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On asymptotic behavior of some record functions

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Abstract. The range and midrange are widely used, particularly in statistical quality control as an estimator of the dispersion tendency. In fact, the range itself is a very simple measure of dispersion, gives a quick and easy to estimate indication about the spread of data. The extremal quotient is used in climatic study, industrial quality control, life testing, small-area variation analysis and the classical heterogeneity of variance situation. In this paper, sufficient conditions for the weak convergence of the record range, record midrange, record extremal quotient and record extremal product are obtained. The classes of non-degenerate limit distribution functions of these statistics are characterized. Illustrative examples are provided, which lend further support to our theoretical results.

1. Introduction

For several distributions, linear functions of order statistics provide good estimators of location and scale parameters. The range may be considered as an important linear function of order statistics. It is widely used, particularly in statistical quality control as an estimator of the dispersion tendency. In fact, the range itself is a very simple measure of dispersion. So many short-cut tests have been based on this statistic. Also, the functions such as midrange, product and quotient are important functions of order statistics. Consider a sequence of independent and identically distributed random variables (rv's) $\{X_n : n \geq 1\}$ with distribution function (df) F. Let $M_n = \max\{X_1, \ldots, X_n\}$ and $L_n = \min\{X_1, \ldots, X_n\}$. The range is the length of the smallest interval, which contains all the data. It is defined by $r_n = M_n - L_n$ and provides an indication of statistical dispersion. The limit laws for the range were fully characterized by de Haan (1974). The midrange point, i.e., the point halfway between the two extremes L_n and M_n, namely defined by $v_n = \frac{1}{2}(M_n + L_n)$, is an indicator of the central tendency of the data. The limit laws for the midrange can be obtained from the work of de Haan (1974). Both range and midrange are not particularly robust for small samples. The extremal quotient is defined by $q_n = M_n/L_n$ (see Galambos and Simoneli, 2004). This statistic is obviously not affected by a change of scale. Therefore, its use may be of interest in cases where the scale plays no role, e.g., in climatic study (see Canard, 1946). The extremal quotient has been used in several fields, most notably in industrial quality control, life testing, small-area variation analysis and the classical heterogeneity of variance situation. For example, a quality engineer might use this statistic as a basic measurement in controlling the roundness of a circular component in a production process. Also, Wong and Wong (1979) used the extremal quotient to test the hypothesis that the population of a sample has an exponential df. Wong and Wong (1982) used this statistic for testing the shape parameter of the Weibull df. The limit laws for the extremal quotient were fully characterized by Barakat (1998). The geometric mean is defined by $\rho_n = \sqrt{|M_n L_n|}$. We henceforth call it the geometric range, while we call its square $p_n = \rho_n^2$ the sample

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extremal product. The latter statistic is often found useful in ranking. When the sample size is large enough the geometric range may be defined by

$$\rho_n = \begin{cases} \sqrt{M_n L_n}, & \text{if } 0 \leq x_0 < x^0 \leq \infty, \text{ or } -\infty \leq x_0 < x^0 \leq 0, \\ \sqrt{-M_n L_n}, & \text{if } -\infty \leq x_0 < 0 < x^0 \leq \infty, \end{cases}$$

where $-\infty \leq \inf\{x : F(x) > 0\} = x_0 < x^0 = \sup\{x : F(x) < 1\} \leq \infty$. The limit laws for the geometric range were fully characterized by Barakat and Nigm (1996).

Record values are found in many situations of daily life as well as in many statistical applications. Often we are interested in observing new records and in recording them, e.g. Olympic records or world records in sports. Record values have been introduced by Chandler (1952) in order to model data of extreme weather conditions. It may also be helpful as a model for successively largest insurance claims in non-life insurance, for highest water-level or highest temperatures. Record values are also used in reliability theory. To be precise, record values are defined by means of record times at which successively largest values appear.

