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Some characterization results based on conditional expectation of dual generalized order statistics

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Abstract. In this paper, a general form of continuous probability distributions $F(x) = a(h(x))^{-c}$ $x \in (\alpha, \beta)$ has been characterized through conditional expectation of $p-th$ power of difference of two dual generalized order statistics. Further, some deductions and related results are also discussed.

1. Introduction

Kamps (1995) introduced the concept of generalized order statistics (gos) to unify several models of ascendingly ordered random variables, e.g. upper order statistics, k-record values, progressively Type-II censored order statistics, Pfeifer records and sequential order statistics. These models can be effectively applied in reliability theory and survival analysis. However, random variables that are decreasingly ordered cannot be comprised into this framework. Further, this model is inappropriate to study, e.g. reversed ordered order statistic and lower record values models. Based on the gos, Burkschat *et al.* (2003) introduced the concept of the dual generalized order statistics ($dgos$) that enables a common approach to study descendingly ordered random variables like reversed ordered order statistics and lower record values.
Let $n \geq 2$ be a given integer and $\tilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in \Re^{n-1},\ k > 0$ be the parameters such that

$$\gamma_i = k + n - i + \sum_{j=i}^{n-1} m_j > 0 \quad \text{for} \quad 1 \leq i \leq n-1.$$

The random variables $X_d(1, n, \tilde{m}, k), X_d(2, n, \tilde{m}, k), \ldots, X_d(n, n, \tilde{m}, k)$ are said to be dual generalized order statistics from an absolutely continuous distribution function *df* $F()$ with the probability density function *pdf* $f()$, if their joint density function is of the form

$$f_{X_d(1,n,\tilde{m},k),\ldots,X_d(n,n,\tilde{m},k)}(x_1, x_2, \ldots, x_n)$$
$$= k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}\big[F(x_i)\big]^{m_i} f(x_i)\right)\big[F(x_n)\big]^{k-1} f(x_n) \tag{1}$$

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for $F^{-1}(1) > x_1 \geq x_2 \geq \ldots \geq x_n > F^{-1}(0)$.

Here we may consider two cases:

Case I: $\gamma_i \neq \gamma_j$; $i \neq j = 1, 2, \ldots, n-1$.

In view of (1) the *pdf* of $r-th$ dual generalized order statistics $X_d(r, n, \tilde{m}, k)$ is

$$f_{X_d(r,n,\tilde{m},k)}(x) = C_{r-1} f(x) \sum_{i=1}^{r} a_i(r) \big[F(x)\big]^{\gamma_i - 1} \tag{2}$$

and the joint *pdf* of $X_d(r, n, \tilde{m}, k)$ and $X_d(s, n, \tilde{m}, k)$, $1 \leq r < s \leq n$ is

$$f_{X_d(r,n,\tilde{m},k),X_d(s,n,\tilde{m},k)}(x,y) = C_{s-1} \left(\sum_{i=r+1}^{s} a_i^{(r)}(s) \left[\frac{F(y)}{F(x)}\right]^{\gamma_i} \right)$$

$$\times \left(\sum_{i=1}^{r} a_i(r) \big[F(x)\big]^{\gamma_i} \right) \frac{f(x)}{F(x)} \frac{f(y)}{F(y)}, \ \alpha \leq y < x \leq \beta. \tag{3}$$

The conditional *pdf* of $X_d(s, n, \tilde{m}, k)$ given $X_d(r, n, \tilde{m}, k) = x$, $1 \leq r < s \leq n$ in view of (2) and (3) is

$$f_{X_d(s,n,\tilde{m},k)|X_d(r,n,\tilde{m},k)}(y|x) = \frac{C_{s-1}}{C_{r-1}} \sum_{i=r+1}^{s} a_i^{(r)}(s) \frac{[F(y)]^{\gamma_i - 1}}{[F(x)]^{\gamma_i}} f(y), \ x > y. \tag{4}$$

where,

$$C_{r-1} = \prod_{i=1}^{r} \gamma_i, \ \gamma_i = k + n - i + \sum_{j=i}^{n-1} m_j > 0,$$

$$a_i(r) = \prod_{\substack{j=1 \\ j \neq i}}^{r} \frac{1}{(\gamma_j - \gamma_i)}, \ \gamma_i \neq \gamma_j, \ 1 \leq i \leq r \leq n$$

and

$$a_i^{(r)}(s) = \prod_{\substack{j=r+1 \\ j \neq i}}^{s} \frac{1}{(\gamma_j - \gamma_i)}, \ \gamma_i \neq \gamma_j, \ r+1 \leq i \leq s \leq n.$$

Case II: $m_i = m$ (say); $i = 1, 2, \ldots, n-1$.

