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L-moments and TL-moments estimation and relationships for moments of progressive type-II right censored order statistics from Frechet distribution

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Abstract. In this paper, we have obtained L-moments and TL-moments of Frechet distribution and used them to find the L-moments and TL-moments estimators of the parameters θ and λ of the Frechet distribution. Also we have obtained recurrence relations satisfied by the single and product moments of progressive Type-II right censored order statistics from Frechet distribution. This will enable one to evaluate the single and product moments in a recursive way.

1. Introduction

Maurice Fréchet (1927) was a French mathematician who had identified one possible limit distribution for the largest order statistic which refers extreme value distribution. The Fréchet distribution is also known as the Extreme Value Type II distribution or Fréchet model. Also, it is one of the probability distributions used to model extreme events (see Kotz and Nadarajah (2000), Fisher and Tippett (1928) and Gumbel (1958)). The extreme value distribution which has importance in risk management, finance, insurance, economics, hydrology, material sciences, telecommunications, and many other industries dealing with extreme events as a suitable model to represent phenomena with usually large maximum observations.

The probability density function (pdf) of Fréchet distribution is given by (see Fig. 1)

$$f(x) = \left(\frac{x}{\lambda}\right)^{-\theta-1}\left(\frac{\theta}{\lambda}\right)\ \exp\left\{-\left(\frac{x}{\lambda}\right)^{-\theta}\right\}, \qquad x > 0, \quad \theta > 0, \quad \lambda > 0, \tag{1}$$

where θ is the shape parameter ($\theta > 0$), and λ is the scale parameter ($\lambda > 0$). This distribution is bounded on the lower side ($x > 0$) and has a heavy upper tail. The cumulative distribution function (cdf) is given by

$$F(x) = \exp\left\{-\left(\frac{x}{\lambda}\right)^{-\theta}\right\}, \quad x > 0, \quad \theta > 0, \quad \lambda > 0. \tag{2}$$

The mean, variance, skewness and kurtosis of the 2-parameter Fréchet distribution, as defined in (1), are

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given by, respectively,

$$\mu = \lambda\,\Gamma\Big(1-\frac{1}{\theta}\Big), \qquad \sigma^2 = \lambda^2\Big[\Gamma\Big(1-\frac{2}{\theta}\Big) - \Big\{\Gamma\Big(1-\frac{1}{\theta}\Big)\Big\}^2\Big], \quad \theta > 2$$

$$\gamma_1 = \frac{\mu_3}{(\mu_2)^{3/2}} = \frac{\Gamma\Big(1-\frac{3}{\theta}\Big) - 3\Gamma\Big(1-\frac{1}{\theta}\Big)\Gamma\Big(1-\frac{2}{\theta}\Big) + 2\Big\{\Gamma\Big(1-\frac{1}{\theta}\Big)\Big\}^3}{\Big[\Gamma\Big(1-\frac{2}{\theta}\Big) - \Big\{\Gamma\Big(1-\frac{1}{\theta}\Big)\Big\}^2\Big]^{3/2}}, \quad \theta > 3$$

and

$$\beta_2 = \frac{\mu_4}{(\mu_2)^2} = \frac{\Gamma\Big(1-\frac{4}{\theta}\Big) + 6\Big\{\Gamma\Big(1-\frac{1}{\theta}\Big)\Big\}^2\Gamma\Big(1-\frac{2}{\theta}\Big) - 4\Gamma\Big(1-\frac{1}{\theta}\Big)\Gamma\Big(1-\frac{3}{\theta}\Big) - 3\Big\{\Gamma\Big(1-\frac{1}{\theta}\Big)\Big\}^4}{\Big[\Gamma\Big(1-\frac{2}{\theta}\Big) - \Big\{\Gamma\Big(1-\frac{1}{\theta}\Big)\Big\}^2\Big]^2}, \quad \theta > 4.$$

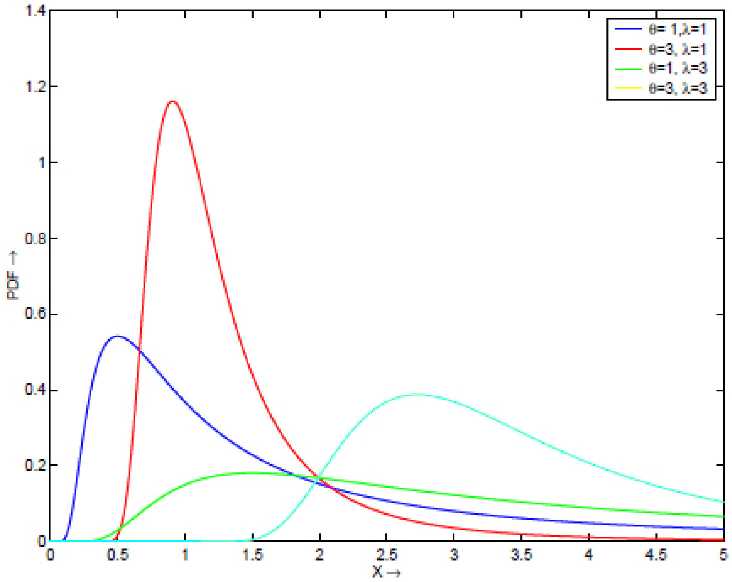

Figure 1: Pdf of Fréchet Distribution

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From equations (1) and (2), one can observe that the characterizing differential equation for Fréchet distribution is given by

$$\Big(\frac{\lambda^\theta}{\theta}\Big) f(x) = x^{-\theta-1}[1-(1-F(x))]. \tag{3}$$

In this paper we have derived L-moments and TL-moments of Fréchet distribution. We have also established several recurrence relations satisfied by the single and product moments of progressive Type-II right censored order statistics from Fréchet distribution defined in (1), which will enable one to compute the single and the product moments in a recursive manner.

