

$$x_k^{'} = 1 - \frac{1}{n_k(\log n_k)^{\frac{(c+\epsilon)}{r}}} \text{ and } A_k = \left(M^*_{r,\tau_k} > x_k^{'}\right).$$

For any $\epsilon' > 0$, one can write

$$\begin{aligned} P(A_k \ i.o.) =& P\left(\left(M^*_{r,\tau_k} > x_k^{'}\right) \cap \left(\left|\frac{\tau_k}{n_k} - 1\right| < \epsilon' \cup \left|\frac{\tau_k}{n_k} - 1\right| \geq \epsilon'\right) \ i.o.\right) \\ =& P\left(\left(M^*_{r,\tau_k} > x_k^{'}\right) \cap \left(\left|\frac{\tau_k}{n_k} - 1\right| < \epsilon'\right) \ i.o.\right) \\ &+ P\left(\left(M^*_{r,\tau_k} > x_k^{'}\right) \cap \left(\left|\frac{\tau_k}{n_k} - 1\right| \geq \epsilon'\right) \ i.o.\right) \\ =& P(B_k \ i.o.) + P(C_k^* \ i.o.), \text{ say.} \end{aligned}$$

Note that,

$$(C_k^* \ i.o.) \subseteq \left(\left|\frac{\tau_k}{n_k} - 1\right| \geq \epsilon' \ i.o.\right) \text{ and } P\left(\left|\frac{\tau_k}{n_k} - 1\right| \geq \epsilon' \ i.o.\right) = 0.$$

Hence, $P(A_k \ i.o.) = P(B_k \ i.o.)$. Also,

$$\begin{aligned} (B_k \ i.o.) =& \left(M^*_{r,\tau_k} > x_k^{'},\ [n_k(1-\epsilon')] < \tau_k < [n_k(1+\epsilon')] \ i.o.\right) \\ \subseteq& \left(M^*_{r,[n_k(1+\epsilon')]} > x_k^{'},\ [n_k(1-\epsilon')] < \tau_k < [n_k(1+\epsilon')] \ i.o.\right) \end{aligned}$$

The fact that $\frac{\tau_k}{n_k} \to 1 \ a.s.$ implies that $[n_k(1-\epsilon')] < \tau_k < [n_k(1+\epsilon')] \ a.s.$.
As such,

$$\begin{aligned} & P\left(M^*_{r,[n_k(1+\epsilon')]} > x_k^{'}, [n_k(1-\epsilon')] < \tau_k < [n_k(1+\epsilon')] \ i.o\right) \\ =& P\left(M^*_{r,[n_k(1+\epsilon')]} > x_k' \ i.o.\right) = P\left(D_k \ i.o.\right), \text{ say.} \end{aligned}$$

Consequently, one gets from the above discussion

$$P(A_k \ i.o.) = P(B_k \ i.o.) \leq P(D_k \ i.o.).$$

To show $P(D_k \ i.o.) = 0$, define $n_k' = [(1+\epsilon')n_k]$. Then,

$$P\left(D_k\right) = P\left(M^*_{r,n_k'} > x_k^{'}\right) = P\left(M^*_{r,n_k'} > 1 - \frac{1}{n_k(\log n_k)^{\frac{c+\epsilon}{r}}}\right). \tag{12}$$

From Lemma 3.1 one can find $c_2 > 0$ and $k_1 > 0$ such that for all $k \geq k_1$,

$$P\left(M^*_{r,n_k'} > 1 - \frac{1}{n_k(\log n_k)^{\frac{(c+\epsilon)}{r}}}\right) \leq \frac{c_2}{(\log n_k)^{c+\epsilon}}.$$

Since $\sum_k \frac{1}{(\log n_k)^{c+\epsilon}} < \infty$, from Borel-Cantelli lemma, one gets $P(D_k \ i.o.) = 0$ which in turn implies that $P(A_k \ i.o.) = 0$.

In order to show (11), Consider

$$P\left(\left(n_k\left(1 - M^*_{r,\tau_k}\right)\right)^{\frac{1}{\log\log n_k}} < e^{-\frac{(c-\epsilon)}{r}} \ i.o.\right) = P\left(M^*_{r,\tau_k} > 1 - \frac{1}{(\log n_k)^{\frac{(c-\epsilon)}{r}}} \ i.o.\right).$$

Let $x_k^{''} = 1 - \dfrac{1}{(\log n_k)^{\frac{(c-\epsilon)}{r}}}$ and define $A_k' = \left(M^*_{r,\tau_k} > x_k''\right)$.

