ProbStat Forum, Volume 08, January 2015, Pages 34–44
ISSN 0974-3235

ProbStat Forum is an e-journal.
For details please visit www.probstat.org.in

Characterization of one-truncation parameter family of distributions through expectation

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Abstract. For characterization of one (left or right)-truncation parameter families of distributions (which includes notably negative exponential distribution, Pareto distribution, power function distribution, uniform distribution and generalized uniform distribution as special case) one needs any arbitrary non-constant function only in place of various approaches such as identical distributions, absolute continuity, constancy of regression of order statistics, continuity and linear regression of order statistics, non-degeneracy available in the literature. Path breaking different approach for characterization of general setup of one-truncation parameter family of distributions through expectation of any arbitrary non constant differentiable function of random variable is obtained. Applications and examples are given for illustrative purpose.

1. Introduction

One-truncation parameter family of distributions with probability density function (pdf)

$$f_j(x;\theta) = \begin{cases} q_j(\theta)h_j(x) & \\ 0, & \text{otherwise,} \end{cases} \tag{1}$$

where $-\infty \leq a < b \leq \infty$ are known constant, $a < \theta < x < b$ for $j = 1$, $a < x < \theta < b$ for $j = 2$, $h_j; (j = 1, 2)$ are positive absolutely continuous functions, $q_j; (j = 1, 2)$ are everywhere differentiable functions is characterized.

Since $h_j(.)$;(j = 1 or 2) is positive and the range is truncated by truncation parameter θ from left or right respectively $q_1^{-1}(b) = q_2^{-1}(a) = 0$.

Using identity of distribution and equality of expectation of function of function, characterization for general set up of one (left or right)-truncation parameter family of distributions defined in (1) through expectation of any arbitrary non-constant differentiable function is given which includes characterization of negative exponential distribution, Pareto distribution as special case of $f_1(x;\theta)$ where as power function distribution, uniform distribution, generalize uniform distribution as special case of $f_2(x;\theta)$.

Keywords. characterization, truncation parameter families of distributions, negative exponential distribution, Pareto distribution, power function distribution, uniform distribution, generalized uniform distribution
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Several characterizations of these distributions by various approaches are available in the literature. Notably for power function distribution independence of suitable function of order statistics and distributional properties of transformation of exponential variable used by Fisz (1958), Basu (1965), Govindarajulu (1966) and Dallas (1976), linear relation of conditional expectation used by Beg and Kirmani (1974), recurrence relations between expectations of function of order statistics used by Ali and Khan (1998), record valves used by Nagraja (1977), lower record statistics used by Faizan and Khan (2011), product of order statistics used by Arslan (2011) and Lorenz curve used by Moothathu (1986) are available in the literature.

Other approaches such as coefficient of correlation of order statistics of sample of size two used by Bartoszynski (1980), Terrel (1983), Fernando and Rebollo (1997), maximal correlation coefficient between order statistics of identically distributed spacings *etc.* [used by Stapleton (1963), Arnold and Meeden (1976), Driscoll (1978), Shimizu and Huang (1983), Abdelhamid (1985)], moment conditions used by Lin (1988), Too and Lin (1989), moments of n-fold convolution modulo one used by Chow and Huang (1999), inequalities of chernoff-type used by Sumitra and Subir (1990) for characterization of uniform distribution.

Various approaches were used for characterization of negative exponential distribution. Amongst many other Fisz (1958), Tanis (1964), Rogers (1963) and Ferguson (1967) used properties of identical distributions, absolute continuity, constant regression of adjacent order statistics, Ferguson (1964, 1965) and Crawford (1966), used linear regression of adjacent order statistics of random, independent and non degenerate random variables, Nagaraja (1977, 1988) used linear regression of two adjacent record values were as Khan *et al.* (2009) used difference of two conditional expectations, conditioned on a non-adjacent order statistics to characterized negative exponential distribution.

