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Inferences on the Parameters of Power Law Distribution

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Abstract. Computational approach test (CAT) based on maximum likelihood estimator is feasable approach to the statistical hypothesis test. It might be because it does not require the knowledge of any sampling distribution. Further, it relies heavily on numerical computations and Monte-Carlo simulation. This paper demonstrates that the CAT can be as good as, if not better than, the uniformly most powerful test (UMP-test) to test the scale parameter in power law distribution where the shape parameter is known. We employ the CAT in power law distribution when the shape parameter is unknown and compare it with an exact method. We also apply the CAT for testing scale parameters in two power law populations and give an example with simulated data. Simulation studies show that the actual sizes and powers of the CAT are satisfactory.

1. Introduction

Advances in the technology of computational tools have significantly affected the statistical inference and estimation. Complex theoretical results can now be better realized through numerical computations and/or Monte-Carlo simulations well before that can be verified analytically. Recently, Pal et al. (2007) developed a simple computational technique, called CAT, for hypothesis testing problems. This method uses the maximum likelihood estimation (MLE) for the purpose of statistical inference on unknown parameters. The CAT is a simple procedure based on a simple set of computational steps which can be implemented easily by applied researchers. The computational mechanism is such that the CAT finds the critical region automatically. Chang et al. (2010) showed that for the one-way ANOVA problem with the usual assumptions, the CAT provides power which is very close to that of the classical F test. Chang and Pal (2008a) applied the CAT to the Behrens-Fisher problem and compared this approach with the five existing methods. Also Chang and Pal (2008b) applied the proposed CAT for testing the common mean of several normal distributions. The CAT does not require the knowledge of any sampling distribution but depends heavily on numerical computations and Monte-Carlo simulation. Hence, the conventional approach had been to use either the asymptotic theory or some sort of approximations to the null distributions of the test statistics.

The power law has attracted particular attention over the years both for its mathematical properties, which sometimes leads to surprising physical consequences, and for its appearance in a wide range of natural and man-made phenomena. The word frequency, citations of scientific papers, web hits, copies of books sold, telephone calls, magnitude of earthquakes, diameter of moon craters, intensity of solar flares, intensity of wars, wealth of the richest people, frequencies of family names, and populations of cities, for example, are all thought to have power-law distributions (Newman, 2005).

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The power law density function with parameters θ and β (pow(θ,β)) is given by

$$f(x;\theta,\beta)=\frac{\beta x^{\beta-1}}{\theta^{\beta}},\quad 0<x<\theta,\quad \beta>0, \tag{1}$$

where θ is the scale parameter and β is the shape parameter. The theoretical mean and variance are derived as (Krishnamoorthy, 2006)

$$E(X)=\frac{\theta\beta}{\beta+1},\quad Var(X)=\frac{\theta^2\beta}{(\beta+2)(\beta+1)^2}.$$

In this paper, we consider testing the scale parameter θ in the power law distribution for two cases: i) the shape parameter, β is known; ii) the shape parameter, β is unknown. In the first case, there is the UMP test. In the second case, there is not UMP-test, however there is an exact F-test. In each case, we apply the CAT for testing the hypothesis about θ, besides, we use simulation studies to show that the CAT can be as good as, if not better than, the other method. In the main section of this paper, we consider testing the equality of scale parameters of two power law distributions when the shape parameters are unknown. In the latter case, as far as we know, there is no exact approach for testing the equality of scale parameters. We proposed a CAT for testing the equality of scale parameters. Simulation studies show that the actual sizes and powers of this approach are satisfactory.

This article is organized as follows: we provide, in Section 2, the general computational framework to handle a scalar valued parameter based on the MLE estimator. In Section 3, we derive the CAT for testing hypotheses on the scale parameter of the power law distribution for two cases and compare the size/power of the CAT with the accepted tests. We also illustrate our approach using a real example. In Section 4, we use the CAT for testing the equality of scale parameters in two power law populations and give an example with simulated data. Also, we study the actual size and power of this approach based on simulation studies.

2. Computational Approach Test (CAT)

Let $X_1,X_2,...,X_n$ be a random sample from distribution $F(x;\theta)$, $\theta\in\Theta$. We present the CAT for two cases: when there is no nuisance parameter, and when there are nuisance parameters.

Case-1: Assume θ is scaler valued, and there is no nuisance parameter. The methodology of the CAT for testing

$$H_0:\theta=\theta_0,\quad vs.\quad H_A:(\theta<\theta_0\ \ or\ \ \theta>\theta_0\ \ or\ \ \theta\neq\theta_0)$$

at a desired level α, is given below in three simple steps:

Step 1. Obtain $\hat\theta$ the MLE of θ.