2. L-moments and TL-moments

It may be mentioned that Hosking (1990) introduced the L-moments as a linear combination of probability weighted moments. Similar to ordinary moments, L-moments can also be used for parameter estimation, interval estimation and hypothesis testing. Hosking has shown that first four L-moments of a distribution measure, respectively, the average, dispersion, symmetry and tail weight (or peakedness) of the distribution. L-moments have turned out to be a popular tool in parametric estimation and distribution identification problems in different scientific areas such as hydrology in the estimation of flood frequency, etc. (see e.g. Hosking (1990), Stedinger et al. (1993), Hosking and Wallis (1997)). In comparison to the conventional moments, L-moments have lower sample variances and are more robust against outliers. For example, L_1 is the same as the population mean, is defined in terms of a conceptual sample of size $r = 1$, while L_2 is an alternative to the population standard deviation , is defined in terms of a conceptual sample of size $r = 2$. Similarly, the L-moments L_3 and L_4 are alternatives to the un-scaled measures of skewness and kurtosis μ_3 and μ_4, respectively (see Sillitto (1969)). Elamir and Seheult (2003) introduced an extension of L-moments and called them TL-moments (trimmed L-moments). TL-moments are more robust than L-moments and exist even if the distribution does not have a mean, for example the TL-moments exist for Cauchy distribution (see Abdul-Moniem and Selim (2009) and Shabri et al. (2011)). Abdul-Moniem (2007) derived L-moments and TL-moments for the exponential distribution. Similar work has been done by Shahzad and Asghar (2013) for Dagum distribution.

2.1. Methodology for L-moments for Fréchet Distribution

Let $X_1, X_2, \ldots, X_n$ be a random sample from Fréchet distribution defined in (1) and $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ denote the corresponding order statistics. Then the r^{th} L-moment defined by Hosking (1990) is as follows:

$$L_r = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k \binom{r-1}{k} E(X_{r-k:r}), \tag{4}$$

where

$$E(X_{i:r}) = \frac{r!}{(i-1)!(r-i)!}\int_0^\infty x[F(x)]^{i-1}[1-F(x)]^{r-i} f(x)dx.$$

For $r = 1, 2, 3, 4$ in (4), the first four L-moments can be derived as follows:

$$L_1 = E(X_{1:1}) \tag{5}$$

$$L_2 = \frac{1}{2}E(X_{2:2} - X_{1:2}) \tag{6}$$

$$L_3 = \frac{1}{3}E(X_{3:3} - 2X_{2:3} + X_{1:3}) \tag{7}$$

$$L_4 = \frac{1}{4}E(X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4}). \tag{8}$$

The L-moments of the Fréchet distribution are obtained by utilizing (5) - (8) as given below:

$$L_1 = \lambda\ \Gamma(1 - \frac{1}{\theta}) \tag{9}$$

$$L_2 = \lambda\ \Gamma(1 - \frac{1}{\theta})\Big[\frac{2}{(2)^{1-(1/\theta)}} - 1\Big] \tag{10}$$

$$L_3 = \lambda\ \Gamma(1 - \frac{1}{\theta})\Big[\frac{6}{(3)^{1-(1/\theta)}} - \frac{6}{(2)^{1-(1/\theta)}} + 1\Big]$$

$$L_4 = \lambda\, \Gamma(1 - \frac{1}{\theta})\Big[\frac{20}{(4)^{1-(1/\theta)}} - \frac{30}{(3)^{1-(1/\theta)}} + \frac{12}{(2)^{1-(1/\theta)}} - 1\Big].$$

In particular, L_1, L_2, L_3 and L_4 are population measures of the location, scale, skewness and kurtosis, respectively.

The L-skewness τ_3 and L-kurtosis τ_4 of Fréchet distribution will be given by (cf. Shabri et al., 2011)

$$\tau_3 = \frac{L_3}{L_2} = \frac{\frac{6}{(3)^{1-(1/\theta)}} - \frac{6}{(2)^{1-(1/\theta)}} + 1}{\frac{2}{(2)^{1-(1/\theta)}} - 1}$$

and

$$\tau_4 = \frac{L_4}{L_2} = \frac{\frac{20}{(4)^{1-(1/\theta)}} - \frac{30}{(3)^{1-(1/\theta)}} + \frac{12}{(2)^{1-(1/\theta)}} - 1}{\frac{2}{(2)^{1-(1/\theta)}} - 1}.$$