By proceeding as above one can show that

$$\begin{aligned} P\left(A_k' \; i.o.\right) =& P\left(M^*_{r,\tau_k} > x_k'', [n_k(1-\epsilon')] < \tau_k < [n_k(1+\epsilon')] \; i.o.\right) \\ \geq& P\left(M^*_{r,[n_k(1-\epsilon')]} > x_k'', [n_k(1-\epsilon')] < \tau_k < [n_k(1+\epsilon')] \; i.o.\right) \end{aligned}$$

From the fact that $\dfrac{\tau_k}{n_k} \to 1 \; a.s.$, note that $[n_k(1-\epsilon')] < \tau_k < [n_k(1+\epsilon')] \; a.s.$. Consequently,

$$P\left(A_k' \; i.o.\right) \geq P\left(M^*_{r,[n_k(1-\epsilon')]} > x_k'' \; i.o.\right) \tag{13}$$

Let $n_k'' = [n_k(1-\epsilon')], k \geq 1$, and let $M^{'}_{r,n_k''}$ denote the r^{th} largest observation

among $X_j, n_{k-1}'' < j \leq n_k'', k \geq 1$. Note that,

$$M^*_{r,[n_k(1-\epsilon')]} = M^*_{r,n_k''} > M^{'}_{r,n_k''}$$

and that $\left(M^{'}_{r,n_k''}\right)$ forms a sequence of mutually independent r.v.s.. Now, in view of (13), the relation (11) will be established, once we show that

$$P\left(M_{r,n_k''} > x_k'' \; i.o.\right) = 1. \tag{14}$$

Note that,

$$\liminf_{k\to\infty} \frac{n_{k+1}}{n_k} > 1$$

implies that there exists $\rho > 1$, such that $\frac{n_{k+1}}{n_k} \geq \rho$ for all k large. Using this fact and proceeding as in lemma 3.1, one can find a $c_3 > 0$ and $k_2 > 0$ such that for all $k \geq k_2$

$$P\left(M^{'}_{r,n_k''} > 1 - \frac{1}{n_k(\log n_k)^{\frac{c-\epsilon}{r}}}\right) \geq \frac{c_3}{(\log n_k)^{(c-\epsilon)}}.$$

Since $\sum_k \frac{1}{(\log n_k)^{(c-\epsilon)}} = \infty$ and $(M^{'}_{r,n_k''})$ are mutually independent, by Borel-Cantelli lemma one gets

$$P\left(M^{'}_{r,n_k''} > 1 - \frac{1}{n_k(\log n_k)^{\frac{c-\epsilon}{r}}} \; i.o.\right) = 1.$$

From the relation $M^{'}_{r,n_k''} \leq M^*_{r,n_k''}$, we have

$$P\left(M^*_{r,n_k''} > 1 - \frac{1}{n_k(\log n_k)^{\frac{c-\epsilon}{r}}} \; i.o.\right) = 1.$$

In turn, $P(A_k' \; i.o.) = 1$. Hence the proof is complete. □

Lemma 3.3.

$$\liminf_{n\to\infty} \left(n(1-M^*_{r,n})\right)^{\frac{1}{\log\log n}} = e^{-\frac{1}{r}} \; a.s. \, .$$

Proof. The theorem is proved once we show that for any $\epsilon \in (0,1)$

$$P\left(\left(n(1-M^*_{r,n})\right)^{\frac{1}{\log\log n}} < e^{-\frac{(1+\epsilon)}{r}} \; i.o.\right) = 0 \tag{15}$$

and

$$P\left(\left(n(1-M^*_{r,n})\right)^{\frac{1}{\log\log n}} < e^{-\frac{(1-\epsilon)}{r}}\ i.o.\right) = 1. \tag{16}$$

or equivalently

$$P\left(\log(n(1-M^*_{r,n})) < \frac{-(1+\epsilon)}{r}\log\log n\ i.o.\right) = 0 \tag{17}$$

and

$$P\left(\log(n(1-M^*_{r,n})) < \frac{-(1-\epsilon)}{r}\log\log n\ i.o.\right) = 1. \tag{18}$$