The upper record values (or simply a record) can be defined as an observation X_j, such that $X_j > \max(X_1, \ldots, X_{j-1})$. By convention X_1 is a record value. The indices at which record values occur are given by the rv's $T_n = \min\{j : j > T_{n-1},\ X_j > X_{j-1},\ n > 1\}$ and $T_1 = 1$. Thus, the record value sequence $\{R_n\}$ is then defined by $\mathcal{R}_n = X_{T_n},\ n \geq 1$. The explicit form of the df of $\mathcal{R}_n$ is given by

$$P(\mathcal{R}_n \leq x) = \begin{cases} 1 - \Gamma_n(Q(x)), & \text{if } n > 1, \\ F(x), & \text{if } n = 1, \end{cases}$$

where $Q(x) = -\log(1 - F(x))$ is the hazard function of the df F (see Arnold et al., 1998). Resnick (1973) showed that the possible limiting record value distributions of the suitably normalized record $\mathcal{R}_n^* = c_n^{-1}(\mathcal{R}_n - d_n),\ c_n > 0,\ d_n \in \Re$, are

$$H_{i,\beta}(x) = \mathcal{N}(-\log(-\log G_{i,\beta}(x))) = \mathcal{N}(-\log(V_{i,\beta}(x))),\ i = 1, 2, 3,$$

where $\mathcal{N}(\cdot)$ is the standard normal distribution and $G_{i,\beta}(x) = \exp(-(V_{i,\beta}(x))$ is the well-known limit distribution of the maximum order statistics, in which the functions $V_{i,\beta},\ i = 1, 2, 3$, are defined as

$$\left.\begin{array}{l} \text{Type I: } V_{1,\ \beta}(x) = \begin{cases} x^{-\beta}, & x > 0, \\ \infty, & x \leq 0, \end{cases} \\[2ex] \text{Type II: } V_{2,\ \beta}(x) = \begin{cases} (-x)^{\beta}, & x \leq 0, \\ 0, & x > 0, \end{cases} \\[2ex] \text{Type III: } V_{3,\ 0}(x) = e^{-x},\ \forall x. \end{array}\right\}.$$

In this case we say that F is in the domain of record attraction of $H_{i,\beta}$ and write $F \in \mathcal{D}_{\mathcal{R}}(H_{i,\beta})$ (i.e., $F \in \mathcal{D}_{\mathcal{R}}(H_{i,\beta})$ means that there are suitable normalizing constants $c_n > 0$ and d_n for which $P(\mathcal{R}_n^* \leq x) = P(c_n^{-1}(\mathcal{R}_n - d_n) \leq x)$ weakly converges to $H_{i,\beta}$).

Remark 1.1. Although, in order that F be attracted to $H_{i,\beta}$, i.e., $F \in \mathcal{D}_{\mathcal{R}}(H_{i,\beta})$, the continuity of the df F in a left neighborhood of the right extremity x^0 is enough (c.f. Arnold et al., 1998), but the continuity of the underlying df is needed to avoid ties in the records. Therefore, we assume that the underlying df is continuous.

Throughout this paper, we will assume that $F \in \mathcal{D}_{\mathcal{R}}(H_{i,\beta})$. The following theorem due to Resnick (1973) (see Arnold et al., 1998) is a basic tool for our study.

Theorem 1.2 (Duality Theorem). *If an associated df F_a is defined by $F_a = 1 - \exp(-\sqrt{Q(x)})$ and $\Psi_F(n) = \inf\{y : F(y) > 1 - e^{-n}\} = F^{-1}(1 - e^{-n}) \underset{n}{\rightarrow} x^0$, then the following limit implications hold:*

(i) $F \in \mathcal{D}_{\mathcal{R}}(H_{1,\beta})$ *if and only if* $F_a \in \mathcal{D}(G_{1,\frac{\beta}{2}})$. *In this case* $F^{-1}(1) = x^0 = \infty$ *and we may use as normalizing constants* $c_n = \Psi_F(n)$ *and* $d_n = 0$;

(ii) $F \in \mathcal{D}_{\mathcal{R}}(H_{2,\beta})$ *if and only if* $F_a \in \mathcal{D}(G_{2,\frac{\beta}{2}})$. *In this case* $F^{-1}(1) = x^0$ *is necessarily finite and we may use as normalizing constants* $c_n = x^0 - \Psi_F(n)$ *and* $d_n = x^0$;

(iii) $F \in \mathcal{D}_{\mathcal{R}}(H_{3,0})$ *if and only if* $F_a \in \mathcal{D}(G_{3,0})$ *and in this case we may use as normalizing constants* $c_n = \Psi_F(n + \sqrt{n}) - \Psi_F(n)$ *and* $d_n = \Psi_F(n)$.