The *pdf* of $r-th$ of $X_d(r, n, m, k)$ is

$$f_{X_d(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!} \big[F(x)\big]^{\gamma_r - 1} f(x) \ g_m^{r-1}\big(F(x)\big) \tag{5}$$

and the joint *pdf* of $X_d(r, n, m, k)$ and $X_d(s, n, m, k)$, $1 \leq r < s \leq n$ is

$$f_{X_d(r,n,m,k),X_d(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!} \ [F(x)]^m f(x) \ g_m^{r-1}\big(F(x)\big)$$

$$\times [h_m\big(F(y)\big) - h_m\big(F(x)\big)]^{s-r-1} [F(y)]^{\gamma_s - 1} f(y), \ \alpha \leq y < x \leq \beta. \tag{6}$$

The conditional *pdf* of $X_d(s,n,m,k)$ given $X_d(r,n,m,k) = x$, $1 \le r < s \le n$ in view of (5) and (6) is

$$f_{X_d(s,n,m,k)|X_d(r,n,m,k)}(y|x) = \frac{C_{s-1}}{C_{r-1}(s-r-1)!(m+1)^{s-r-1}} \frac{[F(y)]^{\gamma_{s-1}}}{[F(x)]^{\gamma_{r+1}}}$$
$$\times \left[[F(x)]^{m+1} - [F(y)]^{m+1}\right]^{s-r-1} f(y), \; x > y. \tag{7}$$

where,

$$h_m(x) = \begin{cases} -\dfrac{1}{m+1}\, x^{m+1} & , \; m \neq -1 \\ -\log x & , \; m = -1 \end{cases}$$

and $\quad g_m(x) = h_m(x) - h_m(1), \quad x \in [0,1).$

If $m = 0$ and $k = 1$, then $X_d(r,n,m,k)$ reduces to the $(n-r+1)-th$ lower order statistic, $X_{n-r+1:n}$ from the sample $X_1, X_2, \ldots, X_n$ [David and Nagaraja (2003)]. If $m = -1$ and $k = 1$, then $X_d(r,n,m,k)$ is the $r-th$ lower record value from an infinite sequence of independent and identically distributed (*iid*) random variables (*rv*) [Ahsanullah (1995)].

The characterization of probability distributions through conditional expectation of order statistics and record values have been seen among others are Nagraja (1977, 1988), Wu and Ouyang (1996), Franco and Ruiz (1995, 1996, 1997), López-Blázquez and Moreno-Rebello (1997), Wesolowski and Ahsanullah (1997), Dembińska and Wesolowski (1998, 2000), Wu and Lee (2001), Raqab (2002), Lee (2003), Athar *et al.* (2003), Khan and Athar (2004), Wu (2004), Noor and Athar (2014), Athar and Akhter (2015) and references therein.

Khan and Abu-Salih (1989) characterized some general forms of distributions through conditional expectation of order statistics fixing adjacent order statistics. Khan and Abouammoh (2000) extended the result of Khan and Abu-Salih (1989) and characterized the distributions when the conditioning is not adjacent. Khan *et al.* (2006) established characterizing relationships for the distributions through *gos* and characterized several distributions through conditional expectation of function of *gos*. Further, Khan *et al.* (2010) characterized several distributions through conditional expectation of function of *dgos*.

For various developments on characterization dealing with *gos* and *dgos* one may refer to Keseling (1999), Bieniek and Szynal (2003), Ahsanullah (2004), Khan and Alzaid (2004), Mbah and Ahsanullah (2007), Bieniek (2007, 2009), Samuel (2008), Khan *et al*. (2012), Noor *et al*. (2014, 2015) and references therein.

2. Characterization theorems

Theorem 2.1: *Let $X_d(i,n,\tilde{m},k)$, $1 \le i \le n$, be dual generalized order statistics based on absolutely continuous df $F(x)$ and pdf $f(x)$ over the support (α,β), where α and β may be finite or infinite, then for $1 \le r < s \le n$,*

$$E\left[\{h(X_d(s,n,\tilde{m},k)) - h(X_d(r,n,\tilde{m},k))\}^p | X_d(r,n,\tilde{m},k) = x\right] = \xi_{r,s,p}(x)$$
$$= a^*(h(x))^p \tag{8}$$

if and only if

$$F(x) = a(h(x))^{-c}, \; a \neq 0, \tag{9}$$

where

$$a^* = \sum_{j=0}^{p} (-1)^{j+p} \binom{p}{j} \prod_{i=r+1}^{s} \left(\frac{c\gamma_i}{c\gamma_i - j}\right), \; c\gamma_i \neq j$$

and $h(x)$ is a monotonic and differentiable function of x and p is a positive integer.

