

Concomitants of Generalized Order Statistics from Bivariate Pareto Distribution

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Abstract

Order statistics, record values and several other model of ordered random variables can be viewed as special case of generalized order statistics [Kamps, 1995]. In this paper the probability density function for single and the joint probability density function of two concomitants of generalized order statistics from bivariate Pareto distribution have been obtained and expressions for single and product moments are derived. Further the results are deduced for $k - th$ upper record values and order statistics.

Keywords: Generalized order statistics; concomitants of generalized order statistics; bivariate Pareto distribution; single and product moments.

AMS Subject Classification: 62G30, 62E10.

1 Introduction

The concept of generalized order statistics (*gos*) have been introduced and extensively studied by Kamps(1995). Begum (2003) obtained the moments of concomitants of order statistics from bivariate Pareto II distribution. In this paper we have obtained the moments of concomitants of generalized order statistics *gos* from bivariate Pareto distribution.

The random variables $X(1, n, \tilde{m}, k), X(2, n, \tilde{m}, k), \dots, X(n, n, \tilde{m}, k)$, $k > 0, m \in \mathbb{R}$ are n *gos* from an absolutely continuous distribution function (*df*) $F()$ and probability density funtion (*pdf*) $f()$, if their joint density function is of the form

$$k \left(\prod_{j=1}^{n-1} \gamma_j \right) \left(\prod_{i=1}^{n-1} [1 - F(x_i)]^{m_i} f(x_i) \right) [1 - F(x_n)]^{k-1} f(x_n) \quad (1.1)$$

on the cone $F^{-1}(0) < x_1 \leq x_2 \leq \dots \leq x_n < F^{-1}(1)$.

where $\gamma_i = k + (n - j)(m + 1)$ for all j , $1 \leq j \leq n$, k is a positive integer and $m \geq -1$.

If $m = 0$ and $k = 1$, then $X(r, n, m, k)$ reduces to the ordinary $r - th$ order statistics and (1.1) will be the joint *pdf* of order statistics $X_{1:n} \leq X_{2:n} \leq \dots \leq X_{n:n}$ from *df* $F()$. If $m = -1$ and $k = 1$, then (1.1) will be the joint *pdf* of the first n upper record values of

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the identically and independently distributed random variables (*iidrvs*) with *df* $F()$ and *pdf* $f()$.

In view of (1.1), the *pdf* of $X(r, n, m, k)$ is

$$f_{r,n,m,k}(x) = \frac{C_{r-1}}{(r-1)!} [\bar{F}(x)]^{\gamma_r-1} f(x) g_m^{r-1}(F(x)) \quad (1.2)$$

and joint *pdf* of $X(s, n, m, k)$ and $X(r, n, m, k)$, is $1 \leq r < s \leq n$.

$$\begin{aligned} f_{r,s,n,m,k}(x, y) &= \frac{C_{s-1}}{(r-1)!(s-r-1)!} [\bar{F}(x)]^m f(x) g_m^{r-1}(F(x)) \\ &\quad \times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [\bar{F}(y)]^{\gamma_s-1} f(y), \quad \alpha \leq x < y \leq \beta \end{aligned} \quad (1.3)$$

where,

$$C_{r-1} = \prod_{i=1}^r \gamma_i, \quad \gamma_i = k + (n-i)(m+1), \quad i = 1, 2, \dots, n$$

$$h_m(x) = \begin{cases} -\frac{1}{m+1} (1-x)^{m+1}, & m \neq -1 \\ -\log(1-x), & m = -1 \end{cases}$$

$$\text{and } g_m(x) = h_m(x) - h_m(0), \quad x \in (0, 1).$$

Let $(X_i, Y_i), i = 1, 2, \dots, n$, be n pairs of independent random variables from some bivariate population with distribution function $F(x, y)$. If we arrange the X variates in ascending order as $X(1, n, m, k) \leq X(2, n, m, k) \leq \dots \leq X(n, n, m, k)$ then Y variates paired (not necessarily in ascending order) with these generalized ordered statistics are called the concomitants of generalized order statistics and are denoted by $Y_{[1,n,m,k]}, Y_{[2,n,m,k]}, \dots, Y_{[n,n,m,k]}$.

The *pdf* of $Y_{[r,n,m,k]}$, the r -th concomitant of generalized order statistics is given as

$$g_{[r,n,m,k]}(y) = \int_{-\infty}^{\infty} f(y|x) f_{r,n,m,k}(x) dx \quad (1.4)$$

and the joint *pdf* of $Y_{[r,n,m,k]}$ and $Y_{[s,n,m,k]}$ are

$$g_{[r,s,n,m,k]}(y_1, y_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{x_1} f(y_1|x_1) f(y_2|x_2) f_{r,s,n,m,k}(x_1, x_2) dx_1 dx_2 \quad (1.5)$$

where $f_{r,s,n,m,k}(x_1, x_2)$ is the joint *pdf* of $X(r, n, m, k), X(s, n, m, k)$, $1 \leq r < s \leq n$.

The most important use of concomitants arises in selection procedures when $k < n$ individuals are chosen on the basis of their X -values. Then the corresponding Y -values

represent performance on an associated characteristic. For example, X might be the score of a candidate on a screening test and Y the score on a later test.

There are vast study on concomitants of order statistics by several authors in literature. An excellent review on concomitants of order statistics is given Bhattacharya (1984) and David and Nagaraja (1998). O'Connell and David (1976) studied order statistics and their concomitants in some double sampling situations. Do and Hall(1991) obtained distribution theory using concomitants of order statistics with application to Monte Carlo simulation for the bootstrap. Tsukabayashi (1998) obtained the joint distribution and moments of an extreme of the dependent variables and the concomitant of an extreme of the independent variables. Balasubramanian and Beg(1996, 1997, 1998) studied the concomitants for Marshall- Olkin bivariate exponential distribution, Morgenstern type bivariate exponential distribution and Gumbel's bivariate exponential distribution and gave the recurrence relation between single and product moment of concomitants of order statistics. Begum and Khan (1997, 1998, 2000) studied the concomitants for Gumbel's bivariate Weibull distribution, bivariate Burr distribution, Marshall and Olkin bivariate Weibull distribution respectively. Ahsanullah and Beg (2006) studied the concomitants for Gumbel's bivariate exponential distribution and derived the recurrence relations between single and product moment of gos. Nayabuddin (2013) studied the concomitants of gos for bivariate Lomax distribution.