Economic variation in reported income and true income used by Krishnaji (1970), Nagesh *et al.* (1974), independence of suitable function of order statistics used by Malik (1970), Ahsanullah and Kabir (1974), Shah and Kabe (1981) and Dimaki and Xekalaki (1993), mean and the extreme observation of the sample used by Srivastava (1965), linear relation of conditional expectation used by Beg and Kirmani (1974), Dallas (1976), recurrence relations between expectations of function of order statistics used by Ali and Khan (1998), exponential and related distributions used by Tavangar and Asadi (2010), for characterization of Pareto distribution.

Necessary and sufficient conditions for pdf $f(x;\theta)$ to be $f_j(x;\theta)$, (j = 1 or 2), defined in (1) is established in section 2. Section 3 is devoted for applications where as section 4 is devoted to examples for illustrative purpose.

2. Characterization theorem

Let X be a random variable (r.v) with distribution function F. Assume that F is continuous on the interval (a,b), where $-\infty \le a < b \le \infty$. Let $g(X)$ be a non-constant differentiable function of X on the interval (a,b), where $-\infty \le a < b \le \infty$ and more over $g(X)$ be non constant. Then $f(x;\theta)$ is $f_j(x;\theta)$, pdf defined in (1) if and only if

$$E\Big[g(X) + \frac{\frac{d}{dX}g(X)}{M(X)}\Big] = g(\theta) \tag{2}$$

where $M(X)$ is finite function.

Proof. Given $f_j(x;\theta)$ defined in (1), for necessity of (2) if $\phi(X)$ is such that $g(\theta) = E[\phi(X)]$ where $g(\theta)$ is differentiable function then

$$g(\theta)=\begin{cases}\int_\theta^b \phi(x)f_1(x;\theta)\mathrm{d}x for j=1\\ \int_a^\theta \phi(x)f_2(x;\theta)\mathrm{d}x for j=2,\end{cases} \tag{3}$$

Differentiating with respect to θ on both sides of (3) and replacing X for θ, and denoting finite function

$$M(X)=\frac{d}{dX}\Big[\log\Big(q_j^{-1}(X)\Big)\Big], \tag{4}$$

and simplifying one gets

$$\phi(X)=g(X)+\frac{\frac{d}{dX}g(X)}{M(X)}, \tag{5}$$

which establishes the necessity of (2). Conversely given (2) let $k_j(x;\theta), j=1,2$ be pdf of r.v X such that

$$g(\theta)=\begin{cases}\int_\theta^b \phi(x)k_1(x;\theta)\mathrm{d}x for j=1\\ \int_a^\theta \phi(x)k_2(x;\theta)\mathrm{d}x for j=2\end{cases} \tag{6}$$

Since $q_1^{-1}(b)=q_2^{-1}(a)=0$ the following identity holds.

$$g(\theta)=\begin{cases}-q_1(\theta)\int_\theta^b \Big[\frac{d}{dx}g(x)q_1^{-1}(x)\Big]\mathrm{d}x for j=1\\ q_2(\theta)\int_\theta^b \Big[\frac{d}{dx}g(x)q_2^{-1}(x)\Big]\mathrm{d}x for j=2\end{cases} \tag{7}$$

Differentiating the integrand of (7) $g(x)q_j^{-1}(x)$, $(j=1,2)$ and taking $\frac{d}{dx}q_j^{-1}(x)$ as one factor one gets

$$g(\theta)=\begin{cases}\int_\theta^b \phi(x)\Big[-q_1(\theta)\frac{d}{dx}q_1^{-1}(x)\Big]\mathrm{d}x for j=1\\ \int_a^\theta \phi(x)\Big[q_2(\theta)\frac{d}{dx}q_2^{-1}(x)\Big]\mathrm{d}x for j=2,\end{cases} \tag{8}$$

where $\phi(X)$ is a function of X derived in (5) for $j=1,2$. From (6) and (8) by uniqueness theorem

$$k_j(x;\theta)=\begin{cases}-q_1(\theta)\frac{d}{dx}q_1^{-1}(x) for j=1\\ q_2(\theta)\frac{d}{dx}q_2^{-1}(x) for j=2.\end{cases} \tag{9}$$