Our aim in this paper is to study the asymptotic behavior of the record range $r_n = \mathcal{R}_n - \mathcal{R}_1 = \mathcal{R}_n - X_1$, the midrange $v_n = \frac{\mathcal{R}_n + \mathcal{R}_1}{2} = \frac{\mathcal{R}_n + X_1}{2}$, the record extremal quotient $q_n = \frac{\mathcal{R}_n}{\mathcal{R}_1} = \frac{\mathcal{R}_n}{X_1}$ and the record extremal product $p_n = \mathcal{R}_n \mathcal{R}_1 = \mathcal{R}_n X_1$. Namely, we derive the possible non-trivial and trivial limit df's of all suitably normalized preceding statistics, the trivial limit is defined when the convergence takes place, such that one of the statistics R_n and $R_1 = X_1$ outweighs the other (see de Haan, 1974). This problem is recently tackled by Barakat et al. (2013) and Barakat et al. (2015), for m-generalized order statistics when $m > -1$, i.e., the record values case was excluded from this study. In this paper we will fill this gap.

2. Weak convergence of the record functions

The following theorem fully characterizes the possible limit non-degenerate df's (trivial and non-trivial) of the statistics r_n, v_n, q_n and p_n.

Theorem 2.1. *Let* $C_{n:t} > 0$ *and* $D_{n:t} \in \Re$, $t = r, v, q, p$, *be suitable normalizing constants. Furthermore, let* $r_n^* = C_{n:r}^{-1}(r_n - D_{n:r})$, $v_n^* = C_{n:v}^{-1}(v_n - D_{n:v})$, $q_n^* = C_{n:q}^{-1}(q_n - D_{n:q})$ *and* $p_n^* = C_{n:p}^{-1}(p_n - D_{n:p})$.
(i) If $F \in \mathcal{D}_{\mathcal{R}}(H_{1,\beta})$, *then*

$$P(r_n^* \leq r) \xrightarrow[n]{w} H_{1,\beta}(r) \text{ (trivial limit law, since } \mathcal{R}_n \text{ outweighed } \mathcal{R}_1 = X_1),$$

$$P(v_n^* \leq v) \xrightarrow[n]{w} H_{1,\beta}(v) \text{ (trivial limit law, since } \mathcal{R}_n \text{ outweighed } \mathcal{R}_1 = X_1),$$

$$P(q_n^* \leq q) \xrightarrow[n]{w} \begin{cases} F(0) + \int_0^\infty \mathcal{N}(\beta \log qx))dF(x), & \text{if } q \geq 0, \\ \int_{-\infty}^0 \mathcal{N}(\beta \log qx))dF(x), & \text{if } q < 0 \end{cases}$$

and

$$P(p_n^* \leq p) \xrightarrow[n]{w} \begin{cases} F(0) + \int_0^\infty \mathcal{N}(\beta \log \frac{p}{x}))dF(x), & \text{if } p \geq 0, \\ \int_{-\infty}^0 \mathcal{N}(\beta \log \frac{p}{x}))dF(x), & \text{if } p < 0, \end{cases}$$

where "$*$"denotes the convolution operator and " $\xrightarrow[n]{w}$ *"means converges weakly as* $n \to \infty$. *The normalizing constants can be chosen such as* $2C_{n:v} = C_{n:r} = C_{n:q} = C_{n:p} = c_n = \Psi_F(n)$ *and* $D_{n:v} = D_{n:r} = D_{n:q} = D_{n:p} = d_n = 0$.
(ii) If (a) $F \in \mathcal{D}_{\mathcal{R}}(H_{2,\beta})$, $x^0 > 0$, *or (b)* $F \in \mathcal{D}_{\mathcal{R}}(H_{3,0})$, $0 < x^0 < \infty$, *then*

