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Recurrence Relations for Single and Product Moments of Upper k-Record Values from Chen Distribution and a Characterization

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Abstract. In this study we give some recurrence relations satisfied by single and product moments of upper k-record values from Chen distribution. Further a characterization of the Chen distribution based on conditional expectation of function of upper k-record values is presented.

Mathematics Subject Classification. ???.

Keywords. Record values, upper k-record, single moments, product moments, reucrrence relations, Chen distribution, characterization, conditional expectation.

1 Introduction

A random variable X is said to have the Chen distribution if its probability density function (pdf) is of the form

$$f(x) = \lambda\beta x^{\beta-1} e^{x^\beta} \exp(\lambda(1-e^{x^\beta})), \quad x, \lambda, \beta > 0 \tag{1.1}$$

and the cumulative distribution function (cdf) is given by

$$F(x) = 1 - \exp(\lambda(1-e^{x^\beta})), \quad x, \lambda, \beta > 0\,. \tag{1.2}$$

The Chen distribution at (1.1) was introduced by Chen (2000). This is a two-parameter lifetime distribution with bathtub shape or increasing failure rate function. It is easy to see from (1.1) and (1.2) that for the Chen distribution

$$f(x) = \beta x^{\beta-1}[\lambda + \{-\log(1-F(x))\}][1-F(x)] \tag{1.3}$$

Remark 1.1. By setting $\beta = 1$ in (1.1). Chen distribution reduces to

$$f(x) = \lambda e^{x^\beta} \exp(\lambda(1-e^{x})), \quad x, \lambda, \beta > 0 \tag{1.4}$$

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Now let us consider Gompertz distribution with parameters α and β. A random variable X is said to have Gompertz distribution if its probability density function (pdf) is of the form

$$f(x) = \beta e^{\alpha x} \exp\left(\frac{\beta}{\alpha}(1 - e^{\alpha x})\right), \quad x, \alpha, \beta > 0. \tag{1.5}$$

Putting $\alpha = 1$ and $\beta = \lambda$ in (1.5) we get (1.4). Thus Gompertz distribution is a particular case of Chen distribution and results for Gompertz distribution can be derived as a special case from Chen distribution.

Suppose $\{X_n, n \geq 1\}$ is an infinite sequence of independent, identically distributed (i.i.d.) random variables with common cdf $F(x)$ and pdf $f(x)$, respectively. Let us assume that $F(\cdot)$ is continuous so that ties are not possible.

Let $Y_n = \max\{X_1, X_2, X_3, \ldots, X_n\}$, $n = 1, 2, \ldots$. We say X_j is an upper record value of this sequence if $Y_j > Y_{j-1}$, $j \geq 2$. The indices at which the upper record values occur are given by the upper record times $\{U(n), n \geq 1\}$, where

$$U(n) = \min\{j \mid j > U(n-1),\, X_j > X_{U(n-1)},\, n > 1\} \text{ with } U(1) = 1.$$

Then $X_{U(n)}$ and $U(n)$ are the sequences of upper record values and upper record times, respectively. Similarly, for a fixed $k \geq 1$, we define the sequence $\{U(n:k), n \geq 1\}$ of upper k-record times of $\{X_n, n \geq 1\}$ as follows:

$$\begin{aligned} &U(1:k) = 1 \\ &U(n+1:k) = \min\{j > U(n:k) : X_{j:j+k-1} > X_{U(n:k):U(n:k)+k-1}\}, \end{aligned}$$

where $X_{j:n}$ denotes the jth order statistic of a sample $(X_1, X_2, \ldots, X_n)$, (see Kamps (1995a, b)). Then $X_{U(n:k)}$ and $U(n:k)$ are called the sequences of upper k-record values and upper k-record times, respectively. For $k = 1$ and $n = 1, 2, \ldots$, we write $U(n:1) = U(n)$ and $X_{U(n:1)} = X_{U(n)}$.

