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A study of inferential problem about the lifetime of homogeneity of several systems under generalized exponential model based on type-II censored sampling design

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Abstract. In this paper, we consider the simultaneous estimation of unknown parameters of the m generalized exponential distributions using the Type II censoring scheme. We obtain maximum likelihood estimators of the unknown parameters with the help of Newton-Raphson method. Extensive simulation study is done to demonstrate the procedure and to get to study of the performance. Further, likelihood ratio test is discussed to test homogeneity of several scale parameters.

1. Introduction

The industrial revolution onwards man has become more and more dependent on machine or system. The failure of the machine make the whole system of work related to the machine stop abruptly. Therefore, a manufacturer of a product is always interesting in assessing the reliability of the product. The reliability of product is usually evaluated based on some characteristics of lifetimes such as mean, median, quantile, survival function and hazard function etc. But, the lifetime experiments are usually much time consuming and expensive. There are several situations where it is neither desirable nor possible to obtain complete sample. In such cases, Type II censoring is the most commonly used sampling plan. In statistics literature we find that of research papers which use the plan for various lifetime models such as normal, exponential and Weibull. For details one can refer: Gupta (1952); Cohen (1965); Mann et al. (1974); Lawless (1982); Sinha (1986); Hossain et al. (2003). In manufacturing setting, the problem of comparing effectiveness of products is important. In this situation after placing several independent samples of units manufactured by the several processes, the reliability engineer would like to make early and efficient decision on the effectiveness of the products under the life test in terms of standard hazard rate function. Balakrishnan and Ng (2006) extensively study the problem of comparing two populations in terms of stochastic ordering. Sharafi et al. (2013) study distribution free test for comparison of hazard rates of two distributions under Type II censoring. Recently, Gupta and Kundu (1999) propose generalized exponential (GE) distribution which has common properties of both gamma and Weibull distributions. Further, it enhances its utility for having closed form expression for its cumulative distribution function. There are many properties of the distribution is discussed in Gupta and Kundu (1999). In this paper, we discuss the inferential problem about the lifetime of homogeneity of several systems under the generalized exponential distribution based on Type II censored sampling design and further, study the reliability characteristics of distributions. We

Keywords. Type II censoring scheme, generalized exponential distribution, Newton-Raphson method, ML estimation, Monte-Carlo Simulation technique, likelihood ratio test

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now consider a design, where we put m types of systems simultaneously on test in which for each type of systems we start with u units and continue the experiment till G^* failures are observed i.e. the total numbers of units put on test are mu and the total number of failures we observe at the end of experiment are $G = mG^*$. Assuming that the lifetime distribution of unit for each type of systems to be generalized exponential with shape parameter α and scale parameters $\beta_i; i = 1, 2, , m$. In the experiment after each failure the failure time is observed, and denoted it by $t_{gi}; g = 1, 2, , G^*; i = 1, 2, , m$. At the end of experiments, we have data $(u, G, t_{gi}; g = 1, 2, , G^*; i = 1, 2, , m)$. The organization of whole paper is as below.

In Section 2 we give the probability density function, the survival function and the hazard rate of the generalized exponential distribution and develop the likelihood for Type II censored sampling design under generalized exponential distribution. In Section 3 we derive the expressions for maximum likelihood estimators of parameters and their asymptotic variance-covariance matrix when shape parameter of the distribution is known and when it is unknown. Section 4 discusses algorithm for generation of data from Type II censored sampling design under generalized exponential distribution and provides iterative procedure for estimation of the parameters through Newton-Raphson method. Further, the tables of ML estimates and their asymptotic standard errors, estimate of reliability and hazard rates and their mean square error at fixed time point are given which are simulated through Monte-Carlo simulation technique for both the cases of shape parameter known and unknown. In Section 5, we discuss likelihood ratio test for simultaneous testing of homogeneity of scale parameters when the shape parameter is known. The cut-off points for the test statistics are obtained through Monte-Carlo simulation. Some concluding remarks are given in Section 6.

2. Generalized Exponential Distribution and Likelihood Function for Type II Censoring Design

Consider an item whose life time is denoted by T. The random variable T is assumed to have generalized exponential distribution (GE), as defined by Gupta and Kundu (1999), with distribution function

$$F(t; \alpha, \beta) = (1 - e^{-\beta t})^\alpha, (t > 0, \alpha > 0, \beta > 0). \tag{1}$$

The corresponding density function is given by

$$f(t; \alpha, \beta) = \alpha\beta(1 - e^{-\beta t})^{\alpha-1}e^{-\beta t}, (t > 0, \alpha > 0, \beta > 0). \tag{2}$$

Here α is a shape parameter, β is a scale parameter. We denote the GE distribution with shape parameter α and scale parameter β as $GE(\alpha, \beta)$. Then the reliability function is

$$\bar{F}(t) = P(T > t) = 1 - (1 - e^{-\beta t})^\alpha \tag{3}$$

and the hazard function is

$$h(t) = \frac{f(t)}{\bar{F}(t)}$$

$$h(t) = \alpha\beta[\frac{e^{-\beta t}}{1 - e^{-\beta t}}][\frac{(1 - e^{-\beta t})^\alpha}{1 - (1 - e^{-\beta t})^\alpha}]. \tag{4}$$