Since q_1 is increasing function of θ with $q_1^{-1}(b)=0$ and q_2 is decreasing function of θ with $q_2^{-1}(a)=0$ integrating (9) on both sides one gets

$$1=\begin{cases}\int_\theta^b k_1(x;\theta)\mathrm{d}x for j=1\\ \int_\theta^b k_2(x;\theta)\mathrm{d}x for j=2.\end{cases} \tag{10}$$

and denoting

$$h_j(x) = (-1)^j \frac{d}{dx} q_j^{-1}(x), \tag{11}$$

one gets (10) as

$$k_j(x;\theta) = \begin{cases} q_j(\theta)h_j(x) & \\ 0, & \text{otherwise,} \end{cases} \tag{12}$$

Hence $k_j(x;\theta)$ reduces to $f_j(x;\theta)$ defined in (1) which establishes sufficiency of (2).

Remark. Using $\phi(X)$ given in (5) one can determine $f_j(x;\theta)$ by

$$M(X) = \frac{\frac{d}{dX} g(X)}{\phi(X) - g(X)}, \tag{13}$$

and pdf is given by

$$f_j(x;\theta) = (-1)^j \frac{\frac{d}{dX} q_j^{-1}(x)}{q_j^{-1}(\theta)}, \tag{14}$$

where $q_j^{-1}(x)$ is decreasing function for $-\infty \leq a < b \leq \infty$ with $q_j^{-1}(b) = 0$ for $j = 1$ and $q_j^{-1}(x)$ is increasing function for $-\infty \leq a < b \leq \infty$ with $q_j^{-1}(a) = 0$ for $j = 2$ such that it satisfies (4) for $j = 1, 2$.

3. Special cases, characterizations of various distributions

As special cases of the characterization theorem following distributions are characterized.

(A) Characterization of negative exponential distribution with pdf

$$f_3(x;\theta) = \begin{cases} \exp -(x-\theta); & a < \theta < x < b, \\ 0, & \text{otherwise.} \end{cases} \tag{15}$$

The sufficient condition in characterization theorem being

$$E\Big[g(X) - \frac{d}{dX} g(X)\Big] = g(\theta) \tag{16}$$

where $g(\theta)$ is non-constant function. From (13) $M(X)$ turns out as -1 and hence using (4)

$$M(X) = \frac{d}{dX}\Big[\log\Big(q_j^{-1}(X)\Big)\Big] = -1 \Rightarrow q_j^{-1}(X) = \exp(-X), \tag{17}$$

which is decreasing function on $-\infty \leq a < b \leq \infty$ with $q_j^{-1}(b) = 0$ therefore $-\infty \leq a < \theta < x < b \leq \infty$ and by using (11)

$$h_j(x) = \exp(-X). \tag{18}$$

Substituting these values in (1) for j=1, $f_1(x;\theta)$ reduces to $f_3(x;\theta)$ defined in (15). Thus negative exponential distribution is characterized.

(B) Characterization of Pareto distribution with pdf

$$f_4(x;\theta) = \begin{cases} \frac{c\theta^c}{x^{c+1}}; & a < \theta < x < b. \\ 0, & \text{otherwise,} \end{cases} \tag{19}$$

The sufficient condition in characterization theorem being

$$E\Big[g(X) - \frac{X}{c}\frac{d}{dX}g(X)\Big] = g(\theta) \tag{20}$$

where $g(\theta)$ is non-constant function. From (13) $M(X)$ turns out as $-c/X$ and hence using (4)

$$M(X) = \frac{d}{dX}\Big[\log\Big(q_j^{-1}(X)\Big)\Big] = -\frac{X}{c} \Rightarrow q_j^{-1}(X) = \frac{1}{cX^c}, \tag{21}$$

which is decreasing function on $-\infty \leq a < b \leq \infty$ with $q_j^{-1}(b) = 0$ therefore $-\infty \leq a < \theta < x < b \leq \infty$ and by using (11)

$$h_j(X) = \frac{1}{X^{c+1}}. \tag{22}$$

Substituting these values in (1) for j=1, $f_1(x;\theta)$ reduces to $f_4(x;\theta)$ defined in (19). Thus Pareto distribution is characterized.