Chandler (1952) introduced record values and record value times. Properties of record values of i.i.d. random variables have been extensively studied in the literature. Various developments on records and related topics have been reviewed by a number of authors including Glick (1978), Nevzorov (1987), Resnick (1987), Nagaraja (1988), Ahsanullah (1995), Arnold and Balakrishnan (1989), Arnold, Balakrishnan and Nagaraja (1998). In this paper, we establish some recurrence relations satisfied by the single and product moments of upper k-record values from the Chen distribution has also been obtained on using the conditional expectation of upper k-record values. Similar results for modified Weibull and Gompertz distribution have been derived by Sultan (2007) and Khan dna Zia (2009). We shall denote

$$\begin{aligned} &\alpha^{(r)}_{(m;k)} = E(X^{(r)}_{U(n:k)}), \quad r, n = 1, 2, \ldots, \\ &\alpha^{(r,s)}_{(m;k)} = E(X^{(r)}_{U(m:k)} X^{(s)}_{U(n:k)}), \quad 1 \leq m \leq n-1 \text{ and } r, s = 1, 2, \ldots, \end{aligned}$$

$$\alpha_{(m,n:k)}^{(r,0)} = E(X_{U(n:k)}^{(r)}) = \alpha_{n:k}^{(r)}, \quad 1 \leq m \leq n-1 \text{ and } r = 1, 2, \ldots,$$
$$\alpha_{(m,n:k)}^{(0,s)} = E(X_{U(m:k)}^{(s)}) = \alpha_{m:k}^{(s)}, \quad 1 \leq m \leq n-1 \text{ and } s = 1, 2, \ldots.$$

2 Relations for Single Moments

The pdf of $X_{U(n:k)}^{(s)}$, $n = 1, 2, \ldots$ is given by

$$f_{X^{U(n:k)}}(x) = \frac{k^n}{(n-1)!}\{-\log(1-F(x))\}^{n-1}[1-F(x)]^{k-1}f(x), \; -\infty < x < \infty. \tag{2.1}$$

(see Dziubdziela and Kopocinski (1976) and Grudzien (1982)).

Theorem 2.1. *For $n \geq 1$, $r = 0, 1, 2, \ldots$ and $\beta > 0$,*

$$\left(\frac{r}{\beta}+1\right)\alpha_{(n:k)}^{(r)} = k\lambda[\alpha_{(n:k)}^{(r+\beta)} - \alpha_{(n-1:k)}^{(r+\beta)}] + [\alpha_{(n+1:k)}^{(r+\beta)} - \alpha_{(n:k)}^{(r+\beta)}] \tag{2.2}$$

Integrating by parts treating $x^{r+\beta-1}$ for integration and the rest of the integrand for differentiaion and simplifying the so obtained relation, we immediately obtain the result of Theorem 2.1.

Remark 2.1. By setting $k = 1$ in (2.2), we get the recurrence relation for single moments of upper record values from Chen distribution.

Remark 2.2. By setting $\beta = 1$ abd $k = 1$ in (2.2) and rearranging. We get the recurrence relation for single moments of upper record values from Gompertz distribution in (1.4), which is in agreement with Khan and Zia (2009).

3 Relations for Product Moments

We have the joint pdf of $X_{U(m:k)}$ and $X_{U(m:k)}$ $1 \leq m \leq n$, as

$$\begin{aligned} f_{X_{U(m;k)},X_{U(n,k)}}(x,y) = {} & \frac{k^n}{(m-1)!(n-m-1)!}\{-\log(1-F(x))\}^{m-1} \\ & \cdot \{-\log(1-F(y)) + \log(1-F(x))\}^{n-m-1} \\ & \cdot [1-F(y)]^{k-1}\frac{f(x)f(y)}{[1-F(x)]}\;; \quad -\infty < x < \infty \end{aligned} \tag{3.1}$$

(see Dziubdziela and Kopocinski (1976) and Grudzien (1982)).