If Y follows $GE(\alpha, 1)$, then the corresponding moment generating function, is given by

$$\begin{aligned} M(s) &= \alpha \int_0^\infty (1 - e^{-y})^{\alpha-1}e^{(s-1)y}\, dy \qquad (5) \\ &= \alpha \int_0^1 (1 - z)^{\alpha-1}z^{-s}dz = \frac{\Gamma(\alpha + 1)\Gamma(1 - s)}{\Gamma(\alpha - s + 1)}\ s < 1. \end{aligned}$$

Differentiating $lnM(s)$ and evaluating at $s = 0$, we get the mean and variance of $GE(\alpha, 1)$ as

$$E(Z) = \psi(\alpha + 1) - \psi(1) \text{ and } var(Z) = \psi^{'}(1) - \psi^{'}(\alpha + 1). \tag{6}$$

where $\psi(\cdot)$ is the digamma function and $\psi^{'}(\cdot)$ is its derivative. If Z follows $GE(\alpha, 1)$ and $T = \frac{1}{\beta}Z$ then T follows $GE(\alpha, \beta)$. Therefore, the mean and variance of T is given by

$$E(T) = \frac{\psi(\alpha + 1) - \psi(1)}{\beta} \text{ and } var(Z) = \frac{\psi^{'}(1) - \psi^{'}(\alpha + 1)}{\beta^2}. \tag{7}$$

The likelihood function for Type II censoring design for i-th type of systems observing G^* failures from u units given in literature as

$$L_i = \frac{u!}{(u - G^*)!}\prod_{g=1}^{G^*} f_i(t_g)[\bar{F}_i(t_{G^*})]^{(u-G^*)} \tag{8}$$

Therefore, the likelihood for whole experiments

$$\begin{aligned} L &= \prod_{i=1}^{m} L_i \\ &= \prod_{i=1}^{m} \{\frac{u!}{(u - G^*)!}\prod_{g=1}^{G^*} f_i(t_g)[\bar{F}_i(t_{G^*})]^{(u-G^*)}\}. \end{aligned} \tag{9}$$

Substitute the equations (2 and 3) in equations (9) we have, the likelihood function

$$\begin{aligned} L = \prod_{i=1}^{m} \{&\frac{u!}{(u - G^*)!}\prod_{g=1}^{G^*} \alpha\beta_i(1 - e^{-\beta_i t_{gi}})^{\alpha-1} e^{-\beta_i t_{gi}} \\ &\times [1 - (1 - e^{-\beta_i t_{G^*i}})^{\alpha}]^{(u-G^*)}\}. \end{aligned} \tag{10}$$

3. Maximum Likelihood Estimation

In this section we obtain maximum likelihood estimates of $\alpha, \beta_i(i = 1, 2, ..., m)$, reliability function, hazard rate and observed information matrix under the design. The log likelihood equation of (10) would be

$$\begin{aligned} l = &\, mln(\frac{u!}{(u - G^*)!}) + mG^* ln\alpha + G^*\sum_{i=1}^{m} ln\beta_i - \sum_{i=1}^{m}\sum_{g=1}^{G^*}\beta_i t_{gi} \\ &+ (\alpha - 1)\sum_{i=1}^{m}\sum_{g=1}^{G^*} ln(1 - e^{-\beta_i t_{gi}}) + (u - G^*)\sum_{i=1}^{m} ln[1 - (1 - e^{-\beta_i t_{G^*i}})^{\alpha}]. \end{aligned} \tag{11}$$

Differentiate (11) with respect to α and $\beta_i(i = 1, 2, ..., m)$ we have

$$\begin{aligned} \frac{\partial l}{\partial \alpha} = &\frac{mG^*}{\alpha} + \sum_{i=1}^{m}\sum_{g=1}^{G^*} ln(1 - e^{-\beta_i t_{gi}}) \\ &- (u - G^*)\sum_{i=1}^{m} \frac{(1 - e^{-\beta_i t_{G^*i}})^{\alpha} \ln(1 - e^{-\beta_i t_{G^*i}})}{1 - (1 - e^{-\beta_i t_{G^*i}})^{\alpha}}. \end{aligned} \tag{12}$$

$$\begin{aligned} \frac{\partial l}{\partial \beta_i} = &\frac{G^*}{\beta_i} + (\alpha - 1)\sum_{g=1}^{G^*}[\frac{t_{gi}e^{-\beta_i t_{gi}}}{1 - e^{-\beta_i t_{gi}}}] \\ &- \sum_{g=1}^{G^*} t_{gi} - \alpha(u - G^*)t_{G^*i}\frac{e^{-\beta_i t_{G^*i}}(1 - e^{-\beta_i t_{G^*i}})^{\alpha-1}}{1 - (1 - e^{-\beta_i t_{G^*i}})^{\alpha}} \text{For } i = 1, 2, ..., m. \end{aligned} \tag{13}$$

The estimates of parameters $\underline{\beta}$ are obtained in two cases when (i) shape parameter α is known and (ii) shape parameter α is unknown.