(C) Characterization of power function distribution with pdf

$$f_5(x;\theta) = \begin{cases} c\theta^{-c}x^{c-1}; & a < x < \theta < b, \\ 0, & \text{otherwise.} \end{cases} \tag{23}$$

The sufficient condition in characterization theorem being

$$E\Big[g(X) + \frac{X}{c}\frac{d}{dX}g(X)\Big] = g(\theta) \tag{24}$$

where $g(\theta)$ is non-constant function. From (13) $M(X)$ turns out as c/X and hence using (4)

$$M(X) = \frac{d}{dX}\Big[\log\Big(q_j^{-1}(X)\Big)\Big] = \frac{X}{c} \Rightarrow q_j^{-1}(X) = \frac{X^c}{c}, \tag{25}$$

which is increasing function on $-\infty \leq a < b \leq \infty$ with $q_j^{-1}(a) = 0$ therefore $-\infty \leq a < x < \theta < b \leq \infty$ and by using (11)

$$h_j(X) = X^{c+1}. \tag{26}$$

Substituting these values in (1) for $j = 2$, $f_2(x;\theta)$ reduces to $f_5(x;\theta)$ defined in (23). Thus power function distribution is characterized.

(D) Characterization of uniform distribution with pdf

$$f_6(x;\theta) = \begin{cases} \frac{1}{\theta} & a < x < \theta < b, \\ 0, & \text{otherwise.} \end{cases} \tag{27}$$

The sufficient condition in characterization theorem being

$$E\Big[g(X) + X\frac{d}{dX}g(X)\Big] = g(\theta) \tag{28}$$

where $g(\theta)$ is non-constant function. From (13) $M(X)$ turns out as $1/X$ and hence using (4)

$$M(X) = \frac{d}{dX}\Big[\log\Big(q_j^{-1}(X)\Big)\Big] = \frac{X}{c} \Rightarrow q_j^{-1}(X) = X, \tag{29}$$

which is increasing function on $-\infty \leq a < b \leq \infty$ with $q_j^{-1}(a) = 0$ therefore $-\infty \leq a < x < \theta < b \leq \infty$ and by using (11)

$$h_j(x) = 1. \tag{30}$$

Substituting these values in (1) for $j = 2$, $f_2(x;\theta)$ reduces to $f_6(x;\theta)$ defined in (27). Thus uniform distribution is characterized.

(E) Characterization of generalized uniform distribution with pdf

$$f_6(x;\theta) = \begin{cases} \frac{\alpha+1}{\theta^{\alpha+1}}x^{\alpha} & a < x < \theta < b, \\ 0, & \text{otherwise.} \end{cases} \tag{31}$$

The sufficient condition in characterization theorem being

$$E\Big[g(X) + \frac{X}{\alpha+1}\frac{d}{dX}g(X)\Big] = g(\theta) \tag{32}$$

where $g(\theta)$ is non-constant function. From (13) $M(X)$ turns out as $\frac{\alpha+1}{X}$ and hence using (4)

$$M(X) = \frac{d}{dX}\Big[\log\Big(q_j^{-1}(X)\Big)\Big] = \frac{X}{c} \Rightarrow q_j^{-1}(X) = \frac{X^{\alpha+1}}{\alpha+1}, \tag{33}$$

which is increasing function on $-\infty \leq a < b \leq \infty$ with $q_j^{-1}(a) = 0$ therefore $-\infty \leq a < x < \theta < b \leq \infty$ and by using (11)

$$h_j(x) = X^{\alpha}. \tag{34}$$

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Substituting these values in (1) for $j = 2$, $f_2(x;\theta)$ reduces to $f_6(x;\theta)$ defined in (31). Thus uniform distribution is characterized.