Now, we derive some simple recurrence relations for the product moments of upper k-record values by using the relation (1.3).

Theorem 3.1. *For $m \geq 1$, $r, s = 0, 1, 2, \ldots$, and $\beta < 0$,*

$$\left(\frac{r}{\beta}+1\right)\alpha^{(r,s)}_{(m,m+1:k)} = \lambda k[\alpha^{(r+\beta+s)}_{(m,k)} - \alpha^{(r+\beta,s)}_{(m-1,m:k)}] + [\alpha^{(r+\beta+s)}_{(m+1:k)} - \alpha^{(r+\beta,s)}_{(m,m+1,n:k)}] \tag{3.2}$$

and

$$\left(\frac{r}{\beta}+1\right)\alpha^{(r,s)}_{(m,n:k)} = \lambda k[\alpha^{(r+\beta,s)}_{(m,n-1,k)} - \alpha^{(r+\beta,s)}_{(m-1,n-1:k)}] + [\alpha^{(r+\beta,s)}_{(m+1,n:k)} - \alpha^{(r+\beta,s)}_{(m,n:k)}] \tag{3.3}$$

Proof. From equation (3.1) for $1 \leq m \leq n-2$ and $r, s = 0, 1, 2, \ldots$ and by using (1.3), we get

$$\alpha^{(r,s)}_{(m,n:k)} = \frac{k^n}{(m-1)!(n-m-1)!}\int_0^\infty y^s f(y)[1-F(y)]^{k-1} I(y)dy, \tag{3.4}$$

where

$$\begin{aligned} I(y) = \beta\lambda \int_0^y & x^{r+\beta-1}\{-\log(1-F(x))\}^{m-1}\{-\log(1-F(y)) \\ & + \log(1-F(x))\}^{n-m-1}dx + \beta\int_0^y x^{r+\beta-1}\{-\log(1-F(x))\}^m \\ & \cdot \{-\log(1-F(y)) + \log(1-F(x))\}^{n-m-1}dt\,. \end{aligned} \tag{3.5}$$

Integrating $I(y)$ by parts treating $x^{r+\beta-1}$ for integration and the rest of the integrand for differentiation, we get

$$\begin{aligned} I(y) = & \frac{\beta\lambda(n-m-1)}{(r+\beta)}\int_0^y x^{r+\beta}\{-\log(1-F(x))\}^{m-1} \\ & \cdot \{-\log(1-F(y)) + \log(1-F(x))\}^{n-m-2}\frac{f(x)}{[1-F(x)]}dx \\ & - \frac{\beta\lambda(m-1)}{(r+\beta)}\int_0^y x^{r+\beta}\{-\log(1-F(x))\}^{m-2} \\ & \cdot \{-\log(1-F(y)) + \log(1-F(x))\}^{n-m-1}\frac{f(x)}{[1-F(x)]}dx \\ & - \frac{\beta(n-m-1)}{(r+\beta)}\int_0^y x^{r+\beta}\{-\log(1-F(x))\}^{m} \\ & \cdot \{-\log(1-F(y)) + \log(1-F(x))\}^{n-m-2}\frac{f(x)}{[1-F(x)]}dx \\ & - \frac{\beta m}{(r+\beta)}\int_0^y x^{r+\beta}\{-\log(1-F(x))\}^{m-1} \\ & \cdot \{-\log(1-F(y)) + \log(1-F(x))\}^{n-m-1}\frac{f(x)}{[1-F(x)]}dx\,. \end{aligned} \tag{3.6}$$

Substituting the above expression in (3.4) and simplifying the resulting equation, we get equation (3.3).