The *pdf* of bivariate Pareto distribution is given as [Johnson and Kotz, 1972]

$$f(x, y) = \frac{p(p+1)(ab)^{p+1}}{(bx + ay - ab)^{p+2}}, \quad p > 0, x > a > 0, y > b > 0 \quad (1.6)$$

and corresponding *df* is

$$F(x, y) = 1 - \frac{b^p}{y^p} - \frac{a^p}{x^p} + \frac{(ab)^{p+1}}{(bx + ay - ab)^{p+2}}, \quad p > 0, x > a > 0, y > b > 0. \quad (1.7)$$

The conditional *pdf* of Y given X is

$$f(y|x) = \frac{a(p+1)(bx)^{p+1}}{(bx + ay - ab)^{p+2}}, \quad y > b > 0, \quad (1.8)$$

the marginal *pdf* of X is

$$f(x) = \frac{pa^p}{x^{p+1}}, \quad p > 0, \quad x > a > 0 \quad (1.9)$$

and marginal *df* of X is

$$F(x) = 1 - \frac{a^p}{x^p}, \quad p > 0, \quad x > a > 0. \quad (1.10)$$

2 Probability Density Function of $Y_{[r,n,m,k]}$

For the bivariate Pareto distribution as given in (1.6), using (1.2) and (1.8) in (1.4), the pdf of $r - th$ concomitants of gos $Y_{[r,n,m,k]}$ for $m \neq -1$ is given as:

$$g_{[r,n,m,k]}(y) = \frac{p(p+1)}{ab} \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \\ \times \int_a^{\infty} \left(\frac{a}{x}\right)^{p\gamma_{r-i}+2} \left[1 + \left(\frac{y}{b}-1\right)\frac{a}{x}\right]^{-p+2} dx. \quad (2.1)$$

Let $z = \frac{a}{x}$, then the R.H.S. of (2.1) reduces to

$$g_{[r,n,m,k]}(y) = \frac{p(p+1)}{ab} \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \\ \times \int_0^1 z^{p\gamma_{r-i}} \left[1 + \left(\frac{y}{b}-1\right)z\right]^{-p+2} dz. \quad (2.2)$$

Note that [Prudnikov *et al*, 1986]

$$(1+z)^{-a} = \sum_{p=0}^{\infty} \frac{(-1)^p (a)_p z^p}{p!}. \quad (2.3)$$

Thus using relation (2.3) in (2.2), we get

$$g_{[r,n,m,k]}(y) = \frac{p(p+1)}{b} \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \\ \times \sum_{l=0}^{\infty} \frac{(-1)^l (p+2)_l \left(\frac{y}{b}-1\right)^l}{l!} \frac{1}{p\gamma_{r-i} + l + 1}. \quad (2.4)$$

To prove that $\int_b^{\infty} g_{[r,n,m,k]}(y) dy = 1$

We have,

$$\int_b^{\infty} g_{[r,n,m,k]}(y) dy = \frac{p(p+1)}{b} \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \\ \times \int_b^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^l (p+2)_l \left(\frac{y}{b}-1\right)^l}{l!} \frac{1}{p\gamma_{r-i} + l + 1} dy. \quad (2.5)$$

Since

$$(\lambda + m) = \frac{\lambda (\lambda + 1)_m}{(\lambda)_m} \quad [\text{Srivastava and Karlsson, 1985}]. \quad (2.6)$$

Thus, using relation (2.6) in (2.5), we get

$$\begin{aligned} &= \frac{p(p+1)}{b} \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \frac{1}{p\gamma_{r-i} + 1} \\ &\times \int_b^\infty \sum_{l=0}^{\infty} \frac{(-1)^l (p+2)_l (p\gamma_{r-i} + 1)_l}{(p\gamma_{r-i} + 2)_l} \frac{\left(\frac{y}{b} - 1\right)^l}{l!} dy. \end{aligned} \quad (2.7)$$

Noting that [Prudnikov et al, 1986]

$${}_2F_1 \left[\begin{matrix} a, & b \\ c & \end{matrix}; -z \right] = \sum_{p=0}^{\infty} \frac{(-1)^p (a)_p (b)_p}{(c)_p} \frac{z^p}{p!} \quad (2.8)$$

Conditionally convergent for $|z| = 1$, $z \neq 1$, if $-1 < \text{Re}(w) \leq 0$.

In view of (2.8),(2.7) becomes

$$\begin{aligned} &= \frac{p(p+1)}{b} \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \frac{1}{p\gamma_{r-i} + 1} \\ &\times \int_b^\infty {}_2F_1 \left[\begin{matrix} p\gamma_{r-i} + 1, & p+2 \\ p\gamma_{r-i} + 2 & \end{matrix}; 1 - \frac{y}{b} \right] dy. \end{aligned} \quad (2.9)$$

Let $t = \frac{y}{b} - 1$, then

$$\begin{aligned} &= p(p+1) \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \frac{1}{p\gamma_{r-i} + 1} \\ &\times \int_0^\infty {}_2F_1 \left[\begin{matrix} p\gamma_{r-i} + 1, & p+2 \\ p\gamma_{r-i} + 2 & \end{matrix}; -t \right] dt. \end{aligned} \quad (2.10)$$

Note that[Prudnikov et al, 1986]

$$\int_0^\infty x^{p-1} {}_2F_1 \left[\begin{matrix} a & b \\ c & \end{matrix}; -m x \right] dx = \frac{(m)^{-p} \Gamma(c) \Gamma(p) \Gamma(a-p) \Gamma(b-p)}{\Gamma(a) \Gamma(b) \Gamma(c-p)}. \quad (2.11)$$

Therefore in view of (2.11),(2.10) reduces to

$$= \frac{C_{r-1}}{(r-1)!(m+1)^r} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} B\left(\frac{k}{m+1} + (n-r) + i, -1\right). \quad (2.12)$$

For real positive α, c and a positive integer b , we have

$$\sum_{a=0}^b (-1)^a \binom{b}{a} B(a + \alpha, c) = B(\alpha, c + b). \quad (2.13)$$

Thus using relations (2.13) in (2.12), we get

$$\begin{aligned} &= \frac{C_{r-1}}{(r-1)!(m+1)^r} B\left(\frac{k}{m+1} + n - r, -r\right) \\ &= 1. \end{aligned}$$

3 Moment of $Y_{[r,n,m,k]}$

In this section, we derive the moments of $Y[r, n, m, k]$ for bivariate Pareto distribution by using the results of the previous section. Using (2.4), the moments of $Y[r, n, m, k]$ is

$$E(Y_{[r,n,m,k]}^{(k_1)}) = \int_b^\infty y^{k_1} g_{[r,n,m,k]}(y) dy \quad (3.1)$$

$$\begin{aligned} &= \frac{p(p+1)}{b} \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \\ &\times \int_b^\infty y^{k_1} \sum_{l=0}^\infty \frac{(-1)^l (p+2)_l \left(\frac{y}{b} - 1\right)^l}{l!} \frac{1}{p\gamma_{r-i} + l + 1} dy. \end{aligned} \quad (3.2)$$

Now using relation (2.6) in (3.2), we get

$$\begin{aligned} &= \frac{p(p+1)}{b} \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \frac{1}{p\gamma_{r-i} + 1} \\ &\times \int_b^\infty y^{k_1} \sum_{l=0}^\infty \frac{(-1)^l (p+2)_l (p\gamma_{r-i} + 1)_l}{(p\gamma_{r-i} + 2)_l} \frac{\left(\frac{y}{b} - 1\right)^l}{l!} dy. \end{aligned} \quad (3.3)$$