4. Examples

Let $\phi_i(X) = X$ and $E[X] = g_i(\theta)$, $i = 1, 2, 3, 4, 5$ where

$$g_i(\theta) = \begin{cases} \frac{c}{c+1}\theta; & \text{for i} = 1, \\ \frac{\theta}{2}; & \text{for i} = 2, \\ \frac{\alpha+1}{\alpha+2}; & \text{for i} = 3, \\ \theta + 1; & \text{for i} = 4, \\ \frac{c}{c-1}\theta; & \text{for i} = 5, \end{cases} \tag{35}$$

be means and let

$$\phi_i(X) = \begin{cases} \frac{c+1}{c} X p^{-\frac{1}{c}}; & \text{for i} = 6, \\ 2Xp; & \text{for i} = 7, \\ \frac{\alpha+2}{\alpha+1} X p^{\frac{1}{\alpha+1}}; & \text{for i} = 8, \\ -\log(1-p) + X - 1; & \text{for i} = 9, \\ \frac{c-1}{c} X (1-p)^{-\frac{1}{c}}; & \text{for i} = 10, \\ -\frac{\left(\frac{c}{t}\right)\left(\frac{t}{X}\right)^{2c}}{\left[\left(\frac{t}{X}\right)^{c} - 1\right]^2}; & \text{for i} = 11, \\ -\frac{t}{(t-X)^2}; & \text{for i} = 12, \\ \frac{(\alpha+1)t\left(\frac{t}{X}\right)^{2\alpha}}{\left[X - t\left(\frac{t}{X}\right)^{\alpha}\right]^2}; & \text{for i} = 13, \end{cases} \tag{36}$$

be such that $E[\phi_i(X)] = g_i(\theta)$ where

$$g_i(\theta) = \begin{cases} \theta p^{\frac{1}{c}}; & \text{for i = 6,} \\ \theta p; & \text{for i = 7,} \\ \theta p^{\frac{1}{\alpha+1}}; & \text{for i = 8,} \\ -\log(1-p)+\theta; & \text{for i = 9,} \\ \theta(1-p)^{-\frac{1}{c}}; & \text{for i = 10,} \\ -\frac{\left(\frac{c}{t}\right)}{\left(\frac{\theta}{t}\right)^{c}-1}; & \text{for i = 11,} \\ \frac{1}{(\theta-t)}; & \text{for i = 12,} \\ \frac{(\frac{\alpha+1}{\theta})\left(\frac{t}{\theta}\right)^{\alpha}}{1-\left(\frac{t}{\theta}\right)^{\alpha+1}}; & \text{for i = 13,} \end{cases} \tag{37}$$

is pth quantile for $i = 6, 7, 8, 9, 10$ and is hazard function for $i = 11, 12, 13$.

Using (13) we get

$$M(X) = \frac{\frac{d}{dX}g(X)}{\phi(X)-g(X)} = \begin{cases} \frac{c}{X}; & \text{for i = 1, 6, 11 ,} \\ \frac{1}{X}; & \text{for i = 2, 7, 12,} \\ \frac{\alpha+1}{X}; & \text{for i = 3, 8, 13,} \\ -1; & \text{for i = 4, 9,} \\ -\frac{c}{X}; & \text{for i = 5, 10.} \end{cases} \tag{38}$$