2.2. Sample L-moments and L-Moment estimators

The L-moments can be estimated from the sample order statistics as follows (see Asquith, 2007):

$$l_r = \frac{1}{r\binom{n}{r}} \sum_{i=1}^{n} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \binom{i-1}{r-1-k} \binom{n-i}{k} X_{i:n}. \tag{11}$$

From equations (9), (10) and (11) the L-moment estimators for parameters θ and λ of Fréchet distribution will be given as

$$l_1 = \frac{1}{n}\sum_{i=1}^{n} X_{i:n} = \bar{X} = \lambda\, \Gamma(1 - \frac{1}{\theta}),$$

$$l_2 = \frac{2}{n(n-1)} \sum_{i=1}^{n} (i-1) X_{i:n} - \bar{X} = \bar{X}\Big[\frac{2}{(2)^{1-(1/\theta)}} - 1\Big].$$

By solving, we get

$$\hat{\theta} = \frac{\log 2}{\log\left[\frac{l_2}{\bar{X}} + 1\right]}, \quad \bar{X} \neq 0,$$

and

$$\hat{\lambda} = \frac{\bar{X}}{\Gamma\Big(1 - \frac{1}{\hat{\theta}}\Big)}.$$

2.3. Methodology for TL-moments for Fréchet Distribution

Elamir and Seheult (2003) introduced some robust modification of Eq. (4) (and called it as TL-moments) in which $E(X_{r-k:r})$ is replaced by $E(X_{r+t_1-k:r+t_1+t_2})$ for each r where t_1 smallest and t_2 largest are trimmed from the conceptual sample. The following formula gives the r^{th} TL-moment (cf. Elamir and Seheult, 2003):

$$L_r^{(t_1,t_2)} = \frac{1}{r}\sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} E(X_{r+t_1-k:r+t_1+t_2}). \tag{12}$$

One can observe that TL-moments are more robust than L-moments and exist even if the distribution does not have a mean, for example the TL-moments exist for Cauchy distribution (cf. Abdul-Moniem and Selim, 2009). TL-moments reduce to L-moments when $t_1 = 0$ and $t_2 = 0$.

TL-moments equation (12) with $t_1 = 1$ and $t_2 = 1$ is defined as $L_r^{(1,1)} = L_r^{(1)}$:

$$L_r^{(1)} = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}E(X_{r+1-k:r+2}). \tag{13}$$

For $r = 1, 2, 3, 4$ in (13), the first four L-moments can be derived as follow:

$$L_1^{(1)} = E(X_{2:3}) \tag{14}$$

$$L_2^{(1)} = \frac{1}{2}E(X_{3:4} - X_{2:4}) \tag{15}$$

$$L_3^{(1)} = \frac{1}{3}E(X_{4:5} - 2X_{3:5} + X_{2:5}) \tag{16}$$

$$L_4^{(1)} = \frac{1}{4}E(X_{5:6} - 3X_{4:6} + 3X_{3:6} - X_{2:6}). \tag{17}$$

The TL-moments of the Fréchet distribution are obtained by utilizing (14) - (17) and are given below:

$$L_1^{(1)} = 6\lambda\ \Gamma(1-\frac{1}{\theta})\Big[\frac{1}{(2)^{1-(1/\theta)}} - \frac{1}{(3)^{1-(1/\theta)}}\Big] \tag{18}$$

$$L_2^{(1)} = 6\lambda\ \Gamma(1-\frac{1}{\theta})\Big[-\frac{2}{(4)^{1-(1/\theta)}} + \frac{3}{(3)^{1-(1/\theta)}} - \frac{1}{(2)^{1-(1/\theta)}}\Big] \tag{19}$$

$$L_3^{(1)} = \frac{10}{3}\lambda\ \Gamma(1-\frac{1}{\theta})\Big[-\frac{20}{(5)^{1-(1/\theta)}} + \frac{10}{(4)^{1-(1/\theta)}} - \frac{12}{(3)^{1-(1/\theta)}} + \frac{2}{(2)^{1-(1/\theta)}}\Big]$$

$$L_4^{(1)} = \frac{30}{4}\lambda\ \Gamma(1-\frac{1}{\theta})\Big[-\frac{14}{(6)^{1-(1/\theta)}} + \frac{35}{(5)^{1-(1/\theta)}} - \frac{30}{(4)^{1-(1/\theta)}} + \frac{10}{(3)^{1-(1/\theta)}} - \frac{1}{(2)^{1-(1/\theta)}}\Big].$$

The TL-skewness $\tau_3^{(1)}$ and TL-kurtosis $\tau_4^{(1)}$ of the Fréchet distribution will be

$$\tau_3^{(1)} = \frac{L_3^{(1)}}{L_2^{(1)}} = \frac{5}{9}\frac{\Big[-\frac{10}{(5)^{1-(1/\theta)}} + \frac{20}{(4)^{1-(1/\theta)}} - \frac{12}{(3)^{1-(1/\theta)}} + \frac{2}{(2)^{1-(1/\theta)}}\Big]}{\Big[-\frac{2}{(4)^{1-(1/\theta)}} + \frac{3}{(3)^{1-(1/\theta)}} - \frac{1}{(2)^{1-(1/\theta)}}\Big]},$$

$$\tau_4^{(1)} = \frac{L_4^{(1)}}{L_2^{(1)}} = \frac{5}{4}\frac{\Big[-\frac{14}{(6)^{1-(1/\theta)}} + \frac{35}{(5)^{1-(1/\theta)}} - \frac{30}{(4)^{1-(1/\theta)}} + \frac{10}{(3)^{1-(1/\theta)}} - \frac{1}{(2)^{1-(1/\theta)}}\Big]}{\Big[-\frac{2}{(4)^{1-(1/\theta)}} + \frac{3}{(3)^{1-(1/\theta)}} - \frac{1}{(2)^{1-(1/\theta)}}\Big]}.$$