Given $U_{1,n} \leq U_{2,n} \leq \dots\dots\dots \leq U_{n,n}$ as the order statistics of random observations $U_1, U_2, U_3, \dots, U_n$ from Uniform $(0,1)$, from Kiefer (1971) note that

$$\liminf_{n\to\infty} \frac{\log nU_{r,n}}{\log\log n} = 1\ a.s. \tag{19}$$

Define $Y_j = 1 - U_j$ and note that Y_j is $U(0,1), j = 1, 2, \dots n$. As such $Y_1, Y_2, \dots\dots, Y_n$ becomes a random sample from $U(0,1)$. Let the order statistics be $Y_{1,n} \leq Y_{2,n} \dots \leq Y_{n-k+1,n} \leq \dots Y_{n,n}$. Then $Y_{n-r+1,n} = 1 - U_{r,n}$ which is same as $M^*_{r,n}$ in our notation. From (19), one gets

$$\liminf_{n\to\infty} \frac{\log n(1-M^*_{r,n})}{\log\log n} = 1\ a.s.$$

which proves the lemma. □

Lemma 3.4. *If* (τ_k) *and* (n_k) *satisfy* (A) *and* (B_2), *then*

$$\liminf_{k\to\infty} \left(n_k(1-M^*_{r,\tau_k})\right)^{\frac{1}{\log\log n_k}} = e^{-\frac{1}{r}} \qquad a.s.\ . \tag{20}$$

Proof. Equivalently we show that for any $\epsilon \in (0,1)$

$$P\left(\left(n_k(1-M^*_{r,\tau_k})\right)^{\frac{1}{\log\log n_k}} < e^{-}\frac{(1+\epsilon)}{r}\ i.o.\right) = 0 \tag{21}$$

and

$$P\left(\left(n_k(1-M^*_{r,\tau_k})\right)^{\frac{1}{\log\log n_k}} < e^{-}\frac{(1-\epsilon)}{r}\ i.o.\right) = 1. \tag{22}$$

Proceeding as in Lemma 3.2, one can show that for $\epsilon' > 0$

$$\begin{aligned}
P&\left(\left(n_k(1-M^*_{r,\tau_k})\right)^{\frac{1}{\log\log n_k}} < e^{-\frac{(1+\epsilon)}{r}}\ i.o.\right)\\
&\leq P\left(\left(n_k(1-M^*_{r,n'_k})\right)^{\frac{1}{\log\log n_k}} < e^{-\frac{(1+\epsilon)}{r}}\ i.o.\right)\\
&= P\left(\log(n_k(1-M^*_{r,n'_k})) < -\frac{(1+\epsilon)}{r}\log\log n_k\right).
\end{aligned}$$

Applying lemma 3.3 and observing $\log\log n'_k \sim \log\log n_k$ one gets,

$$P\left(\log(n_k(1-M^*_{r,n'_k})) < -\frac{(1+\epsilon)}{r}\log\log n_k\ i.o.\right) = 0.$$

Now, we need to show that

$$P\left(n_k(1-M_{r,\tau_k}) < e^{-\frac{(1-\epsilon)}{r}\log\log n_k}\ i.o.\right) = 1.$$

Again arguing as in lemma 3.2, it is sufficient if one shows that

$$P\left(n_k(1-M_{r,n''_k}) < e^{-\frac{(1-\epsilon)}{r}\log\log n_k}\ i.o.\right) = 1$$

or

$$P\left(n_k\left(1-M_{r,n''_k}\right) < (\log n_k)^{-\frac{(1-\epsilon)}{r}}\ i.o.\right) = 1.$$

Define $m_k = \min\{j : n_j > k^k\}$. By condition (B_2), one can find a $\lambda > 1$ such that for all j large, say, $j \geq j_0, n_{j+1} < \lambda n_j$. Consequently, whenever $m_k - 1 \geq j_0$, one gets $n_{m_k} < \lambda n_{m_{k-1}}$. By the definition of m_k, we hence have for $m_k \geq j_0 + 1$,

$$k^k < n_{m_k} < \lambda k^k. \tag{23}$$

In turn, for $m_k \geq j_0 + 1$,

$$\frac{(k+1)^{k+1}}{\lambda k^k} < \frac{n_{m_{k+1}}}{n_{m_k}} < \frac{\lambda(k+1)^{k+1}}{k^k}.$$