Using relation (2.8) in (3.3), we have

$$\begin{aligned}
 &= \frac{p(p+1)}{b} \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \frac{1}{p\gamma_{r-i} + 1} \\
 &\quad \times \int_b^\infty y^{k_1} {}_2F_1 \left[\begin{matrix} p\gamma_{r-i} + 1, & p+2 \\ p\gamma_{r-i} + 2 & \end{matrix}; \quad 1 - \frac{y}{b} \right] dy. \tag{3.4}
 \end{aligned}$$

Let $z = \frac{y}{b} - 1$, then R.H.S. of (3.4) reduces to

$$\begin{aligned}
 &= (b)^{k_1} p(p+1) \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \frac{1}{p\gamma_{r-i} + 1} \\
 &\quad \times \int_0^\infty (1+z)^{k_1} {}_2F_1 \left[\begin{matrix} p\gamma_{r-i} + 1, & p+2 \\ p\gamma_{r-i} + 2 & \end{matrix}; \quad -z \right] dz. \tag{3.5}
 \end{aligned}$$

Expanding $(1+z)^{k_1}$ binomially in (3.5), we get

$$\begin{aligned}
 &= (b)^{k_1} p(p+1) \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \frac{1}{p\gamma_{r-i} + 1} \\
 &\quad \times \sum_{t=0}^{k_1} \binom{k_1}{t} \int_0^\infty z^{k_1-t} {}_2F_1 \left[\begin{matrix} p\gamma_{r-i} + 1, & p+2 \\ p\gamma_{r-i} + 2 & \end{matrix}; \quad -z \right] dz. \tag{3.6}
 \end{aligned}$$

Using relation (2.11) in (3.7), we get

$$E(Y_{[r,n,m,k]}^{(k_1)}) = (b)^{k_1} \sum_{t=0}^{k_1} \binom{k_1}{t} \frac{\Gamma(k_1 + 1 - t)\Gamma(p + t + 1 - k_1)}{\Gamma(p + 1)} \prod_{i=1}^r \left(1 - \frac{k_1 - t}{p\gamma_i}\right)^{-1}. \tag{3.7}$$

Remark 3.1 : Set $m = 0$, $k = 1$ in (3.7), to get moments of concomitants of order statistics from bivariate Pareto distribution as

$$E(Y_{[r:n]}^{(k_1)}) = (b)^{k_1} \sum_{t=0}^{k_1} \binom{k_1}{t} \frac{\Gamma(k_1 + 1 - t)\Gamma(p + t + 1 - k_1)}{\Gamma(p + 1)} \prod_{i=1}^r \left(1 - \frac{k_1 - t}{p(n - i + 1)}\right)^{-1}.$$

Remark 3.2 : At $m = -1$ in (3.7), we get moments of concomitants of $k - th$ upper record value from bivariate Pareto distribution as

$$E\left(Y_{[r,n,-1,k]}^{(k_1)}\right) = (b)^{k_1} \sum_{t=0}^{k_1} \binom{k_1}{t} \frac{\Gamma(k_1 + 1 - t)\Gamma(p + t + 1 - k_1)}{\Gamma(p + 1)} \left(1 - \frac{k_1 - t}{pk}\right)^{-r}.$$

Means of the concomitants of order statistics (Remark 3.1)

Table 3.1

$b = 1.0000$		$p = 3.0000$								
$n \setminus r$	1	2	3	4	5	6	7	8	9	10
1	1.5000									
2	1.4000	1.6000								
3	1.3750	1.4500	1.675							
4	1.3636	1.4091	1.4909	1.7364						
5	1.3571	1.3896	1.4383	1.5259	1.7889					
6	1.3529	1.3782	1.4125	1.4641	1.5569	1.8353				
7	1.3500	1.3706	1.3970	1.4331	1.4873	1.5847	1.8771			
8	1.3478	1.3652	1.3867	1.4143	1.4519	1.5085	1.6101	1.9153		
9	1.3461	1.3612	1.3793	1.4016	1.4303	1.4694	1.5280	1.6336	1.9505	
10	1.3448	1.3580	1.3736	1.3923	1.4154	1.4451	1.4855	1.5462	1.6555	1.9832

Table 3.2

$b = 2.0000$		$p = 4.0000$								
$n \setminus r$	1	2	3	4	5	6	7	8	9	10
1	2.6667									
2	2.5714	2.7619								
3	2.5454	2.6234	2.8312							
4	2.5333	2.5818	2.6649	2.8865						
5	2.5263	2.5614	2.6124	2.6999	2.9332					
6	2.5217	2.5492	2.5858	2.6391	2.7304	2.9738				
7	2.5185	2.5411	2.5695	2.6075	2.6627	2.7574	3.0099			
8	2.5161	2.5352	2.5585	2.5879	2.6271	2.6841	2.7818	3.0425		
9	2.5143	2.5309	2.5505	2.5745	2.6047	2.6450	2.7037	2.8042	3.0722	
10	2.5128	2.5275	2.5445	2.5646	2.5892	2.6202	2.6616	2.7217	2.8248	3.0997

Table 3.3

$n \setminus r$	$b = 3.0000$			$p = 5.0000$						
	1	2	3	4	5	6	7	8	9	10
1	3.75									
2	3.6667	3.8333								
3	3.6429	3.7143	3.8928							
4	3.6316	3.6767	3.7519	3.9398						
5	3.6250	3.6579	3.7049	3.7832	3.9790					
6	3.6207	3.6465	3.6806	3.7292	3.8102	4.0128				
7	3.6176	3.6389	3.6656	3.7006	3.7506	3.8340	4.0426			
8	3.6154	3.6335	3.6553	3.6826	3.7186	3.7699	3.8554	4.0693		
9	3.6136	3.6294	3.6479	3.6702	3.6981	3.7349	3.7874	3.8749	4.0936	
10	3.6122	3.6262	3.6422	3.6611	3.6839	3.7124	3.7499	3.8035	3.8927	4.1159

It may be noted that in Tables 3.1, 3.2 and 3.3, the well known property of order statistics $\sum_{i=1}^n E(X_{i:n}) = nE(X)$ (David and Nagaraja, 2003) is satisfied.