3.1. Maximum Likelihood Estimation When Shape Parameter is Known

The solution of equations (13) can be evaluated numerically by some suitable iterative procedure such as Newton-Raphson method, for given values of $(u, G, t_{gi}; g = 1, 2, ..., G^*; i = 1, 2, ..., m)$. The MLE of $\underline{\beta} = (\beta_1, \beta_2, ..., \beta_m)$ are obtained as $\hat{\underline{\beta}}$ from equations (13). The MLEs of reliability $(R_i(t_i); i = 1, 2, ..., m)$ and hazard rate $(h_i(t_i); i = 1, 2, ..., m)$ can be evaluated using invariance property of MLEs as

$$\hat{R}_i(t_i) = 1 - \left(1 - e^{-\hat{\beta}_i t_i}\right)^{\alpha}. \tag{14}$$

$$\hat{h}_i(t_i) = \alpha\hat{\beta}_i[\frac{e^{-\hat{\beta}_i t_i}}{1 - e^{-\hat{\beta}_i t_i}}][\frac{(1 - e^{-\hat{\beta}_i t_i})^{\alpha}}{1 - (1 - e^{-\hat{\beta}_i t_i})^{\alpha}}].\text{For } i = 1, 2, ..., m \tag{15}$$

3.1.1. Observed Fisher Information Matrix Under Design

To obtain Fisher information matrix we take derivatives of equations (13) with respect to $\beta_i; i = 1, 2, ..., m$. Therefore, we have,

$$\begin{aligned}\frac{\partial^2 l}{\partial \beta_i^2} &= \frac{-G^*}{\beta_i^2} + (\alpha - 1)\sum\nolimits_{g=1}^{G^*}[\frac{t_{gi}^2 e^{-\beta_i t_{gi}}}{\left(1 - e^{-\beta_i t_{gi}}\right)^2}] \\ &- \frac{\alpha(u - G^*)t_{G^*i}^2 e^{-\beta_i t_{G^*i}}\left(1 - e^{-\beta_i t_{G^*i}}\right)^{\alpha-2}}{\left[1 - \left(1 - e^{-\beta_i t_{G^*i}}\right)^{\alpha}\right]^2}\{\alpha e^{-\beta_i t_{G^*i}} - 1 + \left(1 - e^{-\beta_i t_{G^*i}}\right)^{\alpha}\}.\end{aligned} \tag{16}$$

As rate of failures of systems are independent of each type of systems, derivatives of equations (13) with respect to $\beta_j; j \neq i = 1, 2, ..., m$ are

$$\frac{\partial^2 l}{\partial \beta_i \partial \beta_j} = 0.\ \forall j \neq i = 1, 2, ..., m. \tag{17}$$

For $\alpha > 2$, the Generalized Exponential family satisfies all the regularity conditions (See Bain, 1978, pp.86-87) in a similar way to the gamma family and the Weibull family, and therefore, we have the following results.

Theorem 3.1: For $\alpha > 2$ and $\frac{G^*}{u}$ kept constant the maximum likelihood estimators, $\hat{\underline{\beta}}$ of $\underline{\beta}$ are consistent estimators, and $\sqrt{u}(\hat{\underline{\beta}} - \underline{\beta})$ is asymptotically m-variate normal with mean $\underline{0}$ and variance covariance matrix $\mathbf{V^{-1}}$, where $\mathbf{V}$ is expected value of negative of second derivative matrix of log likelihood with respect to $\underline{\beta}$.
Note: Since evaluation of expected value is cumbersome we will use sample information matrix $\hat{\mathbf{V}}$which, under usual regularity conditions, converges asymptotically to Fisher information matrix.

3.2. Maximum Likelihood Estimation When Shape Parameter is Unknown

The solution of equations (12 and 13) can be evaluated numerically by some suitable iterative procedure such as Newton-Raphson method, for given values of $(u, G, t_{gi}; g = 1, 2, .., G^*; i = 1, 2, ..., m)$. The MLE of $(\alpha, \underline{\beta})$ are obtained as $(\hat{\alpha}, \hat{\underline{\beta}})$ from equations (12 and 13). The MLEs of reliability $(R_i(t_i); i = 1, 2, ..., m)$ and hazard rate $(h_i(t_i); i = 1, 2, ..., m)$ can be evaluated using invariance property of MLEs as

$$\hat{R}_i(t_i) = 1 - \left(1 - e^{-\hat{\beta}_i t_i}\right)^{\hat{\alpha}}. \tag{18}$$

$$\hat{h}_i(t_i) = \hat{\alpha}\hat{\beta}_i[\frac{e^{-\hat{\beta}_i}t_i}{1 - e^{-\hat{\beta}_i}t_i}][\frac{(1 - e^{-\hat{\beta}_i}t_i)^{\hat{\alpha}}}{1 - (1 - e^{-\hat{\beta}_i}t_i)^{\hat{\alpha}}}].\text{ For } i = 1, 2, ..., m \tag{19}$$

3.2.1. Observed Fisher Information Matrix Under Design

To obtain Fisher information matrix we take derivatives of equations (12) and (13) with respect to $\alpha, \beta_i; i = 1, 2, ..., m$. Therefore, we have,

$$\frac{\partial^2 l}{\partial \alpha^2} = \frac{-mG^*}{\alpha^2} - (u - G^*)\frac{[ln(1 - e^{-\beta_i t_{G^*i}})]^2 (1 - e^{-\beta_i t_{G^*i}})^\alpha}{[1 - (1 - e^{-\beta_i t_{G^*i}})^\alpha]^2}. \tag{20}$$

$$\frac{\partial^2 l}{\partial \alpha \partial \beta_i} = \sum_{g=1}^{G^*} [\frac{t_{gi} e^{-\beta_i t_{gi}}}{1 - e^{-\beta_i t_{gi}}}] - \frac{(u - G^*)}{[1 - (1 - e^{-\beta_i t_{G^*i}})^\alpha]^2}$$
$$\times \{\frac{1 - (1 - e^{-\beta_i t_{G^*i}})^\alpha}{\alpha} + ln(1 - e^{-\beta_i t_{G^*i}})\} \tag{21}$$