Proceeding in a similar manner for the case $n = m+1$, the recurrence relation given in (3.2) can easily be established. $\square$

Theorem 3.2. *For $m \geq 1$, $r, s = 0, 1, 2, \ldots$, and $\beta < 0$,*

$$\left(\frac{s}{\beta}+1\right)\alpha^{(r,s)}_{(m,m+1:k)} = \lambda[\alpha^{(r,s+\beta)}_{(m,m+1:k)} - \alpha^{(r+s+\beta)}_{(m:k)}] + m[\alpha^{(r+s+\beta)}_{(m+1:k)} - \alpha^{(r,s+\beta)}_{(m+1,m+2:k)}]$$
$$+ [\alpha^{(r,s+\beta)}_{(m,m+2:k)} - \alpha^{(r,s+\beta)}_{(m,m+1:k)}] \tag{3.7}$$

and for $1 \leq m \leq n-2$, $r, s = 0, 1, \ldots$ and $\beta > 0$,

$$\left(\frac{s}{\beta}+1\right)\alpha^{(r,s)}_{(m,n:k)} = \lambda k[\alpha^{(r,s+\beta)}_{(m,n:k)} - \alpha^{(r,s+\beta)}_{(m,n-1:k)}] + m[\alpha^{(r,s+\beta)}_{(m+1,n:k)} - \alpha^{(r,s+\beta)}_{(m+1,n+1:k)}]$$
$$+ (n-m)[\alpha^{(r,s+\beta)}_{(m,n+1:k)} - \alpha^{(r,s+\beta)}_{(m,n:k)}] \tag{3.8}$$

Proof. From equation (3.1) for $1 \leq m \leq n-2$ and $r, s = 0, 1, 2, \ldots$ and by using (1.3), we get

$$\alpha^{(r,s)}_{(m,n:k)} = \frac{k^n}{(m-1)!(n-m-1)!}\int_0^\infty x^r\{-\log(1-F(x))\}^{m-1}$$
$$\cdot \frac{f(x)}{[1-F(x)]} I(x)dx\,, \tag{3.9}$$

where

$$I(x) = \beta[\lambda + \{-\log(1-F(x))\}]\int_x^\infty y^{s+\beta-1}\{-\log(1-F(y))$$
$$+\log(1-F(x))\}^{n-m-1}[(1-F(y))]^k dy + \beta\int_x^\infty y^{s+\beta-1}$$
$$\cdot\{-\log(1-F(y)) + \log(1-F(x))\}^{n-m}[-\log(1-F(y))]^k dy\,. \tag{3.10}$$

Integrating $I(x)$ by parts treating $y^{s+\beta-1}$ for integration and the rest of the integrand for differentiation, we get

$$I(x) = \frac{\beta k[\lambda + \{-\log(1-F(x))\}]}{(s+\beta)}\int_x^\infty y^{s+\beta}\{-\log(1-F(y))$$
$$+\log(1-F(x))\}^{n-m-1}[1-F(y)]^{k-1}f(y)dy$$
$$+\frac{\beta k[\lambda + \{-\log(1-F(x))\}](n-m-1)}{(s+\beta)}\int_x^\infty y^{s+\beta}\{-\log(1-F(y))$$
$$+\log(1-F(x))\}^{n-m-2}[1-F(y)]^{k-1}f(y)dy$$
$$+\frac{\beta k}{(s+\beta)}\int_x^\infty y^{s+\beta}\{-\log(1-F(y)) + \log(1-F(x))\}^{n-m}$$
$$\cdot[1-F(y)]^{k-1}f(y)dy + \frac{\beta(n-m)}{(s+\beta)}\int_x^\infty y^{s+\beta}\{-\log(1-F(y))$$
$$+\log(1-F(x))\}^{n-m-1}[1-F(y)]^{k-1}f(y)dy\,. \tag{3.11}$$

Substituting the above expression in (3.9) and simplifying the resulting equation, we get equation (3.8).

Proceeding in a similar manner for the case $n = m + 1$, the recurrence relation given in (3.7) can easily be established. □

Remark 3.1. By setting $\beta = 1$ and $k = 1$ in Theorem 3.1, we get the recurrence relation for product moments of upper record values from Gompertz distribution, which is in agreements with Khan and Zia (2009).