Consequently,

$$\lim_{k\to\infty} \frac{n_{m_{k+1}}}{n_{m_k}} = \infty.$$

In other words, the subsequence (n_{m_k}) satisfies (B_1). By Lemma 3.2,

$$\liminf_{k\to\infty}\left(n_{m_k}\left(1-M^*_{r,\tau_{m_k}}\right)\right)^{\frac{1}{\log\log n_{m_k}}} = e^{\frac{-c}{r}}\ a.s.$$

where

$$c = \inf\{d : \sum_k \frac{1}{(\log n_{m_k})^d} < \infty\}.$$

From (23) note that $c = 1$. Consequently,

$$P\left(n_j\left(1-M_{r,\tau_j}\right) < (\log n_j)^{\frac{-(1-\epsilon)}{r}}\ i.o.\right)$$
$$\geq P\left(n_{m_k}\left(1-M_{r,n''_{m_k}}\right) < (\log n_{m_k})^{\frac{-(1-\epsilon)}{r}}\ i.o.\right) = 1.$$

which completes the proof. □

Remark 3.5. *Note that* $n_k = 2^{k^2}, k \geq 1$, *gives* $c = \frac{1}{2}$ *and* $n_k = 2^{2^k}, k \geq 1$ *gives* $c = 0$ *in lemma* 3.2. *Hence, when* (n_k) *is atleast geometrically fast,*

$$\liminf_{k\to\infty}(n_k(1-M_{r,n_k}))^{\frac{1}{\log\log n_k}}$$

becomes a function of (n_k), *unlike in the case of atmost geometrically increasing subsequences. As such, when* (n_k) *is rapidly increasing i.e.,* $\frac{n_{k+1}}{n_k} \to \infty$ *as* $k \to \infty$, *as in Gut and Schwabe* (1996), *we obtain L.I.L. by replacing* $\log\log n_k$ *by* $\log k$.

Lemma 3.6. *If* (τ_k) *and* (n_k) *satisfy* (A) *and* (B_3).

$$\liminf_{k\to\infty}\left(n_k(1-M^*_{r,\tau_k})\right)^{\frac{1}{\log k}} = e^{-\frac{1}{r}}\qquad a.s.\ .$$

Proof. Equivalently we show that for any $\epsilon \in (0,1)$

$$P\left(\left(n_k(1-M^*_{r,\tau_k})\right)^{\frac{1}{\log k}} < e^{-\frac{(1+\epsilon)}{r}} \; i.o.\right) = 0 \tag{24}$$

and

$$P\left(\left(n_k(1-M^*_{r,\tau_k})\right)^{\frac{1}{\log k}} < e^{-\frac{(1-\epsilon)}{r}} \; i.o.\right) = 1. \tag{25}$$

As in lemma 3.2, in order to establish (24) it is sufficient if one can show for $\epsilon' > 0$, that

$$P\left(M^*_{r,n'_k} > 1 - \frac{1}{n_k k^{\frac{1+\epsilon}{r}}} \; i.o.\right) = 0$$

Proceeding as in lemma 3.1, one can find $c_4 > 0$ and $k_3 > 0$ such that for all $k \geq k_3$

$$P\left(M^*_{r,n'_k} > 1 - \frac{1}{n_k k^{\frac{1+\epsilon}{r}}}\right) \leq \frac{c_4}{k^{1+\epsilon}}.$$

Since $\sum_k \frac{1}{k^{1+\epsilon}} < \infty$, from Borel-Cantelli lemma, we get

$$P\left(M^*_{r,n'_k} > 1 - \frac{1}{n_k k^{\frac{1+\epsilon}{r}}} \; i.o.\right) = 0 \text{ or } P\left(\left(n_k(1-M^*_{r,n_k})\right)^{\frac{1}{\log k}} < e^{-\frac{(1+\epsilon)}{r}} \; i.o.\right) = 0.$$

We now establish (25). For $\epsilon' > 0$ but small, let $M'_{r,[(1-\epsilon')n_k]}$ denote the r^{th} highest among $\left(X_{n''_{k-1}+1}, \ldots, X_{n''_k}\right)$, where $n''_k = [(1-\epsilon')n_k]$. Note that $M'_{r,n''_k} \leq M^*_{r,n''_k}$ and that (M'_{r,n''_k}) are mutually independent. From the fact that $X'_n s$ are i.i.d., we have

$$P\left(M'_{r,n''_k} > 1 - \frac{1}{n_k k^{\frac{1-\epsilon}{r}}}\right) = P\left(M^*_{r,(n''_k - n''_{k-1})} > 1 - \frac{1}{n_k k^{\frac{1-\epsilon}{r}}}\right).$$

Proceeding as in lemma 3.1 one can find $c_5 > 0$ and k_4 such that for all $k \geq k_4$

$$P\left(M'_{r,n''_k} > 1 - \frac{1}{n_k k^{\frac{1-\epsilon}{r}}}\right) \geq \frac{c_5}{k^{1-\epsilon}}.$$