Means of the concomitants of lower record values(Remark 3.2)

Table 3.4

r	$k = 2.0000$	$k = 3.0000$	$k = 4.0000$
	$b = 1.0000, p = 3.0000$	$b = 2.0000, p = 4.0000$	$b = 3.0000, p = 5.0000$
1	1.4000	2.5454	3.6316
2	1.4800	2.5950	3.6648
3	1.5760	2.6491	3.6998
4	1.6912	2.7081	3.7366
5	1.8294	2.7725	3.7754
6	1.9953	2.8427	3.8162
7	2.1944	2.9194	3.8592
8	2.4333	3.0029	3.9044
9	2.7199	3.0941	3.9520
10	3.0639	3.1936	4.0021
11	3.4767	3.3021	4.0548
12	3.9720	3.4205	4.1104
13	4.5664	3.5496	4.1688
14	5.2797	3.6905	4.2303
15	6.1356	3.8442	4.2951
16	7.1628	4.0118	4.3632
17	8.3953	4.1947	4.4350
18	9.8744	4.3942	4.5105
19	11.6493	4.6119	4.5900
20	13.7792	4.8493	4.6737

4 Joint Probability Density Function of $Y_{[r,n,m,k]}$ and $Y_{[s,n,m,k]}$

For bivariate Pareto distribution as given in (1.6), using (1.3) and (1.8) in (1.5), the joint pdf of $r - th$ and $s - th$ concomitants of gos $Y_{[r,n,m,k]}$ and $Y_{[s,n,m,k]}$ for $m \neq -1$ is

$$\begin{aligned} g_{[r,s,n,m,k]}(y_1, y_2) &= \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} (a)^{\gamma_{r-i}+2} (b)^{2p+2} p^2 (p+1)^2 \\ &\times \sum_{i=0}^{r-1} \sum_{j=0}^{s-r-1} (-1)^{i+j} \binom{r-1}{i} \binom{s-r-1}{j} \\ &\times \int_a^\infty \frac{1}{(x_2)^{p(\gamma_{s-j}-1)}} \frac{1}{(bx_2 + ay_2 - ab)^{p+2}} I(x_2, y_1) dx_2 \end{aligned} \quad (4.1)$$

where,

$$I(x_2, y_1) = \int_a^{x_2} \frac{1}{(x_1)^{p(s-r+i-j)(m+1)-p}} \frac{1}{(bx_1 + ay_1 - ab)^{p+2}} dx_1 \quad (4.2)$$

or

$$I(x_2, y_1) = (\lambda)^{-\beta} \int_a^{x_2} x_1^{-\alpha} (1 + \frac{bx_1}{\lambda})^{-\beta} dx_1, \quad (4.3)$$

where $\alpha = p(s-r+i-j)(m+1) - p$, $\beta = (p+2)$ and $\lambda = ay_1 - ab$.

Using (2.3) in (4.3), after simplification, we get

$$I(x_2, y_1) = (\lambda)^{-\beta} \sum_{u=0}^{\infty} (-1)^u \frac{(\beta)_u (\frac{b}{\lambda})^u}{u!} \frac{1}{(-\alpha + u + 1)} \left[(x_2)^{-(\alpha-u-1)} - (a)^{-(\alpha-u-1)} \right]. \quad (4.4)$$

Subsituting the value of $I(x_2, y_1)$ from (4.4) in (4.1), we get

$$\begin{aligned} g_{[r,s,n,m,k]}(y_1, y_2) &= \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} (a)^{\gamma_{r-i}+2} (b)^{2p+2} p^2 (p+1)^2 \\ &\times \sum_{i=0}^{r-1} \sum_{j=0}^{s-r-1} (-1)^{i+j} \binom{r-1}{i} \binom{s-r-1}{j} (\lambda)^{-\beta} \sum_{u=0}^{\infty} \frac{(-1)^u}{(-\alpha + u + 1)} \frac{(\beta)_u (\frac{b}{\lambda})^u}{u!} \\ &\times \int_a^\infty \frac{1}{(x_2)^{p(\gamma_{s-j}-1)}} \frac{1}{(bx_2 + ay_2 - ab)^{p+2}} \left[(x_2)^{-(\alpha-u-1)} - (a)^{-(\alpha-u-1)} \right] dx_2 \end{aligned} \quad (4.5)$$

$$\begin{aligned} &= \frac{(a)^{p\gamma_{r-i}+2} (b)^{2p+2} p^2 (p+1)^2 C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \sum_{i=0}^{r-1} \sum_{j=0}^{s-r-1} (-1)^{i+j} \binom{r-1}{i} \binom{s-r-1}{j} \\ &\times (\lambda)^{-\beta} (\delta)^{-\beta} \sum_{u=0}^{\infty} \frac{(\beta)_u (-\frac{b}{\lambda})^u}{u!} \sum_{v=0}^{\infty} \frac{(\beta)_v (-\frac{b}{\delta})^v}{v!} \frac{1}{(1-\theta+v)(2-\theta-\alpha+v+u)} \end{aligned} \quad (4.6)$$

where $\delta = ay_2 - ab$ and $\theta = p\gamma_{s-j} - p$.

Putting the value of θ, λ, δ and α in (4.6), we get

$$\begin{aligned} g_{[r,s,n,m,k]}(y_1, y_2) &= \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} (b)^{2p+2} p^2 (p+1)^2 \\ &\times \sum_{i=0}^{r-1} \sum_{j=0}^{s-r-1} (-1)^{i+j} \binom{r-1}{i} \binom{s-r-1}{j} (y_1 - b)^{-\beta} (y_2 - b)^{-\beta} \\ &\times \sum_{u=0}^{\infty} \frac{(\beta)_u \left(\frac{b}{b-y_1}\right)^u}{u!} \sum_{v=0}^{\infty} \frac{(\beta)_v \left(-\frac{b}{b-y_2}\right)^v}{v!} \frac{1}{(v+1+p-p\gamma_{s-j})(v+u+2p+2-p\gamma_{r-i})}. \end{aligned} \quad (4.7)$$

To prove that $\int_b^\infty \int_b^\infty g_{[r,s,n,m,k]}(y_1, y_2) dy_1 dy_2 = 1$

we have,

$$\begin{aligned} \int_b^\infty \int_b^\infty g_{[r,s,n,m,k]}(y_1, y_2) dy_1 dy_2 &= A \int_b^\infty \int_b^\infty (y_1 - b)^{-\beta} (y_2 - b)^{-\beta} \sum_{u=0}^{\infty} \frac{(\beta)_u \left(\frac{b}{b-y_1}\right)^u}{u!} \\ &\times \sum_{v=0}^{\infty} \frac{(\beta)_v \left(-\frac{b}{b-y_2}\right)^v}{v!} \frac{1}{(v+1+p-p\gamma_{s-j})(v+u+2p+2-p\gamma_{r-i})} dy_1 dy_2 \end{aligned} \quad (4.8)$$

where,

$$A = \frac{(b)^{2p+2} p^2 (p+1)^2 C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \sum_{i=0}^{r-1} \sum_{j=0}^{s-r-1} (-1)^{i+j} \binom{r-1}{i} \binom{s-r-1}{j} \quad (4.9)$$

Set $d = 1 - p\gamma_{s-j} + p + 1$ and $g = 2 - p\gamma_{r-i} + 2p + 2$ in (4.9), then

$$\begin{aligned} &= A \int_b^\infty \int_b^\infty (y_1 - b)^{-\beta} (y_2 - b)^{-\beta} \sum_{u=0}^{\infty} \frac{(\beta)_u \left(\frac{b}{b-y_1}\right)^u}{u!} \\ &\times \sum_{v=0}^{\infty} \frac{(\beta)_v \left(-\frac{b}{b-y_2}\right)^v}{v!} \frac{1}{(v+d)(v+u+g)} dy_1 dy_2. \end{aligned} \quad (4.10)$$