Derivatives of equation (13) with respect to $\beta_i; i = 1, 2, ..., m$ and $\beta_j; j \neq i = 1, 2, ..., m$ are given in equations (16) and (17) respectively. Therefore, we have the following result:

Theorem 3.2: For $\alpha > 2$ and $\frac{G^*}{u}$ kept constant the maximum likelihood estimators,$(\alpha, \underline{\hat{\beta}})$ of $(\alpha, \underline{\beta})$ are consistent estimators, and $\sqrt{u}(\hat{\alpha} - \alpha, \underline{\hat{\beta}} - \underline{\beta})$ is asymptotically $(m + 1)$-variate normal with mean $(0, \underline{0})$ and variance covariance matrix $\mathbf{W^{-1}}$, where $\mathbf{W}$ is expected value of negative of second derivative matrix of log likelihood with respect to $(\alpha, \underline{\beta})$.

Note: Since evaluation of expected value is cumbersome we will use sample information matrix $\hat{\mathbf{W}}$ which, under usual regularity conditions, converges asymptotically to Fisher information matrix.

4. Algorithm, Numerical Exploration and Conclusions

In this Section, a Monte-Carlo simulation study is conducted to compare the performance of the estimates developed in the previous sections. Maximum likelihood estimates are obtained for observations generated through the Type II censoring design when numbers of systems to be compared 2 and 3 for known as well as unknown shape parameter having failure distribution is the GED$(\alpha, \beta_i); i = 1, 2, ..., m$. All calculations are performed on the *R*-language version *R*.2.12.0. The simulation study is conducted for both known as well as unknown shape parameter.

4.1. Known Shape Parameter

In this section, we carry out simulation study for two sets of parameter values $m = 2, \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.3$ and for $m = 3, \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.3, \beta_3 = 1.4$. The simulation is carried out for different values of u and G^*. Here we kept total number of failures in whole experiment $G = uG^*$ fixed. We simulate 1000 samples for each case. The simulation results are summarized in Table 1 and Table 2. The value of α are taken larger than 2 as per the suggestions given in Gupta and Kundu (1999). To carry out our objective we proceed through the following algorithm.

step 1 Taking $m = 2, \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.3$ we generate u random numbers from $GE(\alpha, \beta_1, \beta_2, ..., \beta_m)$ for each type of systems. The same is repeated for the parameters $m = 3, \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.3, \beta_3 = 1.4$.

step 2 Generate G^* Type II censored observations for each type of systems. The generated G^* failure times are $(t_{1i}, t_{2i}, ..., t_{G^*i}); i = 1, 2, ..., m$ for each type of systems.

step 3 Obtain initial estimate of parameter $\beta_i; i = 1, 2, ..., m$ by substituting $\alpha = 1$ in equation (13). Therefore, we have,

$$\hat{\beta}_{i0} = \frac{G^*}{\sum_{g=1}^{G^*} t_{gi} + (u - G^*) t_{G^*i}}. \text{ For } i = 1, 2, ..., m$$

step 4 Obtain initial value of sample information matrix $\hat{\mathbf{V}}$ using the value obtained in Step 3 and also obtain the score vector $S' = (\frac{\partial l}{\partial \beta_1}, \frac{\partial l}{\partial \beta_2}, ..., \frac{\partial l}{\partial \beta_m})$

step 5 Use Newton-Raphson iterative method

$$\underline{\hat{\beta}}_{New} = \underline{\hat{\beta}}_{Old} + \hat{\mathbf{V^{-1}}}(\underline{\hat{\beta}}_{Old}) * S$$

step 6 Repeat Step 5 until the $\sum_{i=1}^{m} |\hat{\beta}_{iNew} - \hat{\beta}_{iOld}| < \epsilon$ where ϵ is very small predefined quantity.

step 7 Repeat the procedures in Step 1 to Step 6 for $N = 1000$ times and obtain following quantities.

(a) $EV_i = \frac{\sum_{j=1}^{N} \hat{\beta}_{ij}}{N}$

(b) Mean Squared Error, $MSE_i = \frac{\sum_{j=1}^{N} (\hat{\beta}_{ij} - \beta_i)^2}{N}$ where $\beta_i; i = 1, 2, .., m$ the values of parameters given in Step 1.

(c) Average Variance-Covariance Matrix $\hat{\mathbf{V^{-1}}}$.

(d) Reliability functions $\hat{R_{ij}}(t_i)$ and hazard rate $\hat{h_{ij}}(t_i); i = 1, 2, .., m$; $j = 1, 2, ..., N$ evaluate using equations (3.4-3.5) and corresponding MSEs are $\frac{\sum_{j=1}^{N} (\hat{R_{ij}}(t_i) - R_i(t_i))^2}{N}$ and $\frac{\sum_{j=1}^{N} (\hat{h_{ij}}(t_i) - h_i(t_i))^2}{N}$

Step 8 Obtain Standard Error (SE) of estimates by taking square root of diagonal elements of $\hat{\mathbf{V^{-1}}}$.