Since $\sum_k \frac{1}{k^{1-\epsilon}} = \infty$ and (M'_{r,n_k}) are mutually independent, by Borel-Cantelli lemma we get

$$P\left(M'_{r,n''_k} > 1 - \frac{1}{n_k k^{\frac{1-\epsilon}{r}}} \; i.o\right) = 1.$$

Consequently,

$$P\left(M^*_{r,n''_k} > 1 - \frac{1}{n_k k^{\frac{1-\epsilon}{r}}} \; i.o\right) = 1 \text{ or } P\left(\left(n_k(1-M^*_{r,n_k})\right)^{\frac{1}{\log k}} < e^{-\frac{(1-\epsilon)}{r}} \; i.o.\right) = 1.$$

Hence the proof is complete. □

Lemma 3.7. *For any sequence (n_k) with $n_k \to \infty$ as $k \to \infty$,*

$$\limsup_{k\to\infty} \left(n_k(1-M^*_{r,\tau_k})\right)^{\frac{1}{\log\log n_k}} = 1 \; a.s. \, .$$

Proof. We show that for any $\epsilon > 0$

$$P\left(\left(n_k(1 - M^*_{r,\tau_k})\right)^{\frac{1}{\log\log n_k}} > e^{-\epsilon} \;\; i.o \right) = 1 \tag{26}$$

and

$$P\left(\left(n_k(1 - M^*_{r,\tau_k})\right)^{\frac{1}{\log\log n_k}} > e^{\epsilon} \;\; i.o \right) = 0, \tag{27}$$

which in turn establishes the theorem. Recall that $n'_k = [(1+\epsilon')n_k]$ and $n''_k = [(1-\epsilon')n_k], 0 < \epsilon' < 1$. As in lemma 3.2, the result is proved once it is shown that

$$P\left(\left(n_k(1 - M_{r,n'_k})\right)^{\frac{1}{\log\log n_k}} > e^{-\epsilon} \;\; i.o \right) = 1$$

and

$$P\left(\left(n_k(1 - M_{r,n''_k})\right)^{\frac{1}{\log\log n_k}} > e^{\epsilon} \;\; i.o \right) = 0.$$

From lemma 2.1, one can find $c_6 > 0$ and $k_5 > 0$ such that for all $k \geq k_5$,

$$P\left(M_{r,n'_k} < 1 - \frac{1}{n_k(\log n_k)^{\epsilon}} \right) \geq 1 - \frac{c_6}{(\log n_k)^{r\epsilon}}.$$

Consequently,

$$\lim_{k\to\infty} P\left(M_{r,n'_k} < 1 - \frac{1}{n_k(\log n_k)^{\epsilon}} \right) = 1.$$

Define

$$A_{r,k} = \left(M_{r,n'_k} < 1 - \frac{1}{n_k(\log n_k)^{\epsilon}} \right).$$

Note that

$$P\left(A_{r,k} \; i.o.\right) \geq \lim_{k\to\infty} P(A_{r,k}) = 1.$$

Hence,

$$P\left(\left(n_k(1 - M_{r,n'_k})\right)^{\frac{1}{\log\log n_k}} > e^{-\epsilon} \quad i.o. \right) = 1$$

In turn, (26) follows.
The proof is complete if one shows that

$$P\left(\left(n_k(1 - M_{r,\tau_k})\right)^{\frac{1}{\log\log n_k}} > e^{\epsilon} \quad i.o \right) = 0.$$

Again arguing as in lemma 3.2, it is sufficient if one shows that

$$P\left(\left(n_k(1 - M_{r,n''_k})\right)^{\frac{1}{\log\log n_k}} > e^{\epsilon} \; i.o. \right) = 0. \tag{28}$$

From Theorem 2 of Kiefer (1971), one can note that

$$P\left(\left(n\,(1 - M_{r,n})\right)^{\frac{1}{\log\log n}} > e^{\epsilon} \quad i.o. \right) = 0$$

which implies (27). Hence the result is proved. □

4. Proofs of the theorems presented in section 2

Given that (X_n) is a sequence of i.i.d. r.v.s. with a common continuous d.f. F define $U_n = F(X_n), n \geq 1$, and observe that $\{U_n\}$ is a sequence of i.i.d. Uniform $(0,1)$ r.v.s.. Recall that $M_{r,n}$ is the r^{th} upper extreme of $X_1, X_2, \ldots, X_n$ and that $M^*_{r,n}$ the r^{th} upper extreme of $U_1, U_2, \ldots, U_n$. Note the relation, $M^*_{r,n} = F(M_{r,n})$.