Noting that [Srivastava and Karlsson, 1985]

$$(\lambda + m + n) = \frac{\lambda (\lambda + 1)_{m+n}}{(\lambda)_{m+n}} \quad (4.11)$$

$$(\lambda)_{m+n} = (\lambda)_m(\lambda+m)_n. \quad (4.12)$$

Now using relation (4.11) and (4.12) in (4.10), we have

$$\begin{aligned} &= \frac{A}{gd} \int_b^\infty (y_2 - b)^{-\beta} \sum_{v=0}^{\infty} \frac{(g)_v (\beta)_v (d)_v \left(\frac{b}{b-y_1}\right)^v}{(g+1)_v (d+1)_v v!} dy_2 \\ &\quad \times \int_b^\infty (y_1 - b)^{-\beta} {}_2F_1 \left[\begin{matrix} g+v, & \beta \\ g+v+1 & \end{matrix}; -\frac{b}{y_1-b} \right] dy_1 \end{aligned} \quad (4.13)$$

$$= \frac{A(b)^{-(\beta-1)}}{gd(\beta-1)} \int_b^\infty (y_2 - b)^{-\beta} \sum_{v=0}^{\infty} \frac{(g)_v (\beta)_v (d)_v \left(\frac{b}{b-y_1}\right)^v}{(g+1)_v (d+1)_v v!} \frac{(g+v)}{g+v+1-\beta} dy_2. \quad (4.14)$$

Using relation (2.6) in (4.14), we get

$$\begin{aligned} &= \frac{A(b)^{-(\beta-1)}}{d(\beta-1)(g-\beta+1)} \int_b^\infty (y_2 - b)^{-\beta} \sum_{v=0}^{\infty} \frac{(d)_v (g-\beta+1)_v (\beta)_v \left(\frac{b}{b-y_1}\right)^v}{(d+1)_v (g-\beta+2)_v v!} dy_2 \\ &= \frac{A(b)^{-(\beta-1)}}{d(\beta-1)(g-\beta+1)} \int_b^\infty (y_2 - b)^{-\beta} {}_3F_2 \left[\begin{matrix} d, & g-\beta+1, & \beta \\ d+1, & g-\beta+2 & \end{matrix}; -\frac{b}{y_2-b} \right] dy_2. \end{aligned} \quad (4.15)$$

Noting that [Prudnikov et al, 1986]

$$\begin{aligned} &\int_0^\infty x^{s-1} {}_3F_2 \left[\begin{matrix} (a_1) & (a_2) & (a_3) \\ (b_1) & (b_2) & \end{matrix}; -x \right] dx \\ &= \frac{\Gamma(b_1) \Gamma(b_2) \Gamma(s) \Gamma(a_1-s) \Gamma(a_2-s) \Gamma(a_3-s)}{\Gamma(a_1) \Gamma(a_2) \Gamma(a_3) \Gamma(b_1-s) \Gamma(b_2-s)} \end{aligned} \quad (4.16)$$

$$[0 < Re < s < Re a_j ; \quad j = 1, 2, 3].$$

Thus in view of (4.16), (4.15) becomes

$$= \frac{A \quad d}{(\beta-1)^2 (g+2-2\beta) (d+1-\beta)} \quad (4.17)$$

Now putting the value of g, d, β, A in (4.17), we get

$$g_{[r,s,n,m,k]}(y_1, y_2) = \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^s} \sum_{i=0}^{r-1} \sum_{j=0}^{s-r-1} (-1)^{i+j} \binom{r-1}{i} \binom{s-r-1}{j} \\ \times B\left(\frac{k}{m+1} + (n-r+i), 1\right) B\left(\frac{k}{m+1} + (n-s+j), 1\right) \quad (4.18)$$

$= 1$ in view of (2.13).

5 Product Moments of two concomitants $Y_{[r,n,m,k]}$ and $Y_{[s,n,m,k]}$

Product Moments of two concomitants $Y_{[r,n,m,k]}$ and $Y_{[s,n,m,k]}$ is as

$$E(Y_{[r,n,m,k]}^{(k_1)} Y_{[s,n,m,k]}^{(k_2)}) = \int_b^\infty \int_b^\infty y_1^{k_1} y_2^{k_2} g_{[r,s,n,m,k]}(y_1, y_2) dy_1 dy_2. \quad (5.1)$$

In view of (4.7) and (5.1), we have

$$E(Y_{[r,n,m,k]}^{(k_1)} Y_{[s,n,m,k]}^{(k_2)}) = A \int_b^\infty \int_b^\infty y_1^{k_1} y_2^{k_2} (y_1 - b)^{-\beta} (y_2 - b)^{-\beta} \sum_{u=0}^{\infty} \frac{(\beta)_u (\frac{b}{b-y_1})^u}{u!} \\ \times \sum_{v=0}^{\infty} \frac{(\beta)_v (-\frac{b}{b-y_2})^v}{v!} \frac{1}{(v+1+p-p\gamma_{s-j})(v+u+2p+2-p\gamma_{r-i})} dy_1 dy_2. \quad (5.2)$$

Set $d = 1 - p\gamma_{s-j} + p + 1$ and $g = 2 - p\gamma_{r-i} + 2p + 2$ in (5.2), to get

$$= A \int_b^\infty \int_b^\infty y_1^{k_1} y_2^{k_2} (y_1 - b)^{-\beta} (y_2 - b)^{-\beta} \sum_{u=0}^{\infty} \frac{(\beta)_u (\frac{b}{b-y_1})^u}{u!} \\ \times \sum_{v=0}^{\infty} \frac{(\beta)_v (-\frac{b}{b-y_2})^v}{v!} \frac{1}{(v+d)(v+u+g)} dy_1 dy_2. \quad (5.3)$$

Using relation (2.6), (4.11) and (4.12) in (5.3), we get

$$= \frac{A}{dg} \int_b^\infty y_2^{k_2} (y_2 - b)^{-\beta} \sum_{v=0}^{\infty} \frac{(g)_v}{(g+1)_v} \frac{(\beta)_v (d)_v}{(d+1)_v} \frac{(-\frac{b}{b-y_2})^v}{v!} \\ \times \left\{ \int_b^\infty y_1^{k_1} (y_1 - b)^{-\beta} \sum_{u=0}^{\infty} \frac{(\beta)_u (g+v)_u}{(g+1+v)_u} \frac{((-\frac{b}{b-y_1}))^u}{u!} dy_1 \right\} dy_2 \quad (5.4)$$

$$\begin{aligned}
 &= \frac{A}{dg} \int_b^\infty y_2^{k_2} (y_2 - b)^{-\beta} \sum_{v=0}^{\infty} \frac{(g)_v}{(g+1)_v} \frac{(\beta)_v}{(d+1)_v} \frac{(-\frac{b}{b-y_2})^v}{v!} \\
 &\quad \times \left\{ \int_b^\infty y_1^{k_1} (y_1 - b)^{-\beta} {}_2F_1 \left[\begin{array}{cc} (g+v); & (\beta) \\ (g+v+1) & \end{array}; -\frac{b}{b-y_1} \right] dy_1 \right\} dy_2. \tag{5.5}
 \end{aligned}$$