Table 1: Maximum Likelihood Estimate of Parameters, Reliability and Hazard Rates and their Efficiency Measures $m = 2, \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.3, \underline{t} = (0.8536, 0.9849), R(\underline{t}) = (0.5570, 0.5570), h(\underline{t}) = (0.8291, 0.7185)$

u	G^*		$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{R}_1(t_1)$	$\hat{R}_2(t_2)$	$\hat{h}_1(t_1)$	$\hat{h}_2(t_2)$
12	06	EV	1.5866	1.3562	0.5382	0.5450	0.9459	0.7982
		MSE	0.1831	0.1312	0.0181	0.0167	0.2294	0.1624
		SE	0.3982	0.3404	-	-	-	-
24	12	EV	1.5486	1.3484	0.5443	0.5420	0.8945	0.7804
		MSE	0.0736	0.0578	0.0085	0.0088	0.0895	0.0702
		SE	0.2692	0.2345	-	-	-	-
36	16	EV	1.5262	1.3238	0.5560	0.5504	0.8648	0.7511
		MSE	0.0469	0.0378	0.0057	0.0061	0.0556	0.0450
		SE	0.2154	0.1870	-	-	-	-
48	24	EV	1.5265	1.3182	0.5498	0.5515	0.8635	0.7426
		MSE	0.0364	0.0240	0.0045	0.0040	0.0430	0.0283
		SE	0.1816	0.1606	-	-	-	-
60	30	EV	1.5173	1.3150	0.5526	0.5525	0.8524	0.7386
		MSE	0.0288	0.0206	0.0036	0.0034	0.0338	0.0242
		SE	0.1652	0.1431	-	-	-	-
72	36	EV	1.5131	1.3157	0.5536	0.5519	0.8466	0.7387
		MSE	0.0208	0.0171	0.0027	0.0029	0.0242	0.0201
		SE	0.1501	0.1306	-	-	-	-
84	42	EV	1.5123	1.3138	0.5538	0.5524	0.8456	0.7359
		MSE	0.0194	0.0135	0.0025	0.0023	0.0224	0.0157
		SE	0.1389	0.1206	-	-	-	-
96	48	EV	1.5088	1.3082	0.5548	0.5547	0.8413	0.7300
		MSE	0.0162	0.0135	0.0021	0.0023	0.0187	0.0156
		SE	0.1295	0.1123	-	-	-	-

From Table 1 and Table 2 we observe that for the known shape parameter α, the means of MLEs for scale parameters $\beta_i; i = 1, 2, ..., m$, the reliability characteristics and hazard rates are very close to their true values. At average mean square errors are relatively small. Further, we observe that the estimates and standard/mean square error are decreasing functions of number u of each systems put on test.

4.2. Unknown Shape Parameter

Similar study, with not much change in the algorithm, one can make simulation study for the case of unknown shape parameter. We make simulation studies for the set of parameters values $m = 2, \alpha =$

Table 2: Maximum Likelihood Estimate of Parameters, Reliability and Hazard Rates and their Efficiency Measures
$m = 2, \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.3, \beta_3 = 1.4,$
$\underline{t} = (0.8536, 0.9849, 0.9146)$, $R(\underline{t}) = (0.5570, 0.5570, 0.5570)$, $h(\underline{t}) = (0.8291, 0.7185, 0.7738)$

u	G^*		$\hat{\beta_1}$	$\hat{\beta_2}$	$\hat{\beta_3}$	$\hat{R}_1(t_1)$	$\hat{R}_2(t_2)$	$\hat{R}_3(t_3)$	$\hat{h}_1(t_1)$	$\hat{h}_2(t_2)$	$\hat{h}_3(t_3)$
24	08	EV	1.5751	1.3520	1.4595	0.5373	0.5428	0.5417	0.9251	0.7875	0.8528
		MSE	0.1103	0.0823	0.1004	0.0118	0.0117	0.0123	0.1368	0.1022	0.1239
		SE	0.3185	0.2735	0.2956	-	-	-	-	-	-
36	12	EV	1.5460	1.3430	1.4358	0.5449	0.5436	0.5475	0.8889	0.7735	0.8216
		MSE	0.0679	0.0510	0.0568	0.0079	0.0078	0.0076	0.0816	0.0620	0.0684
		SE	0.2532	0.2199	0.2351	-	-	-	-	-	-
48	16	EV	1.5152	1.3327	1.4300	0.5544	0.5465	0.5487	0.8530	0.7603	0.8136
		MSE	0.0451	0.0363	0.0446	0.0056	0.0058	0.0061	0.0527	0.0432	0.0532
		SE	0.2139	0.1882	0.2021	-	-	-	-	-	-
60	20	EV	1.5171	1.3184	1.4153	0.5531	0.5560	0.5534	0.8533	0.7531	0.7957
		MSE	0.0351	0.0287	0.0309	0.0044	0.0048	0.0044	0.0409	0.0335	0.0363
		SE	0.1911	0.1649	0.1783	-	-	-	-	-	-
72	24	EV	1.5126	1.3171	1.4176	0.5545	0.5519	0.5522	0.8480	0.7415	0.7975
		MSE	0.0323	0.0236	0.0271	0.0040	0.0040	0.0039	0.0378	0.0275	0.0318
		SE	0.1738	0.1513	0.1628	-	-	-	-	-	-
84	28	EV	1.5174	1.3143	1.4188	0.5524	0.5527	0.5513	0.8521	0.7365	0.7980
		MSE	0.0267	0.0194	0.0224	0.0033	0.0032	0.0032	0.0314	0.0228	0.0263
		SE	0.1612	0.1396	0.1507	-	-	-	-	-	-
96	32	EV	1.5175	1.3136	1.4061	0.5522	0.5585	0.5560	0.8520	0.7365	0.7840
		MSE	0.0245	0.0179	0.0195	0.0030	0.0029	0.0028	0.0286	0.0211	0.0226
		SE	0.1507	0.1305	0.1396	-	-	-	-	-	-

$2.5, \beta_1 = 1.5, \beta_2 = 1.3, \underline{t} = (0.8536, 0.9849)$ and for $m = 3, \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.3, \beta_3 = 1.4, \underline{t} = (0.8536, 0.9849, 0.9146)$ by taking $N = 1000$.