$$P(r_n^* \leq r) \xrightarrow[n]{w} 1 - F(-x^0 r),\ r \geq 0, \text{ (trivial limit law, since } \mathcal{R}_1 = X_1 \text{ outweighed } \mathcal{R}_n),$$

$$P(v_n^* \leq v) \xrightarrow[n]{w} F(x^0 v) \text{ (trivial limit law, since } \mathcal{R}_1 = X_1 \text{ outweighed } \mathcal{R}_n),$$

$$P(q_n^* \leq q) \xrightarrow[n]{w} P(\frac{1}{X} \leq q + 1) \text{ (trivial limit law, since } \mathcal{R}_1 = X_1 \text{ outweighed } \mathcal{R}_n),$$

and

$$P(p_n^* \leq p) \xrightarrow[n]{w} P(X_1 \leq p + 1) = F(p + 1) \text{ (trivial limit law, since } \mathcal{R}_1 = X_1 \text{ outweighed } \mathcal{R}_n),$$

where $2C_{n:v}=C_{n:r}=C_{n:q}=C_{n:p}=d_n$ *and* $D_{n:r}=D_{n:v}=D_{n:p}=D_{n:q}=d_n$.
(iii) (a) If $F\in\mathcal{D}_{\mathcal{R}}(H_{3,0})$, $x^0=\infty$ *and* $c_n^{-1}=(\Psi_F(n+\sqrt{n})-\Psi_F(n))^{-1}\underset{n}{\rightarrow}K<\infty$, *then*

$$P(r_n^*\le r)\ \underset{n}{\overset{w}{\rightarrow}}\ \begin{cases} H_{3,0}(r), & \text{if } K=0 \text{ (trivial limit)},\\ H_{3,0}(r)\star(1-F(-\frac{r}{K})), & \text{if } K>0,\end{cases}$$

$$P(v_n^*\le v)\ \underset{n}{\overset{w}{\rightarrow}}\ \begin{cases} H_{3,0}(v), & \text{if } K=0 \text{ (trivial limit)},\\ H_{3,0}(v)\star F(\frac{v}{K})), & \text{if } K>0,\end{cases}$$

where $2C_{n:v}=C_{n:r}=c_n=\Psi_F(n+\sqrt{n})-\Psi_F(n)$ *and* $D_{n:r}=D_{n:v}=d_n=\Psi_F(n)$.
(b) If $F\in\mathcal{D}_{\mathcal{R}}(\overline{H}_{3,0})$, $x^0=\infty$ *and* $\frac{\Psi_F(n+\sqrt{n})}{\Psi_F(n)}\underset{n}{\rightarrow}1$, *then*

$$P(q_n^*\le q)\ \underset{n}{\overset{w}{\rightarrow}}\ P(\frac{1}{X}\le q+1)\ \text{(trivial limit law)},$$

and

$$P(p_n^*\le p)\ \underset{n}{\overset{w}{\rightarrow}}\ P(X_1\le p+1)=F(p+1)\ \text{(trivial limit law)},$$

where $C_{n:q}=C_{n:p}=d_n=\Psi_F(n)$ *and* $D_{n:v}=D_{n:r}=D_{n:q}=D_{n:p}=d_n$.