Now letting $t_1 = \frac{1}{y_1 - b}$ in (5.5), we have

$$\begin{aligned}
 &= \frac{A}{dg} \int_b^\infty y_2^{k_2} (y_2 - b)^{-\beta} \sum_{v=0}^{\infty} \frac{(g)_v}{(g+1)_v} \frac{(\beta)_v}{(d+1)_v} \frac{(-\frac{b}{b-y_2})^v}{v!} \\
 &\quad \times \left\{ \int_0^\infty \left(b + \frac{1}{t} \right)^{k_1} t^{\beta-2} {}_2F_1 \left[\begin{array}{cc} (g+v); & (\beta) \\ (g+v+1) & \end{array}; -bt \right] dt \right\} dy_2. \tag{5.6}
 \end{aligned}$$

Expanding $\left(b + \frac{1}{t} \right)^{k_1}$ binomially in (5.6), we get

$$\begin{aligned}
 &= \frac{A}{dg} \int_b^\infty y_2^{k_2} (y_2 - b)^{-\beta} \sum_{v=0}^{\infty} \frac{(g)_v}{(g+1)_v} \frac{(\beta)_v}{(d+1)_v} \frac{(-\frac{b}{b-y_2})^v}{v!} \\
 &\quad \times \left\{ b^l \sum_{l=0}^{k_1} \binom{k_1}{l} \int_0^\infty t^{\beta+l-k_1-2} {}_2F_1 \left[\begin{array}{cc} (g+v); & (\beta) \\ (g+v+1) & \end{array}; -bt \right] dt \right\} dy_2. \tag{5.7}
 \end{aligned}$$

Using relation (2.9) and (2.6) in (5.7), we have

$$\begin{aligned}
 &= \frac{A}{d \Gamma(\beta)} b^{k_1-\beta+1} \sum_{l=0}^{k_1} \binom{k_1}{l} \frac{\Gamma(\beta+l-k_1-1) \Gamma(k_1-l+1)}{(g+k_1+1-\beta-l)} \\
 &\quad \times \int_b^\infty y_2^{k_2} (y_2 - b)^{-\beta} \sum_{v=0}^{\infty} \frac{(d)_v (\beta)_v (g+k_1+1-l-\beta)_v}{(g+k_1+2-l-\beta)_v} \frac{(-\frac{b}{y_2-b})^v}{v!} dy_2. \tag{5.8}
 \end{aligned}$$

Using relation (4.10) in (5.8), we get

$$\begin{aligned}
 &= \frac{A}{d \Gamma(\beta)} b^{k_1-\beta+1} \sum_{l=0}^{k_1} \binom{k_1}{l} \frac{\Gamma(\beta+l-k_1-1) \Gamma(k_1-l+1)}{(g+k_1+1-\beta-l)} \\
 &\quad \times \int_b^\infty y_2^{k_2} (y_2 - b)^{-\beta} {}_3F_2 \left[\begin{array}{ccc} (d); & (g+k_1+1-l-\beta); & (\beta) \\ (d+1); & (g+k_1+2-l-\beta) & \end{array}; -\frac{b}{y_2-b} \right] dy_2. \tag{5.9}
 \end{aligned}$$

Letting $z = \frac{1}{y_2 - b}$ in (5.9), we get

$$= \frac{A}{d \Gamma(\beta)} b^{k_1 - \beta + 1} \sum_{l=0}^{k_1} \binom{k_1}{l} \frac{\Gamma(\beta + l - k_1 - 1) \Gamma(k_1 - l + 1)}{(g + k_1 + 1 - \beta - l)} \\ \times \int_0^\infty \left(b + \frac{1}{z}\right)^{k_2} z^{\beta - 2} {}_3F_2 \left[\begin{matrix} (d); & (g + k_1 + 1 - l - \beta); & (\beta) \\ (d + 1); & (g + k_1 + 2 - l - \beta) \end{matrix}; -bz \right] dz. \quad (5.10)$$

Expanding $\left(b + \frac{1}{z}\right)^{k_2}$ binomially in (5.10), we get

$$= \frac{A}{d \Gamma(\beta)} b^{k_1 + k_2 - \beta + 1} \sum_{l=0}^{k_1} \binom{k_1}{l} \frac{\Gamma(\beta + l - k_1 - 1) \Gamma(k_1 - l + 1)}{(g + k_1 + 1 - \beta - l)} \\ \times \sum_{l_1=0}^{k_2} \binom{k_2}{l_1} \int_0^\infty z^{\beta + l_1 - k_2 - 2} {}_3F_2 \left[\begin{matrix} (d); & (g + k_1 + 1 - l - \beta); & (\beta) \\ (d + 1); & (g + k_1 + 2 - l - \beta) \end{matrix}; -bz \right] dz. \quad (5.11)$$

Now in view of (4.16) and (5.11), we have

$$= A b^{k_1 + k_2 - 2\beta + 2p + 4} \sum_{l=0}^{k_1} \sum_{l_1=0}^{k_2} \binom{k_1}{l} \binom{k_2}{l_1} \frac{\Gamma(\beta + l - k_1 - 1) \Gamma(k_1 - l + 1)}{\Gamma(\beta)} \\ \times \frac{\Gamma(\beta + l_1 - k_2 - 1) \Gamma(k_2 - l_1 + 1)}{\Gamma(\beta)} \frac{1}{(g + k_1 + k_2 + 2 - l - l_1 - 2\beta)} \frac{1}{(d + k_2 + 1 - \beta - l_1)}. \quad (5.12)$$

Putting the value of A , d , g and β in (5.12), we get

$$= \frac{b^{k_1 + k_2} p^2 (p+1)^2 C_{s-1}}{(r-1)! (s-r-1)! (m+1)^{s-2}} \sum_{l=0}^{k_1} \sum_{l_1=0}^{k_2} \binom{k_1}{l} \binom{k_2}{l_1} \frac{\Gamma(p+l+1-k_1) \Gamma(k_1-l+1)}{\Gamma(p+2)} \\ \times \frac{\Gamma(p+l_1+1-k_2) \Gamma(k_2-l_1+1)}{\Gamma(p+2)} \sum_{i=0}^{r-1} \sum_{j=0}^{s-r-1} (-1)^{i+j} \binom{r-1}{i} \binom{s-r-1}{j} \\ \times \frac{1}{[p\{k+(n-s+j)(m+1)\}-(k_2-l_1)]} \frac{1}{[p\{k+(n-r+i)(m+1)\}-(k_1+k_2-l-l_1)]} \quad (5.13)$$