Proof: To prove the necessary part, we have

$$E\left[\{h(X_d(s,n,\tilde{m},k)) - h(X_d(r,n,\tilde{m},k))\}^p | X_d(r,n,\tilde{m},k) = x\right]$$

$$= \frac{C_{s-1}}{C_{r-1}} \int_\alpha^x (h(y)-h(x))^p \sum_{i=r+1}^{s} a_i^{(r)}(s) \left[\frac{F(y)}{F(x)}\right]^{\gamma_i - 1} \frac{f(y)}{F(x)} dy. \tag{10}$$

Let

$$t = \left[\frac{F(y)}{F(x)}\right], \text{ which implies } (h(y)-h(x))^p = (-1)^p (h(x))^p \left(1 - t^{-1/c}\right)^p.$$

Then the right hand side of (10) reduces to

$$E\left[\{h(X_d(s,n,\tilde{m},k)) - h(X_d(r,n,\tilde{m},k))\}^p | X_d(r,n,\tilde{m},k) = x\right]$$

$$= \frac{C_{s-1}}{C_{r-1}} \sum_{i=r+1}^{s} a_i^{(r)}(s) \int_0^1 (-1)^p (h(x))^p \; (1 - t^{-1/c})^p \; t^{\gamma_i - 1} dt$$

$$= \frac{C_{s-1}}{C_{r-1}} (h(x))^p \sum_{i=r+1}^{s} a_i^{(r)}(s) \left(\sum_{j=0}^{p} (-1)^{j+p} \binom{p}{j} \int_0^1 t^{\gamma_i - \frac{j}{c} - 1} dt \right)$$

$$= (h(x))^p \sum_{j=0}^{p} (-1)^{j+p} \binom{p}{j} \left(\frac{C_{s-1}}{C_{r-1}} \sum_{i=r+1}^{s} a_i^{(r)}(s) \left(\frac{c}{c\gamma_i - j} \right) \right).$$

This proves the necessary part.

To prove sufficiency part, let

$$E\left[\{h(X_d(s,n,\tilde{m},k)) - h(X_d(r,n,\tilde{m},k))\}^p | X_d(r,n,\tilde{m},k) = x\right] = \xi_{r,s,p}(x).$$

Therefore,

$$\frac{C_{s-1}}{C_{r-1}} \int_\alpha^x (h(y)-h(x))^p \sum_{i=r+1}^{s} a_i^{(r)}(s) \frac{[F(y)]^{\gamma_i - 1}}{[F(x)]^{\gamma_i}} f(y) dy = \xi_{r,s,p}(x).$$

Differentiating both sides $w.r.t.$ x, we get

$$-ph'(x) \frac{C_{s-1}}{C_{r-1}} \int_\alpha^x (h(y)-h(x))^{p-1} \sum_{i=r+1}^{s} a_i^{(r)}(s) \frac{[F(y)]^{\gamma_i - 1}}{[F(x)]^{\gamma_i}} f(y) dy$$

$$- \frac{C_{s-1}}{C_{r-1}} \int_\alpha^x (h(y)-h(x))^p \sum_{i=r+1}^{s} \gamma_i \; a_i^{(r)}(s) \frac{[F(y)]^{\gamma_i - 1}}{[F(x)]^{\gamma_i}} \frac{f(x)}{F(x)} f(y) dy = \xi'_{r,s,p}(x),$$

or,

$$-ph'(x)\xi_{r,s,p-1}(x) - \frac{C_{s-1}}{C_{r-1}} \int_\alpha^x (h(y)-h(x))^p \sum_{i=r+1}^{s} \gamma_i \; a_i^{(r)}(s) \frac{[F(y)]^{\gamma_i - 1}}{[F(x)]^{\gamma_i}} \frac{f(x)}{F(x)} f(y) dy = \xi'_{r,s,p}(x).$$

Since,

$$a_i^{(r+1)}(s) = (\gamma_{r+1} - \gamma_i) a_i^{(r)}(s) \quad \text{and} \quad C_r = \gamma_{r+1} C_{r-1}.$$