2.4. Sample TL-moments and TL-moment estimators

The TL-moments can be estimated from a sample as linear combination of order statistics. Elamir and Seheult (2003) presented the following estimator for TL-moments:

$$l_r^{(t_1,t_2)} = \frac{1}{r\binom{n}{r+t_1+t_2}}\sum_{i=t_1+1}^{n-t_2}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\binom{i-1}{r+t_1-k-1}\binom{n-i}{k+t_2}X_{i:n}. \tag{20}$$

If we take $t_1 = t_2 = t$ in Eq. (20), we get $l_r^{(t,t)} \equiv l_r^{(t)}$:

$$l_r^{(t)} = \frac{1}{r\binom{n}{r+2t}} \sum_{i=t+1}^{n-t} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \binom{i-1}{r+t-k-1} \binom{n-i}{k+t} X_{i:n}. \tag{21}$$

From Eqs. (18), (19) and (21), for $t = 1$ and $r = 1, 2$, the TL-moment estimators for parameters λ and θ can be obtained and are given by $l_r^{(1,1)} \equiv l_r^{(1)},\ r = 1, 2$:

$$l_1^{(1)} = \frac{6}{n(n-1)(n-2)} \sum_{i=2}^{n-1} (i-1)(n-i) X_{i:n} = 6\lambda\, \Gamma(1 - \frac{1}{\theta}) \Big[\frac{1}{(2)^{1-(1/\theta)}} - \frac{1}{(3)^{1-(1/\theta)}}\Big] \tag{22}$$

and

$$\begin{aligned} l_2^{(1)} &= \frac{12}{n(n-1)(n-2)(n-3)} \Big(\sum_{i=3}^{n-1} \binom{i-1}{2} \binom{n-i}{1} X_{i:n} - \sum_{i=2}^{n-2} \binom{i-1}{1} \binom{n-i}{2} X_{i:n} \Big) \\ &= 6\lambda\, \Gamma(1 - \frac{1}{\theta}) \Big[- \frac{2}{(4)^{1-(1/\theta)}} + \frac{3}{(3)^{1-(1/\theta)}} - \frac{1}{(2)^{1-(1/\theta)}} \Big]. \end{aligned} \tag{23}$$

By solving (22) and (23) the TL-moment estimators for the parameters λ and θ of Fréchet distribution can be obtained.

3. Progressive Type-II Right Censored Order Statistics

Progressive censoring sampling scheme is very useful in reliability and life time studies. Its allowance for removal of live units from the test at various stages during the experiment will potentially save the experimenter cost while still allowing for the observation of some extreme data. Numerous authors have discussed inference problems for a wide range of distributions under this sampling scheme. See, for example, Aggarwala and Balakrishnan (1996, 1998), Balakrishnan and Aggarwala (2000), Cohen (1963, 1976, 1991), Mann (1969, 1971), Balakrishnan and Sandhu (1995), Balakrishnan, Gupta and Panchapakesan (1995) and Saran and Pushkarna (2001).

Suppose n independent items are put on test with continuous identically distributed failure times $X_1, X_2, \ldots, X_n$. Suppose further that a censoring scheme $(R_1, R_2, \ldots, R_m)$ is fixed such that immediately following the first failure, R_1 surviving items are removed from the experiment at random; immediately following the first failure after that point; i.e., after second observed failure, R_2 surviving items are removed from the experiment at random; this process continues until, at the m^{th} observed failure, R_m items are removed from the test at random. Thus, in this type of sampling, we observe in all m failures and $\sum_{i=1}^m R_i$ items are progressively censored so that $n = m + \sum_{i=1}^m R_i$.

Let $X_{1:m:n}^{(R_1,R_2,\ldots,R_m)} < X_{2:m:n}^{(R_1,R_2,\ldots,R_m)} < \ldots < X_{m:m:n}^{(R_1,R_2,\ldots,R_m)}$ be the m ordered observed failure times in a sample of size n from the Fréchet distribution defined in (1) under progressive Type-II right censoring scheme $(R_1, R_2, \ldots, R_m)$. Then the joint pdf of $X_{1:m:n}^{(R_1,R_2,\ldots,R_m)}, X_{2:m:n}^{(R_1,R_2,\ldots,R_m)}, \ldots, X_{m:m:n}^{(R_1,R_2,\ldots,R_m)}$ is given by (Balakrishnan and Sandhu, 1995)

$$f_{1,2,\ldots,m:m:n}(x_1, x_2, \ldots, x_m) = A(n, m-1) \prod_{i=1}^{m} f(x_i)[1 - F(x_i)]^{R_i}, \quad 0 \le x_1 < x_2 < \cdots < x_m < \infty, \tag{24}$$