Which after simplification, yields

$$E\left(Y_{[r,n,m,k]}^{(k_1)} Y_{[s,n,m,k]}^{(k_2)}\right) = b^{k_1+k_2} \sum_{l=0}^{k_1} \sum_{l_1=0}^{k_2} \binom{k_1}{l} \binom{k_2}{l_1} \frac{\Gamma(p+l+1-k_1)\Gamma(k_1-l+1)}{\Gamma(p+1)} \\ \times \frac{\Gamma(p+l_1+1-k_2)\Gamma(k_2-l_1+1)}{\Gamma(p+1)} \prod_{i=1}^r \left(1 - \frac{k_1+k_2-l-l_1}{p\gamma_i}\right)^{-1} \cdot \prod_{j=r+1}^s \left(1 - \frac{k_2-l}{p\gamma_i}\right)^{-1}. \quad (5.14)$$

Remark 5.1 : Set $m = 0$, $k = 1$ in (5.14), to get product moments of concomitants of order statistics from bivariate Pareto distribution as

$$E\left(Y_{[r:n]}^{(k_1)} Y_{[s:n]}^{(k_2)}\right) = b^{k_1+k_2} \sum_{l=0}^{k_1} \sum_{l_1=0}^{k_2} \binom{k_1}{l} \binom{k_2}{l_1} \frac{\Gamma(p+l+1-k_1)\Gamma(k_1-l+1)}{\Gamma(p+1)} \\ \times \frac{\Gamma(p+l_1+1-k_2)\Gamma(k_2-l_1+1)}{\Gamma(p+1)} \prod_{i=1}^r \left(1 - \frac{k_1+k_2-l-l_1}{p(n-i+1)}\right)^{-1} \prod_{j=r+1}^s \left(1 - \frac{k_2-l}{p(n-i+1)}\right)^{-1}.$$

Remark 5.2 : At $m = -1$, in (5.15), we get product moment of concomitants of $k - th$ upper record value from bivariate Paretodistribution as

$$E\left(Y_{[r,n,-1,k]}^{(k_1)} Y_{[s,n,-1,k]}^{(k_2)}\right) = b^{k_1+k_2} \sum_{l=0}^{k_1} \sum_{l_1=0}^{k_2} \binom{k_1}{l} \binom{k_2}{l_1} \frac{\Gamma(p+l+1-k_1)\Gamma(k_1-l+1)}{\Gamma(p+1)} \\ \times \frac{\Gamma(p+l_1+1-k_2)\Gamma(k_2-l_1+1)}{\Gamma(p+1)} \left(1 - \frac{k_1+k_2-l-l_1}{p k}\right)^{-r} \left(1 - \frac{k_2-l}{pk}\right)^{-(s-r)}.$$

Product moments between the concomitants of order statistics (Remark 5.1)**Table 5.1**

$\alpha_2 = 2.0000$		$k_2 = 5.0000$				$c = 2.0000$			
n	$s \setminus r$	1	2	3	4	5	6	7	8
1	1	2.3333							
2	1	1.9667							
	2	2.2500	2.7000						
3	1	1.8928							
	2	1.9964	2.1143						
	3	2.3071	2.4464	2.9928					
4	1	1.8606							
	2	1.9227	1.9896						
	3	2.0345	2.1057	2.2389					
	4	2.3700	2.4540	2.6129	3.2442				
5	1	1.8425							
	2	1.8866	1.9331						
	3	1.9528	2.0009	2.0744					
	4	2.0719	2.1233	2.2016	2.3486				
	5	2.4293	2.4901	2.5833	2.7599	3.4680			
6	1	1.8308							
	2	1.8650	1.9005						
	3	1.9116	1.9480	1.9981					
	4	1.9814	2.0192	2.0713	2.1507				
	5	2.1071	2.1475	2.2031	2.2880	2.4476			
	6	2.4842	2.5321	2.5984	2.7000	2.8929	3.6721		
7	1	1.8228							
	2	1.8506	1.8793						
	3	1.8864	1.9157	1.9535					
	4	1.9351	1.9652	2.0041	2.0576				
	5	2.0083	2.0395	2.0800	2.1357	2.2205			
	6	2.1399	2.1733	2.2166	2.2762	2.3672	2.5384		
	7	2.5349	2.5747	2.6264	2.6977	2.8072	3.0153	3.8611	
8	1	1.8168							
	2	1.8403	1.8644						
	3	1.8693	1.8938	1.9241					
	4	1.9065	1.9315	1.9625	2.0025				
	5	1.9573	1.9830	2.0148	2.0560	2.1126			
	6	2.0335	2.0602	2.0934	2.1362	2.1952	2.2853		
	7	2.1706	2.1992	2.2347	2.2806	2.3439	2.4406	2.6228	
	8	2.5821	2.6162	2.6587	2.7138	2.7899	2.9067	3.1292	4.0379

**Product moments between the concomitants of $k - th$ upper record statistics
(Remark 5.2)**

Table 5.2

$b = 1.0000$		$p = 3.0000$			$k = 2.0000$						
$s \setminus r$	1	2	3	4	5	6	7	8	9	10	
1	1.1667										
2	1.2000	1.2500									
3	1.2400	1.3000	1.3750								
4	1.2880	1.3600	1.4500	1.5625							
5	1.3456	1.4320	1.5400	1.6750	1.8437						
6	1.4147	1.5184	1.6480	1.8100	2.0125	2.2656					
7	1.4977	1.6221	1.7776	1.9720	2.2150	2.5187	2.8984				
8	1.5972	1.7465	1.9331	2.1664	2.4580	2.8225	3.2781	3.8477			
9	1.7166	1.8958	2.1197	2.3997	2.7496	3.1870	3.7337	4.4172	5.2712		
10	1.8599	2.0749	2.3437	2.6796	3.0995	3.6244	4.2805	5.1006	6.1258	7.4072	

Table 5.3

$b = 1.0000$		$p = 4.0000$			$k = 3.0000$						
$s \setminus r$	1	2	3	4	5	6	7	8	9	10	
1	1.0750										
2	1.0818	1.0900									
3	1.0893	1.0982	1.1080								
4	1.0974	1.1071	1.1178	1.1296							
5	1.1062	1.1168	1.1285	1.1414	1.1555						
6	1.1159	1.1275	1.1402	1.1542	1.1696	1.1866					
7	1.1264	1.1390	1.1529	1.1683	1.1851	1.2036	1.2239				
8	1.1379	1.1516	1.1668	1.1835	1.2019	1.2221	1.2443	1.2687			
9	1.1504	1.1655	1.1820	1.2002	1.2203	1.2423	1.2665	1.2932	1.3225		
10	1.1641	1.1805	1.1986	1.2184	1.2403	1.2643	1.2907	1.3198	1.3518	1.3870	

Table 5.4

$s \setminus r$	$b = 2.0000$		$p = 5.0000$			$k = 4.0000$					
	1	2	3	4	5	6	7	8	9	10	
1	4.1777										
2	4.1871	4.1975									
3	4.1969	4.2079	4.2194								
4	4.2073	4.2188	4.2310	4.2438							
5	4.2182	4.2303	4.2431	4.2567	4.2709						
6	4.2297	4.2425	4.2559	4.2702	4.2852	4.3010					
7	4.2418	4.2552	4.2694	4.2844	4.3002	4.3169	4.3345				
8	4.2545	4.2687	4.2836	4.2994	4.3160	4.3335	4.3521	4.3716			
9	4.2679	4.2828	4.2985	4.3151	4.3326	4.3511	4.3706	4.3912	4.4129		
10	4.2820	4.2977	4.3142	4.3317	4.3501	4.3696	4.3901	4.4118	4.4347	4.4588	