In the presence of unknown shape parameter α, from Table 3 and Table 4 it is seen that the MLEs of scale parameters $\beta_i; i = 1, 2, ..., m$, the reliability characteristics and hazard rates are reaching close to their true values. However the convergence rate is slow compared to the convergence rate when shape parameter is known. Perhaps, it may be the effect of estimate of unknown shape parameter α. Further, we can say, somewhat large sample size is required than what we consider for the estimates to reach their true values.

5. Testing of Hypotheses

The proposed design will have significance only when we are able to ascertain that the m type of systems are not all have identical life time. This can be done by developing ANOVA approach for the proposed design. However, we will utilize likelihood approach to develop a test. The testing hypothesis problem is to test

$$H_0 : \beta_1 = \beta_2 = ... = \beta_m = \beta \text{ against } H_1 : \beta_i \neq \beta_j \text{ for at least one pair } (i, j), i \neq j \tag{22}$$

As we are considering maximum likelihood estimation, the use of likelihood ratio test is much convenient. The test statistic is

$$\lambda_{LR} = \frac{max_{\alpha,\beta} L(\underline{t}, \beta, \alpha)}{max_{\alpha,\underline{\beta}} L(\underline{t}, \underline{\beta}, \alpha)}$$

The test based on $-2ln(\lambda_{LR})$ rejects H_0 in support of H_1 if it is larger than upper α-th cut-off point of chi-square distribution $(m - 1)$ degrees of freedom.

Table 3: Maximum Likelihood Estimate of Parameters, Reliability and Hazard Rates and their Efficiency Measures $m = 2, \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.3, \underline{t} = (0.8536, 0.9849), R(\underline{t}) = (0.5570, 0.5570), h(\underline{t}) = (0.8291, 0.7185)$

u	G^*		$\hat{\alpha}$	$\hat{\beta_1}$	$\hat{\beta_2}$	$\hat{R}_1(t_1)$	$\hat{R}_2(t_2)$	$\hat{h}_1(t_1)$	$\hat{h}_2(t_2)$
12	06	EV	3.6307	1.8729	1.6556	0.5331	0.5230	1.1589	1.0473
		MSE	6.0714	0.6009	0.5327	0.0199	0.0232	0.5369	0.5088
		SE	2.1642	0.6795	0.6023	-	-	-	-
24	12	EV	3.0577	1.7022	1.4842	0.5442	0.5419	1.0048	0.8848
		MSE	1.7537	0.2595	0.2296	0.0099	0.0110	0.2191	0.2026
		SE	1.0701	0.4482	0.3917	-	-	-	-
36	18	EV	2.8157	1.6382	1.4158	0.5438	0.5457	0.9533	0.8231
		MSE	0.6793	0.1448	0.1179	0.0066	0.0065	0.1206	0.0999
		SE	0.7395	0.3570	0.3088	-	-	-	-
48	24	EV	2.7277	1.5942	1.3898	0.5496	0.5465	0.9123	0.8009
		MSE	0.4482	0.0997	0.0860	0.0047	0.0049	0.0807	0.0718
		SE	0.6076	0.3029	0.2645	-	-	-	-
60	30	EV	2.6567	1.5687	1.3574	0.5506	0.5517	0.8914	0.7716
		MSE	0.3026	0.0745	0.0642	0.0040	0.0039	0.0618	0.0532
		SE	0.5196	0.2683	0.2323	-	-	-	-
72	36	EV	2.6308	1.5572	1.3516	0.5515	0.5510	0.8801	0.7665
		MSE	0.2484	0.0621	0.0518	0.0029	0.0033	0.0487	0.0422
		SE	0.4669	0.2435	0.2116	-	-	-	-
84	42	EV	2.5873	1.5339	1.3306	0.5544	0.5542	0.8602	0.7477
		MSE	0.1878	0.0461	0.0405	0.0024	0.0037	0.0358	0.0323
		SE	0.4211	0.2229	0.1936	-	-	-	-
96	48	EV	2.5830	1.5353	1.3320	0.5539	0.5533	0.8620	0.7486
		MSE	0.1608	0.0458	0.0343	0.0023	0.0024	0.0307	0.0277
		SE	0.3915	0.2086	0.1811	-	-	-	-

5.1. Computation of Likelihood Under H_0

The log likelihood ln_L under null hypothesis from equation (11) we have,

$$\begin{aligned} l &= mln(\frac{u!}{(u-G^*)!}) + mG^* ln\alpha + mG^* ln\beta - \beta \sum_{i=1}^{m} \sum_{g=1}^{G^*} t_{gi} \\ &+ (\alpha - 1) \sum_{i=1}^{m} \sum_{g=1}^{G^*} ln(1 - e^{\beta t_{gi}}) + (u - G^*) \sum_{i=1}^{m} ln[1 - (1 - e^{-\beta t_{G^* i}})^{\alpha}]. \end{aligned} \tag{23}$$