where $A(n, m-1) = \prod_{i=0}^{m-1}(n - S_i - i)$ with $S_0 = 0$ and $S_i = R_1 + R_2 + \cdots + R_i = \sum_{k=1}^{i} R_k$ for $1 \le i \le m-1$. Here all the factors in $A(n, m-1)$ are positive integers. Similarly, for convenience in notation, let us define for $q = 0, 1, \ldots, p-1$,

$$A(p, q) = p(p - R_1 - 1)(p - R_1 - R_2 - 2) \ldots (p - R_1 - R_2 - \cdots - R_q - q)$$

with all the factors being positive integers. Thus, for $1 \le r \le m$, we have

$$\mu_{r:m:n}^{(R_1,R_2,\ldots,R_m)^{(k)}} = E\Big[X_{r:m:n}^{(R_1,R_2,\ldots,R_m)}\Big]^k = A(n,m-1) \int_{0<x_1<x_2<\ldots<x_m<\infty} \int \cdots \int x_r^k \prod_{i=1}^{m} f(x_i)[1-F(x_i)]^{R_i} dx_i. \quad (25)$$

In Subsections 3.1 and 3.2, utilizing the characterizing differential equation (3), we will derive recurrence relations for the single and the product moments of progressive Type-II right censored order statistics from Fréchet distribution.

3.1. Single Moments

Theorem 3.1. *For $2 \le m \le n-1$ and $k > \theta > 0$,*

$$\begin{aligned}(k-\theta)\frac{\lambda^\theta}{\theta}\mu_{1:m:n}^{(R_1,R_2,\ldots,R_m)^{(k)}} &= a_1\mu_{2:m-1:n-1}^{(R_1+R_2,R_3,\ldots,R_m)^{(k-\theta)}} + a_2R_1\mu_{1:m:n-1}^{(R_1-1,R_2,\ldots,R_m)^{(k-\theta)}} \\ &\quad -a_3\mu_{2:m-1:n}^{(R_1+R_2+1,R_3,\ldots,R_m)^{(k-\theta)}} - (R_1+1)\mu_{1:m:n}^{(R_1,R_2,\ldots,R_m)^{(k-\theta)}},\end{aligned} \quad (26)$$

where $a_1 = \frac{c}{(n-1)\prod_{i=2}^{m-1}[n-S_i-i]}$, $a_2 = \frac{c}{(n-1)\prod_{i=1}^{m-1}[n-S_i-i]}$, $a_3 = (n-S_1-1)$

and $c = n(n-S_1-1)(n-S_2-2)\ldots(n-S_{m-1}-(m-1))$.

And, for $m = 1, n = 1, 2, \ldots$ and $k > \theta > 0$,

$$(k-\theta)\frac{\lambda^\theta}{\theta}\mu_{1:1:n}^{(n-1)^{(k)}} = (n-1)\mu_{1:1:n-1}^{(n-2)^{(k-\theta)}} - n\mu_{1:1:n}^{(n-1)^{(k-\theta)}}. \quad (27)$$

Proof. Relations in (26) and (27) may be proved by following exactly the same steps as those in proving Theorem 3.2, which is presented below. □

Theorem 3.2. *For $2 \le r \le m-1$, $m \le n-1$ and $k > \theta > 0$,*

$$\begin{aligned}(k-\theta)\frac{\lambda^\theta}{\theta}\mu_{r:m:n}^{(R_1,R_2,\ldots,R_m)^{(k)}} &= b_1\mu_{r:m-1:n-1}^{(R_1,R_2,\ldots,R_r+R_{r+1},\ldots,R_m)^{(k-\theta)}} - b_2\mu_{r-1:m-1:n-1}^{(R_1,\ldots,R_{r-1}+R_r,\ldots,R_m)^{(k-\theta)}} \\ &\quad +b_3R_r\mu_{r:m:n-1}^{(R_1,R_2,\ldots,R_r-1\ldots,R_m)^{(k-\theta)}} - b_4\mu_{r:m-1:n}^{(R_1,\ldots,R_r+R_{r+1}+1,\ldots,R_m)^{(k-\theta)}} \\ &\quad +b_5\mu_{r-1:m-1:n}^{(R_1,\ldots,R_{r-1}+R_r+1,\ldots,R_m)^{(k-\theta)}} - (R_r+1)\mu_{r:m:n}^{(R_1,\ldots,\ldots,R_m)^{(k-\theta)}}\end{aligned} \quad (28)$$

where $b_1 = \frac{c}{\prod_{i=0}^{r-1}[(n-1)-S_i-i]\prod_{t=r+1}^{m-1}[n-S_t-t]}$, $b_2 = \frac{c}{\prod_{i=0}^{r-2}[(n-1)-S_i-i]\prod_{t=r}^{m-1}[n-S_t-t]}$,

$b_3 = \frac{c}{\prod_{i=0}^{r-1}[(n-1)-S_i-i]\prod_{t=r}^{m}[n-S_t-t]}$, $b_4 = (n-S_r-r)$ *and* $b_5 = (n-S_{r-1}-(r-1))$.