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References

- [1] Ahsanullah, M. and Beg, M. I. (2006): Concomitants of generalized order statistics from Gumbel's bivariate exponential distribution, *J. Statist. Theory and Application.*, **6(2)**, 118 - 132.
- [2] Begum, A. A. and Khan, A. H. (1997): Concomitants of order statistics from Gumbel's bivariate Weibull distribution, *Cal. Statist. Assoc. Bull.*, **47**, 131 - 140.
- [3] Begum, A. A. and Khan, A. H. (1998): Concomitants of order statistics from bivariate Burr distribution, *J. Appl. Statist. Sci.*, **7 (4)**, 255 - 265.
- [4] Begum, A. A. and Khan, A. H. (2000): Concomitants of order statistics from Marshall and Olkin bivariate Weibull distribution, *Cal. Statist. Assoc. Bull.*, **50**, 65 - 70.
- [5] Begum, A. A. (2003): Concomitants of order statistics from bivariate Pareto II distribution, *International Journal of Statistical Sciences*, **2**, 27 - 35.
- [6] Balasubramnian, K. and Beg, M. I. (1996): Concomitants of order statistics in bivariate exponential distribution of Marshall and Olkin, *Cal. Statist. Assoc. Bull.*, **46**, 109 - 115.

- [7] Balasubramnian, K. and Beg, M. I. (1997): Concomitants of order statistics in Morgenstern type bivariate exponential distribution, *J. App. Statist. Sci.*, **54** (4), 233 - 245.
- [8] Balasubramnian, K. and Beg, M. I. (1998): Concomitants of order statistics in Gumbel's bivariate exponential distribution, *Sankhya B*, **60**, 399 - 406.
- [9] Bhattacharya, P.K. (1984): Induced order statistics: Theory and Applications. In: Krishnaiah, P.R. and Sen, P.K. (Eds.), *Hand Book of Statistics.*, **4**, 383 - 403, Elsevier Science.
- [10] David, H.A and Nagaraja, H.N. (1998): Concomitants of order statistics In: N.Balakrishnan and C.R.Rao (eds), *Hand Book of Statistics.*, **16**, 487 - 513, Elsevier Science.
- [11] David, H.A., and Nagaraja, H.N. (2003): Order Statistics, *John Wiley, New York.*
- [12] Do, K. A. and Hall, P. (1991): Distribution theory using concomitants of order statistics with application to Monte Carlo simulation for the bootstrap, *J. Roy. Statist. Soc. B* , **54**, 595–607.
- [13] Johnson, N.L. and Kotz, S. (1972) : Distributions in Statistics, Continuous Multivariate Distributions, *John Wiley, New York.*
- [14] Kamps, U. (1995): *A concept of generalized order statistics.* B.G. Teubner Stuttgart, Germany.
- [15] Nayabuddin (2013): Concomitants of generalized order statistics from bivariate Lomax distribution, *ProbStat Forum*, **6**, 73–88.
- [16] O'Connell, M. J. and David, H. A. (1976): Order statistics and their concomitants in some double sampling situations, *Essays in Probability and Statistics.* , 451-466, Tokyo Shinko Tsusho.
- [17] Prudnikov, A. P., Brychkov, Yu. A. and Marichev(1986): *Integral and series*, Vol. 3, More Special Functions, Gordon and Breach Science Publisher, New York.
- [18] Srivastava, H. M. and Karlsson (1985): *Multiple Gaussian Hypergeometric series*, *John Wiley, New York.*
- [19] Tsukibayashi, S. (1998): The joint distribution and moments of an extreme of the dependent variable and the concomitant of an extreme of the independent variable, *Commun. Statist. Theor. Meth.*, **27**(7), 639–651.

Appendix: Clarification of equation (2.1) in the text.

We have the *pdf* of $X(r, n, m, k)$ as

$$f_r(x) = \frac{C_{r-1}}{(r-1)!} [\bar{F}(x)]^{\gamma_r-1} f(x) g_m^{r-1}(F(x)) \quad (1)$$

and the *pdf* of $Y_{[r,n,m,k]}$, the r^{th} concomitant of generalized order statistics is

$$g_{[r,n,m,k]} = \int_{-\infty}^{\infty} f_{Y|X}(y|x) f_{(r,n,m,k)}(x) dx \quad (2)$$

Using (1) in (2), we get

$$g_{[r,n,m,k]} = \frac{C_{r-1}}{(r-1)!} \int_{-\infty}^{\infty} f_{Y|X}(y|x) [\bar{F}(x)]^{\gamma_{r-1}} f(x) g_m^{r-1}(F(x)) dx \quad (3)$$

Now expanding $g_m^{r-1}(F(x)) = \left[\frac{1}{m+1} \left\{ 1 - (F(x))^{m+1} \right\} \right]^{r-1}$ binomially in (3), we get

$$g_{[r,n,m,k]}(y) = \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \int_{-\infty}^{\infty} [\bar{F}(x)]^{\gamma_{r-i}-1} f(x) dx \quad (4)$$

Since for bivariate pareto distribution *pdf* is

$$f(x, y) = \frac{p(p+1)(ab)^{p+1}}{(bx + ay - ab)^{p+2}}, \quad p > 0, x > a > 0, y > b > 0 \quad (5)$$

and corresponding *df* is

$$F(x, y) = 1 - \frac{b^p}{y^p} - \frac{a^p}{x^p} + \frac{(ab)^{p+1}}{(bx + ay - ab)^{p+2}}, \quad p > 0, x > a > 0, y > b > 0 \quad (6)$$

The conditional *pdf* of Y given X is

$$f(y|x) = \frac{a(p+1)(bx)^{p+1}}{(bx + ay - ab)^{p+2}}, \quad y > b > 0, y > 0, \quad (7)$$

The marginal *pdf* of X is

$$f(x) = \frac{pa^p}{x^{p+1}}, \quad p > 0, x > a > 0 \quad (8)$$

The marginal *df* of X is

$$F(x) = 1 - \frac{a^p}{x^p}, \quad p > 0, x > a > 0 \quad (9)$$

Now using (7),(8) and (9) in (4), we get after simplification

$$\begin{aligned} g_{[r,n,m,k]}(y) &= \frac{p(p+1)}{ab} \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \\ &\times \int_0^{\infty} \left(\frac{a}{x} \right)^{p\gamma_{r-i}+2} \left[1 + \left(\frac{y}{b} - 1 \right) \frac{a}{x} \right]^{-(p+2)} dx \end{aligned} \quad (10)$$

which is the expression (2.1) in the manuscript.