Proof. First, we notice that the condition $x^0>0$, in Part (ii), implies that the scale normalizing constant $2C_{n:v}=C_{n:r}=C_{n:q}=C_{n:p}=d_n$ will be positive (at least for large n, namely, $d_n=x^0>0$, in Part (a) and $2C_{n:v}=C_{n:r}=C_{n:q}=C_{n:p}=d_n=\Psi_F(n)\underset{n}{\rightarrow}x^0>0$, in Part (b)). Now, it is easy to check the validity of the representations

$$r_n^*\ \underset{n}{\overset{w}{=}}\ \begin{cases}\mathcal{R}_n^*-\frac{X_1}{c_n}, & \text{if } C_{n:r}=c_n,\ D_{n:r}=d_n,\\ c_nd_n^{-1}\mathcal{R}_n^*-\frac{X_1}{d_n}, & \text{if } C_{n:r}=d_n,\ D_{n:r}=d_n,\end{cases}\tag{1}$$

$$v_n^*\ \underset{n}{\overset{w}{=}}\ \begin{cases}\mathcal{R}_n^*+\frac{X_1}{c_n}, & \text{if } 2C_{n:v}=c_n,\ D_{n:v}=d_n,\\ c_n\ d_n^{-1}\mathcal{R}_n^*+\frac{X_1}{d_n}, & \text{if } 2C_{n:v}=d_n,\ D_{n:v}=d_n,\end{cases}\tag{2}$$

$$q_n^*\ \underset{n}{\overset{w}{=}}\ \begin{cases}\frac{R_n^*}{X_1}, & \text{if } C_{n,q}=c_n,\ D_{n,q}=d_n=0,\\ \frac{c_nd_n^{-1}R_n^*-(X_1-1)}{X_1}, & \text{if } C_{n,q}=d_n,\ D_{n,q}=d_n,\end{cases}\tag{3}$$

$$p_n^*\ \underset{n}{\overset{w}{=}}\ \begin{cases}\mathcal{R}_n^*\ X_1, & \text{if } C_{n:p}=c_n,\ D_{n:p}=d_n=0,\\ c_nd_n^{-1}\mathcal{R}_n^*X_1+(X_1-1), & \text{if } C_{n:p}=d_n,\ D_{n:p}=d_n,\end{cases}\tag{4}$$

where $X_n\underset{n}{\overset{w}{=}}Y_n$ means that the rv's X_n and Y_n have the same limit df. The implication (i) follows from the first part of (1)–(4), Theorem 1.2 and from the independency between $\mathcal{R}_r$ and $\mathcal{R}_s$, if $s-r\underset{n}{\rightarrow}\infty$ (see Barakat, 2007). The implication (ii) follows from the second part of (1)–(4) and Theorem 1.2 (note that Theorem 1.2 implies $c_nd_n^{-1}\underset{n}{\rightarrow}0$, in Parts (a) and (b). On the other hand, the implication (iii), Part (a), follows from the second part of (3)–(4) and Theorem 1.2, where the condition $\frac{\Psi_F(n+\sqrt{n})}{\Psi_F(n)}\underset{n}{\rightarrow}1$ implies $c_nd_n^{-1}\underset{n}{\rightarrow}0$. Finally, the implication (iii), Part (b), follows from the first part of (1)–(2) and Theorem 1.2, where the condition $c_n^{-1}=(\Psi_F(n+\sqrt{n})-\Psi_F(n))^{-1}\underset{n}{\rightarrow}0$ implies the trivial limit (where $\mathcal{R}_n^*$ outweighes $\frac{X_1}{c_n}$), while when the condition $c_n^{-1}=(\Psi_F(n+\sqrt{n})-\Psi_F(n))^{-1}\underset{n}{\rightarrow}K$ implies the given non-trivial limit law. □

3. Illustrative examples and concluding remarks

Illustrative examples are provided in this section, which lend further support to our theoretical results.