Therefore, we have

$$-\gamma_{r+1} \frac{C_{s-1}}{C_{r-1}} \frac{f(x)}{F(x)} \int_\alpha^x (h(y) - h(x))^p \sum_{i=r+1}^{s} a_i^{(r)}(s) \frac{[F(y)]^{\gamma_i - 1}}{[F(x)]^{\gamma_i}} f(y) dy$$

$$+\gamma_{r+1} \frac{C_{s-1}}{C_r} \frac{f(x)}{F(x)} \int_\alpha^x (h(y) - h(x))^p \sum_{i=r+2}^{s} a_i^{(r+1)}(s) \frac{[F(y)]^{\gamma_i - 1}}{[F(x)]^{\gamma_i}} f(y) dy$$

$$= \xi'_{r,s,p}(x) + p h'(x) \xi_{r,s,p-1}(x). \tag{11}$$

Rearranging the terms of (11), we get

$$\frac{f(x)}{F(x)} = -\frac{1}{\gamma_{r+1}} \frac{\xi'_{r,s,p}(x) + p h'(x) \xi_{r,s,p-1}(x)}{[\xi_{r,s,p}(x) - \xi_{r+1,s,p}(x)]}. \tag{12}$$

Now consider

$$\xi'_{r,s,p}(x) + p\, h'(x) \xi_{r,s,p-1}(x)$$

$$= p\, h'(x)(h(x))^{p-1} \sum_{j=0}^{p} (-1)^{j+p} \binom{p}{j} \prod_{i=r+1}^{s} \left(\frac{c\gamma_i}{c\gamma_i - j} \right)$$

$$+ p\, h'(x)(h(x))^{p-1} \sum_{j=0}^{p-1} (-1)^{j+p-1} \binom{p-1}{j} \prod_{i=r+1}^{s} \left(\frac{c\gamma_i}{c\gamma_i - j} \right)$$

$$= p\, h'(x)(h(x))^{p-1} \left[\sum_{j=0}^{p} (-1)^{j+p} \binom{p}{j} \prod_{i=r+1}^{s} \left(\frac{c\gamma_i}{c\gamma_i - j} \right) - \sum_{j=0}^{p-1} (-1)^{j+p} \binom{p}{j} \frac{(p-j)}{p} \prod_{i=r+1}^{s} \left(\frac{c\gamma_i}{c\gamma_i - j} \right) \right]$$

$$= h'(x)(h(x))^{p-1} \left[\sum_{j=0}^{p} (-1)^{j+p} \binom{p}{j} j \prod_{i=r+1}^{s} \left(\frac{c\gamma_i}{c\gamma_i - j} \right) \right] \tag{13}$$

and

$$\xi_{r,s,p}(x) - \xi_{r+1,s,p}(x)$$

$$= (h(x))^p \left[\sum_{j=0}^{p} (-1)^{j+p} \binom{p}{j} \prod_{i=r+1}^{s} \left(\frac{c\gamma_i}{c\gamma_i - j} \right) - \sum_{j=0}^{p} (-1)^{j+p} \binom{p}{j} \prod_{i=r+2}^{s} \left(\frac{c\gamma_i}{c\gamma_i - j} \right) \right]$$

$$= (h(x))^p \left[\sum_{j=0}^{p} (-1)^{j+p} \binom{p}{j} \prod_{i=r+2}^{s} \left(\frac{c\gamma_i}{c\gamma_i - j} \right) \left(\frac{c\gamma_{r+1}}{c\gamma_{r+1} - j} - 1 \right) \right]$$

$$= (h(x))^p \left[\frac{1}{c\gamma_{r+1}} \sum_{j=0}^{p} (-1)^{j+p} \binom{p}{j} j \prod_{i=r+1}^{s} \left(\frac{c\gamma_i}{c\gamma_i - j} \right) \right]. \tag{14}$$

Therefore in view of (12), we get

$$\frac{f(x)}{F(x)} = -\frac{1}{\gamma_{r+1}} \frac{h'(x)(h(x))^{p-1}}{(h(x))^p} \frac{\left[\sum_{j=0}^{p}(-1)^{j+p}\binom{p}{j} j \prod_{i=r+1}^{s}\left(\frac{c\gamma_i}{c\gamma_i - j}\right)\right]}{\left[\frac{1}{c\gamma_{r+1}}\sum_{j=0}^{p}(-1)^{j+p}\binom{p}{j} j \prod_{i=r+1}^{s}\left(\frac{c\gamma_i}{c\gamma_i - j}\right)\right]}$$

$$= -\frac{c\, h'(x)}{h(x)}.$$

Implying that

$$F(x) = a(h(x))^{-c}.$$

Similarly, the characterization result for case II may be obtained on the lines of Theorem 2.1.