Proof. From equation (25), we have

$$\frac{\lambda^\theta}{\theta}\mu_{r:m:n}^{(R_1,R_2,\ldots,R_m)^{(k)}} = A(n,m-1)\frac{\lambda^\theta}{\theta} \int_{0<x_1<x_2<\ldots<x_m<\infty} \int \cdots \int x_r^k \prod_{t=1}^{m} f(x_t)[1-F(x_t)]^{R_t} dx_t$$

$$= A(n,m-1) \int_{0<x_1<x_2<\cdots<x_{r-1}<x_{r+1}<\ldots<x_m<\infty} \int \cdots \int I(x_{r-1},x_{r+1}) \prod_{t=1,t\neq r}^{m} f(x_t)[1-F(x_t)]^{R_t} dx_t, \quad (29)$$

where

$$I(x_{r-1},x_{r+1}) = \frac{\lambda^\theta}{\theta}\int_{x_{r-1}}^{x_{r+1}} x_r^k f(x_r)[1-F(x_r)]^{R_r} dx_r.$$

Using equation (3), we get

$$\begin{aligned} I(x_{r-1}, x_{r+1}) &= \int_{x_{r-1}}^{x_{r+1}} x_r^{k-\theta-1}[1-F(x_r)]^{R_r} dx_r - \int_{x_{r-1}}^{x_{r+1}} x_r^{k-\theta-1}[1-F(x_r)]^{R_r+1} dx_r \\ &= E_0(x_{r-1}, x_{r+1}) - E_1(x_{r-1}, x_{r+1}), \end{aligned} \tag{30}$$

where

$$E_a(x_{r-1}, x_{r+1}) = \int_{x_{r-1}}^{x_{r+1}} x_r^{k-\theta-1}[1-F(x_r)]^{R_r+a} dx_r, \quad a = 0, 1.$$

Integrating by parts yields,

$$\begin{aligned} E_a(x_{r-1}, x_{r+1}) = \frac{1}{k-\theta}\Big[& x_{r+1}^{k-\theta}[1-F(x_{r+1})]^{R_r+a} - x_{r-1}^{k-\theta}[1-F(x_{r-1})]^{R_r+a} \\ & +(R_r+a)\int_{x_{r-1}}^{x_{r+1}} x_{r-1}^{k-\theta}[1-F(x_{r-1})]^{R_r+a-1} f(x_r) dx_r\Big]. \end{aligned} \tag{31}$$

Upon substituting for $E_a(x_{r-1}, x_{r+1})$ for $a = 0, 1$ from (31) into (30) and then substituting the resultant expression for $I(x_{r-1}, x_{r+1})$ in equation (29) and simplifying, it leads to (28). □

Proceeding on similar lines, one can derive the following recurrence relation:

Theorem 3.3. *For $2 \le m \le n-1$ and $k > \theta > 0$, $m > 1$,*

$$\begin{aligned} (k-\theta)\frac{\lambda^\theta}{\theta}\mu_{m:m:n}^{(R_1,R_2,\ldots,R_m)^{(k)}} = & -c_1\mu_{m-1:m-1:n-1}^{(R_1,R_2,\ldots,R_{m-1}+R_m)^{(k-\theta)}} + c_2 R_m \mu_{m:m:n-1}^{(R_1,\ldots,R_m-1)^{(k-\theta)}} \\ & + c_3\mu_{m-1:m-1:n}^{(R_1,\ldots,R_{m-1}+R_m+1)^{(k-\theta)}} + (R_m+1)\mu_{m:m:n}^{(R_1,\ldots,R_m)^{(k-\theta)}}, \end{aligned} \tag{32}$$

where
$c_1 = \frac{c}{\prod_{i=0}^{m-2}[(n-1)-S_i-i][n-S_m-m]}$, $c_2 = \frac{c}{\prod_{i=0}^{m-1}[(n-1)-S_i-i][n-S_m]}$
and $c_3 = [n - S_{m-1} - (m-1)]$.

3.2. Product Moments

To obtain the recurrence relations for the product moments of progressive Type-II right censored order statistics from Fréchet distribution, we have from equation (24), for $1 \le r < s \le m$,

$$\begin{aligned} \mu_{r,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(i,j)}} &= E\Big[\Big\{X_{r:m:n}^{(R_1,R_2,\ldots,R_m)}\Big\}^i \Big\{X_{s:m:n}^{(R_1,R_2,\ldots,R_m)}\Big\}^j\Big] \\ &= A(n, m-1) \int\int_{0<x_1<x_2<\ldots<x_m<\infty}\cdots\int x_r^i x_s^j \prod_{t=1}^{m} f(x_t)[1-F(x_t)]^{R_t} dx_t. \end{aligned} \tag{33}$$

Theorem 3.4. *For $m \le n-1$,*

$$\begin{aligned} (j-\theta)\frac{\lambda^\theta}{\theta}\mu_{1,2:m:n}^{(R_1,R_2,\ldots,R_m)^{(j,k)}} = & \, d_1\mu_{2:m-1:n-1}^{(R_1+R_2,R_3,\ldots,R_m)^{(j-\theta,k)}} + d_2\mu_{1,2:m:n-1}^{(R_1-1,R_2,\ldots,R_m)^{(j-\theta,k)}} \\ & - d_3\mu_{2:m-1:n}^{(R_1+R_2+1,R_3,\ldots,R_m)^{(j-\theta,k)}} - (R_1+1)\mu_{1,2:m-1:n}^{(R_1,\ldots,R_m)^{(j-\theta,k)}}, \end{aligned} \tag{34}$$

where $d_1 = \frac{c}{(n-1)\prod_{i=2}^{m-1}[n-S_i-i]}$, $d_2 = \frac{c}{(n-1)\prod_{i=1}^{m-1}[n-S_i-i]}$ and $d_3 = (n - S_1 - 1)$.