Step 2. Assume that H_0 is true, i.e., set $H_0:\theta=\theta_0$. Generate artificial sample $Y_1,Y_2,...,Y_n$ *i.i.d.* from density $f(x,\theta_0)$ a large number of times (say, M times). For each of these replicated samples, recalculated the MLE of θ (pretending that θ were unknown). Let these recalculated MLE values of θ be $\hat\theta_{01},\hat\theta_{02},...,\hat\theta_{0M}$. Let $\hat\theta_{0(1)}\le\hat\theta_{0(2)}\le...\le\hat\theta_{0(M)}$ be the ordered values of $\hat\theta_{0l}$, $1\le l\le M$.

Step 3. (i) For testing $H_0:\theta=\theta_0$ vs. $H_A:\theta<\theta_0$ at level α, define $\hat\theta_L=\hat\theta_{0(\alpha M)}$. Reject H_0 if $\hat\theta<\hat\theta_L$. Alternatively, calculate the p-value as

$$p=\frac{1}{M}\sum_{l=1}^{M}I_{(\hat\theta_{0(l)}<\hat\theta)}.$$

(ii) For testing $H_0 : \theta = \theta_0$ vs. $H_A : \theta > \theta_0$ at level α, define $\hat{\theta}_U = \hat{\theta}_{0((1-\alpha)M)}$. Reject H_0 if $\hat{\theta} > \hat{\theta}_U$. Alternatively, calculate the p-value as

$$p = \frac{1}{M} \sum_{l=1}^{M} I_{(\hat{\theta}_{0(l)} > \hat{\theta})}.$$

(iii) For testing $H_0 : \theta = \theta_0$ vs. $H_A : \theta \neq \theta_0$ define $\hat{\theta}_L = \hat{\theta}_{0((\alpha/2)M)}$ and $\hat{\theta}_U = \hat{\theta}_{0((1-\alpha/2)M)}$. Reject H_0 if $\hat{\theta}_L < \hat{\theta}$ or $\hat{\theta} > \hat{\theta}_U$. Alternatively, the p-value is computed as:

$$p = 2\min(p_1, 1 - p_1),$$

where $p_1 = \frac{1}{M} \sum_{l=1}^{M} I_{(\hat{\theta}_{0(l)} < \hat{\theta})}$.

Case-2: Assume θ is scaler valued and there are nuisance parameters. Assume that $\theta = (\theta^{(1)}, \theta^{(2)}) \in \Theta$, where $\theta^{(2)}$, is the nuisance parameter, and $\theta^{(1)}$ is the parameter of interest. The methodology of the CAT for testing

$$H_0 : \theta^{(1)} = \theta_0^{(1)} \quad vs. \quad H_A : (\theta^{(1)} < \theta_0^{(1)} \;\; or \;\; \theta^{(1)} > \theta_0^{(1)} \;\; or \;\; \theta^{(1)} \neq \theta_0^{(1)}),$$

at a desired level α, is given through the following steps:

Step 1. Obtain $\hat{\theta} = (\hat{\theta}^{(1)}, \hat{\theta}^{(2)})$, the MLE of θ.

Step 2. (i) Assume that H_0 is true, i.e., set $H_0 : \theta^{(1)} = \theta_0^{(1)}$. Then find the MLE of $\theta^{(2)}$ from the data again. Call this as the 'restricted MLE of $\theta^{(2)}$' under H_0, denoted by $\hat{\theta}_R^{(2)}$.
(ii) Generate artificial sample $Y_1, Y_2, ..., Y_n$ i.i.d. from density $f(x, \theta_0^{(1)}, \hat{\theta}_R^{(2)})$ a large number of times (say, M times). For each of these replicated samples, recalculated the MLE of $\theta = (\theta^{(1)}, \theta^{(2)})$ (pretending that θ were unknown). Let these recalculated MLE values of $\theta^{(1)}$ be $\hat{\theta}_{01}^{(1)}, \hat{\theta}_{02}^{(1)}, ..., \hat{\theta}_{0M}^{(1)}$.
(iii) Let $\hat{\theta}_{0(1)}^{(1)} \leq \hat{\theta}_{0(2)}^{(1)} \leq ... \leq \hat{\theta}_{0(M)}^{(1)}$ be the ordered values of $\hat{\theta}_{0l}^{(1)}$, $1 \leq l \leq M$.