Corollary 2.1: *Under the condition as stated in Theorem 2.1,*

$$E\left[h(X_d(s,n,\tilde{m},k)) | X_d(r,n,\tilde{m},k) = x\right] = h(x) \prod_{i=r+1}^{s} \left(\frac{c\gamma_i}{c\gamma_i - 1} \right), \ c\gamma_i \neq 1, \tag{15}$$

and consequently

$$E\left[h(X_d(r+1,n,\tilde{m},k)) | X_d(r,n,\tilde{m},k) = x\right] = h(x) \left(\frac{c\gamma_{r+1}}{c\gamma_{r+1} - 1} \right), \ c\gamma_{r+1} \neq 1 \tag{16}$$

if and only if

$$F(x) = a(h(x))^{-c}, \quad a \neq 0,$$

where $h(x)$ is a monotonic and differentiable function of x.

Proof: Expression (15) can be proved in view of Theorem 2.1 at $p = 1$ and (16) can be obtained at $s = r + 1$ in (15).

Remark 2.1: Let $m_i = m = 0$, $i = 1, 2, \ldots, n-1$ and $k = 1$, then, characterization result for lower order statistics is given as

$$E[\{h(X_{n-s+1:n}) - h(X_{n-r+1:n})\}^p | X_{n-r+1:n} = x] = b^*(h(x))^p$$

if and only if

$$F(x) = a(h(x))^{-c}, \ a \neq 0,$$

where

$$b^* = \sum_{j=0}^{p} (-1)^{j+p} \binom{p}{j} \prod_{i=r+1}^{s} \left(\frac{c\,(n-i+1)}{c\,(n-i+1) - j} \right), \ c\,(n-i+1) \neq j.$$

Remark 2.2: At $m = -1$ and $k = 1$, the characterization result for lower record will be

$$E[\{h(X_{L(s)}) - h(X_{L(r)})\}^p | X_{L(r)} = x] = c^*(h(x))^p$$

if and only if

$$F(x) = a(h(x))^{-c}, \ a \neq 0,$$

where

$$c^* = \sum_{i=0}^{p} (-1)^{i+p} \binom{p}{i} \left(\frac{c}{c - i} \right)^{s-r}, \ c \neq i.$$

3. Examples

(i) **Power function distribution**

$F(x) = x^{\nu}, \quad 0 < x \leq 1, \; \nu > 0.$

Then $F(x)$ is given by (9) with $a = 1, \; c = -\nu$ and $h(x) = x$,

(ii) **Inverse Power distribution**

$$F(x) = \left(\frac{x-\alpha}{\beta-\alpha}\right)^{\theta}, \quad \alpha < x < \beta, \; \theta > 0, \; \alpha, \beta \in \mathbb{R}, \; \alpha < \beta.$$

Then $F(x)$ is given by (9) with $a = 1, \; c = -\theta$ and $h(x) = (\frac{x-\alpha}{\beta-\alpha})$.

(iii) **Reflected exponential distribution**

$F(x) = e^{\lambda(x-\mu)}, \quad -\infty < x < \mu, \; \lambda > 0.$

Then $F(x)$ is given by (9) with $a = 1, \; c = -1$ and $h(x) = e^{\lambda(x-\mu)}$.

(iv) **Inverse Weibull distribution**

$F(x) = e^{-\theta x^{-\nu}}, \quad 0 < x < \infty, \; \theta, \nu > 0.$

Then $F(x)$ is given by (9) with $a = 1, \; c = \theta$ and $h(x) = e^{x^{-\nu}}$.

(v) **Inverse exponential distribution**

$F(x) = e^{-\frac{\theta}{x}}, \quad x > 0, \; \theta > 0.$

Then $F(x)$ is given by (9) with $a = 1, \; c = \theta$ and $h(x) = e^{\frac{1}{x}}$.

(vi) **Gumbel distribution**

$F(x) = e^{-e^{-x}}, \quad -\infty < x < \infty.$

Then $F(x)$ is given by (9) with $a = 1, \; c = 1$ and $h(x) = e^{e^{-x}}$.

(vii) **Burr type X distribution**

$F(x) = \left(1 - e^{-x^2}\right)^k, \quad 0 < x < \infty, \; k > 0.$

Then $F(x)$ is given by (9) with $a = 1, \; c = -\dfrac{k}{q}$ and $h(x) = \left(1 - e^{-x^2}\right)^q, \; q \neq 0.$

Similarly, characterization results for other distributions may be obtained with proper choice of a and $h(x)$. One may refer to Khan *et al.* (2007).

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