Differentiate (23) with respect to α and β we have

$$\begin{aligned} \frac{\partial l}{\partial \alpha} &= \frac{mG^*}{\alpha} + \sum_{i=1}^{m} \sum_{g=1}^{G^*} ln(1 - e^{-\beta t_{gi}}) \\ &- (u - G^*) \sum_{i=1}^{m} \frac{(1 - e^{-\beta t_{G^* i}})^{\alpha} ln[1 - e^{-\beta t_{G^* i}}]}{1 - (1 - e^{-\beta t_{G^* i}})^{\alpha}} \end{aligned} \tag{24}$$

$$\begin{aligned} \frac{\partial l}{\partial \beta} &= \frac{mG^*}{\beta} + (\alpha - 1) \sum_{i=1}^{m} \sum_{g=1}^{G^*} [\frac{t_{gi} e^{-\beta t_{gi}}}{1 - e^{-\beta t_{gi}}}] - \sum_{i=1}^{m} \sum_{g=1}^{G^*} t_{gi} \\ &- \alpha (u - G^*) \sum_{i=1}^{m} t_{G^* i} \frac{e^{-\beta t_{G^* i}} (1 - e^{-\beta t_{G^* i}})^{\alpha - 1}}{1 - (1 - e^{-\beta t_{G^* i}})^{\alpha}} \end{aligned} \tag{25}$$

Table 4: Maximum Likelihood Estimate of Parameters, Reliability and Hazard Rates and their Efficiency Measures $m = 2, \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.3, \beta_3 = 1.4, \underline{t} = (0.8536, 0.9849, 0.9146), R(\underline{t}) = (0.5570, 0.5570, 0.5570), h(\underline{t}) = (0.8291, 0.7185, 0.7738)$

u	G^*		$\hat{\alpha}$	$\hat{\beta_1}$	$\hat{\beta_2}$	$\hat{\beta_3}$	$\hat{R}_1(t_1)$	$\hat{R}_2(t_2)$	$\hat{R}_3(t_3)$	$\hat{h}_1(t_1)$	$\hat{h}_2(t_2)$	$\hat{h}_3(t_3)$
24	08	EV	3.0431	1.7862	1.5512	1.6477	0.5222	0.5224	0.5297	1.1030	0.9636	1.0092
		MSE	1.2550	0.3499	0.2997	0.3148	0.0146	0.0160	0.0143	0.3414	0.3499	0.3100
		SE	0.9760	0.5281	0.4596	0.4814	-	-	-	-	-	-
36	12	EV	2.8822	1.6976	1.4792	1.5871	0.5339	0.5320	0.5338	1.0140	0.8906	0.9511
		MSE	0.7321	0.2090	0.1787	0.1971	0.0011	0.0105	0.0101	0.1922	0.1696	0.1881
		SE	0.7309	0.4153	0.3628	0.3891	-	-	-	-	-	-
48	16	EV	2.7466	1.6383	1.4097	1.5222	0.5391	0.5436	0.5414	0.9621	0.8240	0.8907
		MSE	0.4266	0.1409	0.1165	0.1264	0.0073	0.0075	0.0068	0.1365	0.1079	0.1137
		SE	0.5862	0.3524	0.3034	0.3272	-	-	-	-	-	-
60	20	EV	2.7173	1.6185	1.4000	1.5179	0.5422	0.5437	0.5396	0.9408	0.8137	0.8871
		MSE	0.3482	0.1132	0.0939	0.1083	0.0057	0.0058	0.0058	0.1007	0.0847	0.0976
		SE	0.5147	0.3119	0.2699	0.2925	-	-	-	-	-	-
72	24	EV	2.6625	1.5744	1.3762	1.4915	0.5514	0.5466	0.5427	0.8976	0.7909	0.8611
		MSE	0.2335	0.0848	0.0665	0.0730	0.0044	0.0045	0.0043	0.0740	0.0584	0.0646
		SE	0.4542	0.2787	0.2437	0.2638	-	-	-	-	-	-
84	28	EV	2.6475	1.5768	1.3736	1.4772	0.5483	0.5456	0.5466	0.9002	0.7887	0.8474
		MSE	0.2008	0.0699	0.0582	0.0702	0.0037	0.0037	0.0040	0.0604	0.0500	0.0622
		SE	0.4164	0.2586	0.2253	0.2424	-	-	-	-	-	-
96	32	EV	2.6076	1.5589	1.3467	1.4554	0.5495	0.5514	0.5493	0.8856	0.7637	0.8272
		MSE	0.1668	0.0608	0.0497	0.0533	0.0032	0.0034	0.0032	0.0517	0.0425	0.0455
		SE	0.3812	0.2402	0.2076	0.2243	-	-	-	-	-	-