Proof. Relation (34) may be proved by following exactly the same steps as those in proving Theorem 3.5, which is presented next. □

Theorem 3.5. *For $1 \le r < s < m$ and $m \le n-1$,*

$$
\begin{aligned}
(j-\theta)\frac{\lambda^\theta}{\theta}\mu_{r,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(j,k)}} &= e_1\mu_{r,s:m-1:n-1}^{(R_1,R_2,\ldots,R_r+R_{r+1},\ldots,R_m)^{(j-\theta,k)}} - e_2\mu_{r-1,s:m-1:n-1}^{(R_1,R_2,\ldots,R_{r-1}+R_r,\ldots,R_m)^{(j-\theta,k)}} \\
&+e_3R_r\mu_{r,s:m:n-1}^{(R_1,R_2,\ldots,R_{r-1},R_r-1,\ldots,R_m)^{(j-\theta,k)}} - e_4\mu_{r,s:m-1:n}^{(R_1,R_2,\ldots,R_r+R_{r+1}+1,\ldots,R_m)^{(j-\theta,k)}} \\
&+e_5\mu_{r-1,s:m-1:n}^{(R_1,R_2,\ldots,R_{r-1}+R_r+1,\ldots,R_m)^{(j-\theta,k)}} - (R_r+1)\mu_{r,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(j-\theta,k)}}
\end{aligned}
\tag{35}
$$

where $e_1 = \frac{c}{\prod_{i=0}^{r-1}[(n-1)-S_i-i]\prod_{t=r+1}^{m-1}[n-S_t-t]}$, $e_2 = \frac{c}{\prod_{i=0}^{r-2}[(n-1)-S_i-i]\prod_{t=r}^{m-1}[n-S_t-t]}$,

$e_3 = \frac{c}{\prod_{i=0}^{r-1}[(n-1)-S_i-i]\prod_{t=r}^{m}[n-S_t-t]}$, $e_4 = (n - S_r - r)$ *and* $e_5 = (n - S_{r-1} - (r-1))$.

Proof. From equation (33), we have

$$
\frac{\lambda^\theta}{\theta}\mu_{r,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(j,k)}} = A(n,m-1)\frac{\lambda^\theta}{\theta}\underset{0<x_1<x_2<\ldots<x_m<\infty}{\int\int\cdots\int} x_r^j x_s^k \prod_{t=1,t\neq r}^{m} f(x_t)[1-F(x_t)]^{R_t}dx_t
$$

$$
= A(n,m-1)\underset{0<x_1<x_2<\cdots<x_{r-1}<x_{r+1}<\ldots<x_m<\infty}{\int\int\cdots\int} x_s^k I(x_{r-1},x_{r+1})\prod_{t=1,t\neq r}^{m} f(x_t)[1-F(x_t)]^{R_t}dx_t, \tag{36}
$$

where

$$
I(x_{r-1},x_{r+1}) = \frac{\lambda^\theta}{\theta}\int_{x_{r-1}}^{x_{r+1}} x_r^j[1-F(x_r)]^{R_r}f(x_r)dx_r.
$$

Using equation (3), we get

$$
I(x_{r-1},x_{r+1}) = \int_{x_{r-1}}^{x_{r+1}} x_r^{j-\theta-1}[1-F(x_r)]^{R_r}dx_r - \int_{x_{r-1}}^{x_{r+1}} x_r^{j-\theta-1}[1-F(x_r)]^{R_r+1}dx_r. \tag{37}
$$

Consider,

$$
E_a(x_{r-1},x_{r+1}) = \int_{x_{r-1}}^{x_{r+1}} x_r^{j-\theta-1}[1-F(x_r)]^{R_r+a}dx_r, \quad a = 0,1.
$$

Integrating by parts yields,

$$
\begin{aligned}
E_a(x_{r-1},x_{r+1}) = \frac{1}{j-\theta}\Big[& x_{r+1}^{j-\theta}[1-F(x_{r+1})]^{R_r+a} - x_{r-1}^{j-\theta}[1-F(x_{r-1})]^{R_r+a} \\
&+(R_r+a)\int_{x_{r-1}}^{x_{r+1}} x_{r-1}^{j-\theta}[1-F(x_{r-1})]^{R_r+a-1}f(x_r)dx_r\Big]
\end{aligned}
\tag{38}
$$

Upon substituting for $E_a(x_{r-1},x_{r+1})$ for $a = 0,1$ from (38) into (37) and then substituting the resultant expression for $I(x_{r-1},x_{r+1})$ in equation (36) and simplifying, it leads to (35). □

Proceeding on similar lines, one can derive the recurrence relations given in the following two theorems.