Example 3.1. For the logistic distribution, $F(x) = P(X \leq x) = \frac{e^x}{1+e^x}$, $\forall x$, we can easily show that $\Psi_{F_2}(u) = \log(e^u - 1)$. Therefore $\frac{\Psi_F(n+\sqrt{n})}{\Psi_F(n)} = \frac{\log(e^{n+\sqrt{n}}-1)}{\log(e^n-1)} \underset{n}{\rightarrow} 1$. Moreover, $\Psi_F(n+\sqrt{n}) - \Psi_F(n) = \log\frac{e^{n+\sqrt{n}}-1}{e^n-1} \underset{n}{\rightarrow} \infty$. Then, from Theorem 1.2 and Theorem 2.1, we get $P(r_n^* \leq r) \overset{w}{\underset{n}{\rightarrow}} H_{3,0}(r)$, $P(v_n^* \leq v) \overset{w}{\underset{n}{\rightarrow}} H_{3,0}(r)$, $P(q_n^* \leq q) \overset{w}{\underset{n}{\rightarrow}} P(\frac{1}{X} \leq q+1)$ and $P(p_n^* \leq p) \overset{w}{\underset{n}{\rightarrow}} P(X_1 \leq p+1) \overset{w}{\underset{n}{\rightarrow}} \frac{e^{p+1}}{1+e^{p+1}}$, $\forall p$.

Example 3.2. For the Weibull distribution, $F(x) = P(X \leq x) = 1 - e^{-x^c}$, $x,\ c > 0$, we can easily show that $\Psi_{F_1}(u) = u^{\frac{1}{c}}$. Therefore $\frac{\Psi_F(n+\sqrt{n})}{\Psi_F(n)} = (1+\frac{1}{\sqrt{n}})^{\frac{1}{c}} \underset{n}{\rightarrow} 1$. Moreover, $\Psi_F(n+\sqrt{n}) - \Psi_F(n) = (n+\sqrt{n})^{\frac{1}{c}} - n^{\frac{1}{c}} = n^{\frac{1}{c}}\frac{1}{c\sqrt{n}}(1+\circ(1)) = \frac{1}{c}n^{\frac{1}{c}-\frac{1}{2}}$. Thus $\Psi_F(n+\sqrt{n}) - \Psi_F(n) \underset{n}{\rightarrow} \frac{1}{c}$, if $c = 2$ and $\Psi_F(n+\sqrt{n}) - \Psi_F(n) \underset{n}{\rightarrow} \infty$, if $c > 2$. Then, from Theorem 1.2 and Theorem 2.1, we get

$$P(r_n^* \leq r) \overset{w}{\underset{n}{\rightarrow}} \begin{cases} H_{3,0}(r), & \text{if } c > 2, \\ H_{3,0}(r) \star (e^{-\frac{r^2}{4}} \mathrm{I}_{(-\infty,0)}(r)), & \text{if } c = 2, \end{cases}$$

where $\mathrm{I}_A(x)$ is the usual indicator function,

$$P(v_n^* \leq v) \overset{w}{\underset{n}{\rightarrow}} \begin{cases} H_{3,0}(v), & \text{if } c > 2, \\ H_{3,0}(v) \star (1 - e^{-\frac{v^2}{4}}) \mathrm{I}_{(0,\infty)}(v)), & \text{if } c = 2, \end{cases}$$

$$P(q_n^* \leq q) \overset{w}{\underset{n}{\rightarrow}} P(\frac{1}{X} \leq q+1) = e^{-(q+1)^{-c}},$$

and

$$P(p_n^* \leq p) \overset{w}{\underset{n}{\rightarrow}} P(X_1 \leq p+1) \overset{w}{\underset{n}{\rightarrow}} 1 - e^{-(p+1)^c}.$$

Concluding remarks

Record values arise naturally in many practical problems and there are several situations pertaining to meteorology, hydrology, sporting and athletic events wherein only record values may be recorded. In this paper we study the asymptotic behavior of some functions of record values, which have important applications. Theorem 2.1, as well as Examples 3.1 and 3.2, show that in most cases of the convergence to a non-degenerate limit, each of the statistics r_n, v_n, q_n and p_n is asymptotically equivalent to the statistic $a\mathcal{R}_n + bX_1$, as $n \rightarrow \infty$, where $-\infty < a, b < \infty$ and $\max(\mid a \mid, \mid b \mid) > 0$. Moreover, in all cases, both of the statistics q_n and p_n (as well as r_n and v_n) converge weakly together to a non-trivial (or trivial) type. Moreover in some cases, some of the four statistics, converge to the same limit df. This fact has considerable practical importance.

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