4.1. Proof of (1) of Theorem 2.1

We need to show that for $\epsilon \in (0, c)$

$$P\left(\frac{(\log n_k)}{(\log\log n_k)}\left(\frac{M_{r,\tau_k}}{V(\log n_k)} - 1\right) > \frac{c+\epsilon}{r\gamma} \;\; i.o.\right) = 0. \tag{29}$$

and

$$P\left(\frac{(\log n_k)}{(\log\log n_k)}\left(\frac{M_{r,\tau_k}}{V(\log n_k)} - 1\right) > \frac{c-\epsilon}{r\gamma} \;\; i.o.\right) = 1. \tag{30}$$

In order to show (29), one can proceed on lines similar to lemma 3.2. Recall that for $\epsilon' \in (0,1), n'_k = [(1+\epsilon)n_k]$ and $n''_k = [(1-\epsilon)n_k]$. Then,

$$P\left(\frac{(\log n_k)}{(\log\log n_k)}\left(\frac{M_{r,\tau_k}}{V(\log n_k)} - 1\right) > \frac{c+\epsilon}{r\gamma} \; i.o.\right)$$

$$\leq P\left(\frac{(\log n_k)}{(\log\log n_k)}\left(\frac{M_{r,n'_k}}{V(\log n_k)} - 1\right) > \frac{c+\epsilon}{r\gamma} \; i.o.\right).$$

By (12) we have

$$P\left(\left(1 - M^*_{r,n'_k}\right) < \frac{1}{n_k(\log n_k)^{\frac{c+\epsilon}{r}}} \;\; i.o\right) = 0. \tag{31}$$

From the relation $M^*_{r,n'_k} = F(M_{r,n'_k})$, note that

$$\begin{aligned}
1 - M^*_{r,n'_k} < \frac{1}{n_k(\log n_k)^{\frac{c+\epsilon}{r}}} &\Leftrightarrow 1 - F(M_{r,n'_k}) < \frac{1}{n_k(\log n_k)^{\frac{c+\epsilon}{r}}} \\
&\Leftrightarrow -\log(1 - F(M_{r,n'_k})) > \log(n_k(\log n_k)^{\frac{c+\epsilon}{r}}) \\
&\Leftrightarrow \; U(M_{r,n'_k}) > \log n_k + \frac{c+\epsilon}{r}\log\log n_k \\
&\Leftrightarrow M_{r,n'_k} > V\left(\log n_k + \frac{c+\epsilon}{r}\log\log n_k\right)
\end{aligned}$$

$$\Leftrightarrow M_{r,n'_k} - V(\log n_k) > V\left(\log n_k + \frac{c+\epsilon}{r}\log\log n_k\right) - V(\log n_k). \tag{32}$$

From condition (2), we have as $k \to \infty$,

$$V\left(\log n_k\left(1 + \frac{c+\epsilon}{r}\frac{\log\log n_k}{\log n_k}\right)\right) - V(\log n_k) \sim \frac{(c+\epsilon)\log\log n_k}{\gamma(r\log n_k)}V(\log n_k).$$

Consequently, from (32), for k large, we have

$$1 - M^*_{r,n'_k} < \frac{1}{n_k(\log n_k)^{\frac{c+\epsilon}{r}}}$$

$$\Leftrightarrow M_{r,n'_k} - V(\log n_k) > \frac{(c+\epsilon)\log\log n_k}{\gamma(r\log n_k)} V(\log n_k)$$

$$\Leftrightarrow \frac{\log n_k}{\log\log n_k}\left(\frac{M_{r,n'_k}}{V(\log n_k)} - 1\right) > \frac{c+\epsilon}{r\gamma}.$$

Hence from (31), (29) follows
Again from lemma 3.2, recalling that

$$P\left(\left(1 - M^*_{r,n''_k}\right) < \frac{1}{n_k(\log n_k)^{\frac{c-\epsilon}{r}}} \;\; i.o\right) = 1$$

and proceeding on the above lines, (30) can be established. The details are omitted.
Proofs of (2) and (3) of Theorem 2.1 and Theorem 2.2 can be obtained using lemmas 3.4, 3.6 and 3.7 respectively and proceeding on the above lines. Hence the details are omitted.