Remark 3.2. By setting $k = 1$ in Theorem 3.1 and 3.2, we get the recurrence relation for product moments of upper record values from Chen distribution.

4 Characterization

Let $X_{U(m;k)}$ and $X_{U(n;k)}$ be the m-th and n-th upper k-record values, then the conditional pdf of $X_{U(n;k)} = y$, $1 lelen$

$$
\begin{aligned}
f_{X_{U(m;k)}|X_{U(n,k)}}(x|y) = {}& \frac{(n-1)!}{(m-1)!(n-m-1)!}\{-\log(1-F(x))\}^{m-1} \\
& \cdot \{-\log(1-F(y)) + \log(1-F(x))\}^{n-m-1} \\
& \cdot \frac{f(x)}{\{-\log(1-F(y))\}^{-1}[1-F(x)]} \; ; \quad -\infty < x < y < \infty
\end{aligned} \tag{4.1}
$$

and the conditional pdf of $X_{U(n;k)}$ given $X_{U(n;k)} = x$, $1 \leq mMn$

$$
\begin{aligned}
f_{X_{U(m;k)}|X_{U(n,k)}}(y|x) = {}& \frac{k^{n-m}}{(n-m-1)!}\{-\log(1-F(y)) + \log(1-F(x))\}^{n-m-1} \\
& \cdot \frac{f(y)[1-F(y)]^{k-1}}{[1-F(x)]^k} \; ; \quad -\infty < x < y < \infty
\end{aligned} \tag{4.2}
$$

Theorem 4.1. *Let X be an absolutely continuous rv with pdf $f(x)$ and cdf $F(x)$ on the support $(0, \infty)$ then*

$$F(x) = 1 - \exp(\lambda(1 - e^{x^\beta})), \quad \lambda, \beta > 0$$

if and only if

$$E[\exp(-\lambda e^{x^\beta_{U(r+1;k)}})|X_{U(r;k)} = x] = \left(\frac{k}{k+1}\right)\exp(-\lambda e^{x^\beta}). \tag{4.3}$$

Proof. From equation (4.2), for $n = r + 1$ and $m = r$, we have

$$
\begin{aligned}
E[\exp(-\lambda e^{x^\beta_{U(r+1;k)}})|X_{U(r;k)} = x] = {}& \frac{k}{[1-F(x)]^k}\int_x^\infty \exp(-\lambda e^{y^\beta}) \\
& \cdot [1-F(y)]^{k-1} f(y)dy\,.
\end{aligned} \tag{4.4}
$$

After using (1.1) and calculating the integral, we obtain (4.3), To prove sufficient part, we have from (4.4)

$$k\int_x^{\infty} \exp(-\lambda e^{y^\beta})[1-F(y)]^{k-1}f(y)dy = \left(\frac{k}{k+1}\right)\exp(-\lambda e^{x^\beta})[1-F(x)]^k \tag{4.5}$$

Differentiating both the sides with respect to x and rearranging, we get

$$\frac{f(x)}{[1-F(x)]} = \lambda\beta x^{\beta-1}e^{x^\beta},$$

which leads to (1.2). □

Remark 4.1. By setting $\beta = 1$ and $k = 1$ in Theorem 4.1, we get the Characterization property of Gompertz distribution, which is in agreements with Khan and Zia (2009).

Remark 4.2. By setting $k = 1$ in Theorem 4.1, we get the Characterization property of Chen distribution based on conditonal expectation of function of upper record values.

5 Discussion

Khan and Zia (2009) have obtained recurrence relations for single and product moments of upper record values from Gompertz distrubition and a characterization. In this paper we have derived recurrence relations for single and product moments of upper k-record values from Chen distribution and a characterization. The results of Khan and Zia for Gompertz distribution can be obtained as a special case from Chen distribution.

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