Differentiate (24) and (25) with respect to (α, β) and β respectively, we have

$$\frac{\partial^2 l}{\partial \alpha^2} = \frac{-mG^*}{\alpha^2} - (u - G^*)\sum_{i=1}^{m}\{\frac{[ln(1 - e^{-\beta t_{G^*i}})]^2(1 - e^{-\beta t_{G^*i}})^\alpha}{[1 - (1 - e^{-\beta t_{G^*i}})^\alpha]^2}\} \tag{26}$$

$$\begin{aligned}\frac{\partial^2 l}{\partial \alpha \partial \beta} &= \sum_{i=1}^{m}\sum_{g=1}^{G^*}[\frac{t_{gi}e^{-\beta t_{gi}}}{1 - e^{-\beta t_{gi}}}] \\ &\quad - (u - G^*)\sum_{i=1}^{m}\{\frac{\frac{1-(1-e^{-\beta t_{G^*i}})^\alpha}{\alpha} + ln(1 - e^{-\beta t_{G^*i}})}{[1 - (1 - e^{-\beta t_{G^*i}})^\alpha]^2}\}\end{aligned} \tag{27}$$

$$\begin{aligned}\frac{\partial^2 l}{\partial \beta^2} &= \frac{-mG^*}{\beta^2} - (\alpha - 1)\sum_{i=1}^{m}\sum_{g=1}^{G^*}[\frac{t_{gi}^2 e^{-\beta t_{gi}}}{(1 - e^{-\beta t_{gi}})^2}] \\ &\quad - \alpha(u - G^*)\sum_{g=1}^{G^*}\frac{t_{G^*i}^2 e^{-\beta t_{G^*i}}(1 - e^{-\beta t_{G^*i}})^{\alpha-2}}{[1 - (1 - e^{-\beta t_{G^*i}})^\alpha]^2} \\ &\quad \times \{\alpha e^{-\beta t_{G^*i}} - 1 + (1 - e^{-\beta t_{G^*i}})^\alpha\}.\end{aligned} \tag{28}$$

The likelihood equation (25) is not mathematically tractable for known as well as unknown shape parameter we use the Newton-Rapshon method to obtain the estimate of parameter β. Here we deal with only known shape parameter. We demonstrate the test procedure for $m = 2$ and $m = 3$. We generate data under our design for the parameter values under $H_1 : \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.3$ and $H_1 : \alpha = 2.5, \beta_1 = 1.9, \beta_2 = 1.5, \beta_3 = 1$ respectively. Then carry out the test procedure as suggested above. The procedure is repeated for the different choices of u and G^*. The results are produced in the Table 5 and Table 6 respectively.

From the Table 5, we infer that the power the test is poor for small sizes. However as the sample size becomes 48(total number failures observed irrespective type of systems) it exhibits its power in identifying the alternative. It requires more sample size to detect small departure from homogeneity. From Table 6, it can reveal that for comparing homogeneity of three systems, as sample size becomes 36 it exhibits its power

Table 5: Likelihood Ratio Test for Testing $H_0 : \beta_1 = \beta_2 = \beta$ vs $H_1 : \beta_1 \neq \beta_2$ when $\alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.3$

u	G^*	$\hat{\beta}$	$\hat{\beta_1}$	$\hat{\beta_2}$	LL_{H_0}	LL_{H_1}	χ^2	$p-value$
12	06	1.3296	1.8135	1.2086	13.84	14.15	0.6096	0.4349
24	12	1.2681	1.5701	1.1866	31.22	31.56	0.6765	0.4108
36	18	1.2457	1.5548	1.2282	70.75	71.68	1.8739	0.1710
48	24	1.1657	1.3578	1.1889	105.91	106.83	1.8307	0.1760
60	30	1.1580	1.4717	1.1160	145.15	146.34	2.3923	0.1219
72	36	1.2923	1.6252	1.2725	194.85	196.60	3.5058	0.0612
84	42	1.1948	1.5914	1.1563	235.97	238.30	4.6467	0.0311
96	48	1.2138	1.6765	1.1844	281.69	285.32	7.2498	0.0071

Table 6: Likelihood Ratio Test for Testing $H_0 : \beta_1 = \beta_2 = \beta_2 = \beta$ vs $H_1 : \beta_i \neq \beta_j (i \neq j = 1, 2, 3)$ when $\alpha = 2.5, \beta_1 = 1.9, \beta_2 = 1.5, \beta_2 = 1$

u	G^*	$\hat{\beta}$	$\hat{\beta_1}$	$\hat{\beta_2}$	$\hat{\beta_3}$	LL_{H_0}	LL_{H_1}	χ^2	$p-value$
24	08	1.3736	2.2309	1.9322	1.1655	41.01	43.89	5.6399	0.0596
36	12	1.2177	1.8818	1.7630	1.0006	70.12	73.77	7.2916	0.0261
48	16	1.2274	1.9504	1.6411	1.0563	104.12	108.22	8.1909	0.0166
60	20	1.2253	2.0577	1.5246	1.0721	140.79	145.01	8.5668	0.0137
72	24	1.2419	1.8800	1.4837	1.1245	184.35	189.18	9.7386	0.0076
84	28	1.2376	1.9139	1.5230	1.1044	226.71	232.17	10.92	0.0042
96	32	1.2303	1.8886	1.5768	1.0726	276.41	283.00	13.1624	0.0014

to identify magnitude of departure from homogeneity. The consistency of the test is also inferred as sample size tends to 96 the p -value becomes almost zero up to two digits in both Table 5 and Table 6.

6. Concluding Remarks

In this article, we have studied fitting of generalized exponential distribution model for several systems when sample observations are drawn based on Type II censoring scheme. Further, we carried out simulation study to demonstrate the performance of the estimators in terms of their MSE and SE. Finally, we provided likelihood ratio test for homogeneity of lifetimes of several of systems.

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