4.2. Proof of (1) Theorem 2.6.

The theorem is proved once it is shown that for $0 < \epsilon < c$

$$P\left(M_{r,\tau_k} > B_{n_k}(\log n_k)^{\frac{c+\epsilon}{r\gamma}} \;\; i.o.\right) = 0$$

and

$$P\left(M_{r,\tau_k} > B_{n_k}(\log n_k)^{\frac{c-\epsilon}{r\gamma}} \;\; i.o.\right) = 1.$$

Arguing as in lemma 3.2, it is sufficient if one shows that for $\epsilon' > 0$

$$P\left(M_{r,n'_k} > B_{n_k}(\log n_k)^{\frac{c+\epsilon}{r\gamma}} \;\; i.o.\right) = 0$$

and

$$P\left(M_{r,n''_k} > B_{n_k}(\log n_k)^{\frac{c-\epsilon}{r\gamma}} \;\; i.o.\right) = 1.$$

By lemma 3.2, we have for $\epsilon \in (0, c)$,

$$P\left(1 - M^*_{r,n'_k} < \frac{1}{n_k(\log n_k)^{\frac{c+\epsilon}{r}}} \;\; i.o.\right) = 0 \tag{33}$$

and

$$P\left(1 - M^*_{r,n''_k} < \frac{1}{n_k(\log n_k)^{\frac{c-\epsilon}{r}}} \;\; i.o.\right) = 1. \tag{34}$$

Using the relations

$$M^*_{r,n'_k} = F(M_{r,n'_k}) \;\text{ and }\; U^*(x) = 1 - F(x) \sim x^{-\gamma}L(x),$$

from (12) one gets,

$$P\left(U^*(M_{r,n'_k}) < \frac{1}{n_k(\log n_k)^{\frac{c+\epsilon}{r}}} \;\; i.o.\right) = 0. \tag{35}$$

By (3) note that

$$U^*(M_{r,n'_k}) < \frac{1}{n_k(\log n_k)^{\frac{c+\epsilon}{r}}} \Leftrightarrow \qquad V^*(U^*(M_{r,n'_k})) > V^*\left(\frac{1}{n_k(\log n_k)^{\frac{c+\epsilon}{r}}}\right)$$

$$\Leftrightarrow M_{r,n'_k} > n_k^{\frac{1}{\gamma}}(\log n_k)^{\frac{c+\epsilon}{r\gamma}} l\left(n_k(\log n_k)^{\frac{c+\epsilon}{r}}\right). \tag{36}$$

From the properties of a s.v. function, by Seneta (1976) we have for any $\delta > 0$,

$$\lim_{k\to\infty}(\log n_k)^{\delta}\frac{l\left(n_k(\log n_k)^{\frac{c+\epsilon}{r}}\right)}{l(n_k)} = \infty.$$

Choosing $\delta = \frac{\epsilon}{2r\gamma}$, one can find a k_6 such that for all $k \geq k_6$,

$$l\left(n_k(\log n_k)^{\frac{c+\epsilon}{r}}\right) \geq \frac{l(n_k)}{(\log n_k)^{\frac{\epsilon}{2r\gamma}}}. \tag{37}$$

Also, $n(1 - F(B_n)) \simeq 1$ implies that $B_n = n^{\frac{1}{\gamma}} l(n)$, since $1 - F(x)$ is regularly varying with index $-\gamma$. Consequently, using (37) in (36), we note that for $k \geq k_6$,

$$\left(U^*(M^*_{r,n'_k}) < \frac{1}{n_k(\log n_k)^{\frac{c+\epsilon}{r}}}\right) \supseteq \left(M_{r,n'_k} > B_{n_k}(\log n_k)^{\frac{c+\frac{\epsilon}{2}}{r\gamma}}\right).$$

Now, (35) implies that

$$P\left(M_{r,n'_k} > B_{n_k}(\log n_k)^{\frac{c+\frac{\epsilon}{2}}{r\gamma}} \ \ i.o.\right) = 0. \tag{38}$$

By proceeding on similar lines and using the fact that for $\delta > 0$

$$\lim(\log n_k)^{-\delta}\frac{l\left(n_k(\log n_k)^{\frac{c-\epsilon}{r}}\right)}{l(n_k)} = 0,$$

(see, Seneta (1976)), choosing $\delta = \frac{\epsilon}{2r\gamma}$, one can show that (34) implies

$$P\left(M_{r,n''_k} > B_{n_k}(\log n_k)^{\frac{c-\frac{\epsilon}{2}}{r\gamma}} \ \ i.o.\right) = 1. \tag{39}$$

Now (38) and (39) together establish the theorem.
Proofs of (2) and (3) of Theorem 2.6 and Theorem 2.7 can be obtained on similar lines by applying lemmas $3.4, 3.6$ and 3.7 respectively. The details are omitted.