Theorem 3.6. *For $1 \leq r < s < m$ and $m \leq n-1$,*

$$\begin{aligned}(k-\theta)\frac{\lambda^\theta}{\theta}\mu_{r,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(j,k)}} &= f_1\mu_{r,s+1:m-1:n-1}^{(R_1,R_2,\ldots,R_s+R_{s+1},\ldots,R_m)^{(j,k-\theta)}} - f_2\mu_{r,s-1:m-1:n-1}^{(R_1,R_2,\ldots,R_{r-1}+R_r,\ldots,R_m)^{(j,k-\theta)}}\\ &+f_3R_s\mu_{r,s:m:n-1}^{(R_1,R_2,\ldots,R_{s-1},R_s-1,\ldots,R_m)^{(j,k-\theta)}} - f_4\mu_{r,s+1:m-1:n}^{(R_1,R_2,\ldots,R_s+R_{s+1}+1,\ldots,R_m)^{(j,k-\theta)}}\\ &+f_5\mu_{r,s-1:m-1:n}^{(R_1,R_2,\ldots,R_{s-1}+R_s+1,\ldots,R_m)^{(j,k-\theta)}} - (R_s+1)\mu_{r,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(j,k-\theta)}},\end{aligned}$$

where
$f_1 = \frac{c}{\prod_{i=0}^{s-1}[(n-1)-S_i-i]\prod_{t=s+1}^{m-1}[n-S_t-t]}$, $f_2 = \frac{c}{\prod_{i=0}^{s-2}[(n-1)-S_i-i]\prod_{t=s}^{m-1}[n-S_t-t]}$,
$f_3 = \frac{c}{\prod_{i=0}^{s-1}[(n-1)-S_i-i]\prod_{t=s}^{m}[n-S_t-t]}$, $f_4 = (n-S_s-s)$ *and* $f_5 = (n-S_{s-1}-(s-1))$.

Theorem 3.7. *For $m \leq n-1$,*

$$\begin{aligned}(k-\theta)\frac{\lambda^\theta}{\theta}\mu_{r,m:m:n}^{(R_1,R_2,\ldots,R_m)^{(j,k)}} &= -g_1\mu_{r,m-1:m-1:n-1}^{(R_1,R_2,\ldots,R_{m-1}+R_m)^{(j,k-\theta)}} + g_2\mu_{r,m:m:n-1}^{(R_1,R_2,\ldots,R_{m-1},R_m-1)^{(j,k-\theta)}}\\ &+g_3\mu_{r,m-1:m-1:n}^{(R_1,R_2,\ldots,R_{m-1}+R_m+1)^{(j,k-\theta)}} - (R_m+1)\mu_{r,m:m:n}^{(R_1,R_2,\ldots,R_m)^{(j,k-\theta)}},\end{aligned}$$

where
$g_1 = \frac{c}{\prod_{i=0}^{m-2}[(n-1)-S_i-i][n-S_m-m]}$, $g_2 = \frac{c}{\prod_{i=0}^{m-1}[(n-1)-S_i-i][n-S_m]}$, $g_3 = [n-S_{m-1}-(m-1)]$.

4. Recursive Algorithm

Utilizing the knowledge of recurrence relations obtained in Sections 3.1 and 3.2, one can calculate single and product moments of progressive Type-II right censored order statistics from Fréchet distribution, for the case when the domain of θ is the set of natural numbers. We are giving the algorithm for the case $\theta = 1$ as follows:

Eq. (27) gives the value of $\mu_{1:1:n}^{(n-1)^{(k)}} = \mu_{1:n}^{(k)}$ $\forall\ k, n$ by the knowledge of $\mu_{1:1:n}^{(n-1)^{(1)}}$ $\forall\ n$. Now, for $m = 2$, we require $\mu_{1:2:n}^{(n-2)^{(1)}}\forall\ n$ to evaluate $\mu_{1:2:n}^{(R_1,R_2)^{(k)}}$ for all R_1, R_2, k and n from eq. (26) in a simple recursive manner. Similarly, eq. (32) can be used to determine $\mu_{2:2:n}^{(R_1,R_2)^{(k)}}$ for $m = 2$, we require the knowledge of $\mu_{2:2:n}^{(R_1,R_2)^{(1)}}$ $\forall\ R_1, R_2$ and $n \geq 2$. Eq. (28) can now be used to determine $\mu_{1:3:n}^{(R_1,R_2,R_3)^{(k)}}$ by the knowledge of $\mu_{1:3:n}^{(R_1,R_2,R_3)^{(1)}}$ $\forall\ R_1, R_2, R_3$ and $n \geq 3$. Similarly, in the same manner we can calculate $\mu_{2:3:n}^{(R_1,R_2)^{(k)}}$ by the required knowledge of $\mu_{2:3:n}^{(R_1,R_2)^{(1)}}$ in an iterative way. A similar algorithm can be written for higher values of θ.

By utilizing the results of Section 3.2, one can obtain in a recursive manner, all the product moments for all sample sizes and all censoring schemes from the Fréchet distribution.

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