Since $q_j^{-1}(X)$ is decreasing function for $-\infty \leq a < b \leq \infty$ with $q_j^{-1}(b) = 0$ for $j = 1$ and $q_j^{-1}(X)$ is increasing function for $-\infty \leq a < b \leq \infty$ with $q_j^{-1}(a) = 0$ for $j = 2$, using (4) in (38) it follows that

$$q_j^{-1}(X) = \begin{cases} \frac{X^c}{x}; & \text{for i = 1, 6, 11 and j = 2,} \\ X; & \text{for i = 2, 7, 12, and j = 2,} \\ \frac{X^{\alpha+1}}{\alpha+1}; & \text{for i = 3, 8, 13, and j = 2,} \\ \exp(-X); & \text{for i = 4, 9, and j = 1,} \\ -\frac{X^{-c}}{c}; & \text{for i = 5, 10 and j = 1,} \end{cases} \tag{39}$$

and by using (11) one gets

$$h_j(x) = (-1)^j \frac{d}{dX} q_j^{-1}(x) = \begin{cases} x^{c-1}; & \text{for i = 1, 6, 11 and j = 2,} \\ 1; & \text{for i = 2, 7, 12, and j = 2,} \\ x^{\alpha}; & \text{for i = 3, 8, 13, and j = 2,} \\ \exp(-x); & \text{for i = 4, 9, and j = 1,} \\ x^{-c-1}; & \text{for i = 5, 10 and j = 1.} \end{cases} \quad (40)$$

Using method described in the remark the pdf $f_j(x, \theta)$ defined in (1) can be characterized through expectation of a function of random variable; $E[\phi_i(X)] = g_i(\theta)$; $i = 1, 2, ..., 13$ non constant functions by substituting M(X) defined in (13) and using $q_j^{-1}(X)$ as appear in (4) and using (11) for (14) as follows :

j	i	$M(X) = \frac{\frac{d}{dX}g(X)}{\phi(X)-g(X)}$	$h_j(X) = (-1)^j \cdot \frac{d}{dX}g(X)$	$q_j^{-1}(X) \ni M(X) = \ni \frac{d}{dX}\log\left(q_j^{-1}(X)\right)$	$f_j(x,\theta) = q_j(\theta)h_j(x) = (-1)^j \frac{\frac{d}{dx}q_j^{-1}(X)}{q_j^{-1}(\theta)}$
1	4, 9	- 1	$\exp(-x)$	$\exp(-x)$	$f_3(x,\theta) = \begin{cases} \exp-(x-\theta); \\ \quad a<\theta<x<b, \\ 0, \text{ otherwise.} \end{cases}$ pdf of negative exponential distribution
1	5, 10	$-\frac{c}{x}$	$\frac{x^{-c}}{c}$	x^{-c-1}	$f_4(x,\theta) = \begin{cases} \frac{c\theta^c}{x^{c+1}}; \\ \quad a<\theta<x<b, \\ 0, \text{ otherwise.} \end{cases}$ pdf of Pareto distribution
2	1, 6, 11	- 1	$\frac{c}{x}$	$\frac{x^c}{c}$	$f_5(x,\theta) = \begin{cases} c\theta^{-c}x^{c-1}; \\ \quad a<x<\theta<b, \\ 0, \text{ otherwise.} \end{cases}$ pdf of power function distribution
2	2, 7, 12	$\frac{1}{x}$	x	1	$f_6(x,\theta) = \begin{cases} \frac{1}{\theta}; \\ \quad a<x<\theta<b, \\ 0, \text{ otherwise.} \end{cases}$ pdf of uniform distribution
2	3, 8, 13	$\frac{\alpha+1}{x}$	$\frac{x^{\alpha+1}}{x^{\alpha+1}}$	$x^{\alpha+1}$	$f_7(x,\theta) = \begin{cases} \frac{\alpha+1}{\theta^{\alpha+1}}x^{\alpha}; \\ \quad a<x<\theta<b, \\ 0, \text{ otherwise.} \end{cases}$ pdf of generalize uniform distribution

Acknowledgement

This work was supported by UGC Major Research Project No : F.No.42-39/2013(SR).

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