4.3. Proof of (1) Theorem 2.11

For F with $\omega(F) < \infty$, from section 2, recall the relation,

$$Z_n = \frac{1}{\omega(F) - X_n}, n \geq 1,$$

where Z_n has d.f. $F^* \in DA.(H_{1,\gamma})$. Consequently, for any $y > 0$,

$$F^*(y) = P\left(Z_n \leq y\right) = P\left(X_n \leq \omega(F) - \frac{1}{y}\right) = F\left(\omega(F) - \frac{1}{y}\right).$$

Note that, $F \in DA.(H_{2,\gamma})$ iff $F^* \in DA.(H_{1,\gamma})$. Also, recall that $M_{r,n}$ is the r^{th} upper extreme of $(X_1, X_2, \ldots, X_n)$. Define $M^{''}_{r,n}$ as the r^{th} upper extreme of $(Z_1, Z_2, \ldots, Z_n), n \geq 1$. Observe that,

$$M^{''}_{r,n_k} = \frac{1}{\omega(F) - M_{r,n_k}}.$$

Since $F^* \varepsilon DA.(H_{1,\gamma})$, from Theorem 2.6 we have,

$$\limsup_{k\to\infty} \left(\frac{M^{''}_{r,n_k}}{B_{n_k}} \right)^{\frac{1}{\log\log n_k}} = e^{\frac{c}{r\gamma}} \quad a.s.$$

Substituting,

$$M^{''}_{r,n_k} = \frac{1}{\omega(F) - M_{r,n_k}},$$

one gets the required result.
The proofs of (2) and (3) of Theorem 2.11 and Theorem 2.12 follow on the above lines. The details are omitted.

References

[1] Chover, J. (1967). A law of iterated logarithm for stable summands, *Proc. Amer. Math.*, **17 (2)**, 441-443.
[2] de Han and Hordijk. (1972). The rate of growth of sample maxima. The Ann. Math. Stat., **43 (4)**, 1185-1196.
[3] Galambos, J. (1978). *The asymptotic theory of extreme order statistics*, Wiley, New York.
[4] Gut, A. (1986). Law of the iterated logarithm for subsequences, *Prob. Math. Stat.*, **7**, 27-58.
[5] Gut, A. (2009). *Stopped random walks:Limit theorems and applications*, 2nd edition, Springer.
[6] Hall, P. (1979). On the relative stability of large order statistics, *Math. Proc. Camb. Phil. Soc.*, **86**, 467-475.
[7] Hartman, P. and Wintner, A. (1941). On the law of the iterated logarithm. Amer. Jou. Math., **63 (1)**, 169-176.
[8] Hüsler, J. (1985). The growth rate of partial maxima on subsequences, *Stat. Prob. Lett.*, **3**, 245-249.
[9] Kiefer, J. (1971). Iterated logarithm analogues for sample quantiles when $p_n \downarrow 0$, *Proc. Sixth Berkeley Symp.*, **1**, 227-244.
[10] Pakes, A. G. (2000). The number and sum of near-maxima for thin tailed populations, *Adv. in Appl. Probab.*, **32**, 1100-1116.
[11] Schwabe, R. and Gut, A. (1996). On the law of the iterated logarithm for rapidly increasing subsequences, *Math. Nachr.*, **178**, 309-332.
[12] Seneta, E. (1976). *Regularly Varying Functions*, Lecture Notes in Mathematics. **508**, Springer, Berlin.
[13] Torrang, I. (1987). Law of the iterated logarithm-Cluster points of deterministic and random subsequences, *Prob. Math. Stat.* , **8**, 133-141.
[14] Vasudeva, R., and Savitha, S. (1992). Law of the iterated logarithm for random subsequences, *Jou. of. Ind. Stat. Assoc.*, **30**, 13-24.
[15] Vasudeva, R., and Divanji, G. (2006). Chover's form of Law of iterated logarithm for rapidly increasing subsequences, *Jou. of. Ind. Stat. Assoc.*, **44 (2)**, 185-195.
[16] Vasudeva, R., and Moridani, A. Y. (2011). Kiefer's law of the iterated logarithm for the vector of upper order statistics, *Probab. and Math. Statist.*, **31 (2)**, 331-347.