Step 3. (i) For testing $H_0 : \theta^{(1)} = \theta_0^{(1)}$ vs. $H_A : \theta^{(1)} < \theta_0^{(1)}$ at level α, define $\hat{\theta}_L^{(1)} = \hat{\theta}_{0(\alpha M)}^{(1)}$. Reject H_0 if $\hat{\theta}^{(1)} < \hat{\theta}_L^{(1)}$. Alternatively, calculate the p-value as

$$p = \frac{1}{M} \sum_{l=1}^{M} I_{(\hat{\theta}_{0(l)}^{(1)} < \hat{\theta}^{(1)})}.$$

(ii) For testing $H_0 : \theta^{(1)} = \theta_0^{(1)}$ against $H_A : \theta^{(1)} > \theta_0^{(1)}$ at level α, define $\hat{\theta}_U^{(1)} = \hat{\theta}_{0((1-\alpha)M)}^{(1)}$. Reject H_0 if $\hat{\theta}^{(1)} > \hat{\theta}_U^{(1)}$. Alternatively, calculate the p-value as

$$p = \frac{1}{M} \sum_{l=1}^{M} I_{(\hat{\theta}_{0(l)}^{(1)} > \hat{\theta}^{(1)})}.$$

(iii) For testing $H_0 : \theta^{(1)} = \theta_0^{(1)}$ against $H_A : \theta^{(1)} \neq \theta_0^{(1)}$ define the cut-off points as $\hat{\theta}_L^{(1)} = \hat{\theta}_{0((\alpha/2)M)}^{(1)}$ and $\hat{\theta}_U^{(1)} = \hat{\theta}_{0((1-\alpha/2)M)}^{(1)}$. Reject H_0 if $\hat{\theta}^{(1)} < \hat{\theta}_L^{(1)}$ or $\hat{\theta}^{(1)} > \hat{\theta}_U^{(1)}$. Alternatively, the p-value is computed as:

$$p = 2\min(p_1, 1 - p_1),$$

where $p_1 = \frac{1}{M} \sum_{l=1}^{M} I_{(\hat{\theta}_{0(l)}^{(1)} > \hat{\theta}^{(1)})}$.

3. The CAT Results for Power Law Distribution

Let $X_1, X_2, ..., X_n$ be a random sample from the power law distribution with density function (1). The MLE of the parameters θ and β are (Krishnamoorthy, 2006)

$$\hat{\theta} = \max\{X_i\},\ i = 1, 2, .., n, \qquad \hat{\beta} = \frac{1}{\log(\hat{\theta}) - \frac{1}{n}\sum_{i=1}^{n} \log(X_i)},$$

with

$$\frac{2n\beta}{\hat{\beta}} \sim \chi^2_{(2n-2)}, \qquad -2n\beta\left[\log(\hat{\theta}) - \log(\theta)\right] \sim \chi^2_{(2)},$$

where $\chi^2_{(m)}$ denotes a central chi-square distribution with m degrees of freedom (df).

In this Section, our interest is to test the hypothesis

$$H_0 : \theta = \theta_0, \quad vs. \quad H_1 : \theta \neq \theta_0, \tag{2}$$

in two cases: β is known and β is unknown. In what follows, we will first present the exiting method and then propose the CAT, in each case.

3.1. β is known

In the power law distribution when the shape parameter β is known, there is the UMP-test for testing the hypothesis (2). Therefore, we compare our approach with UMP-test method. Recall that the $100(1-\alpha)\%$ shortest confidence interval for θ is $(\max\{X_i\}, \frac{\max\{X_i\}}{\sqrt[n\beta]{\alpha}})$, and the critical regions of UMP-test for classical hypotheses on the scale parameter, θ are derived using the left and right limits of this shortest confidence interval (Casella and Berger, 1990).

In the following, we present the CAT for testing the scale parameter θ and compare the actual sizes and powers of this approach with UMP-test method. The CAT is performed using the following steps for testing the hypotheses in (2):

a. In Step 1, we derive $\hat{\theta} = \max\{X_i\}$, $i = 1, ..., n$, the MLE of θ.

b. In Step 2, set $H_0 : \theta = \theta_0$, and generate

$$(1^{\text{st}} \text{ replication})\ Y_1^{(1)}, ..., Y_n^{(1)}\ i.i.d\ \text{pow}(\theta_0, \beta);\ \text{get}\ \hat{\theta}_{01} = \max\{Y_i^{(1)}\}$$

$$\vdots$$

$$(M^{\text{th}} \text{ replication})\ Y_1^{(M)}, ..., Y_n^{(M)}\ i.i.d\ \text{pow}(\theta_0, \beta);\ \text{get}\ \hat{\theta}_{0M} = \max\{Y_i^{(M)}\}.$$

The values $\hat{\theta}_{01}, \hat{\theta}_{02}, ..., \hat{\theta}_{0M}$ are ordered as $\hat{\theta}_{0(1)} \leq \hat{\theta}_{0(2)} \leq ... \leq \hat{\theta}_{0(M)}$.

c. In Step 3, for testing two-sided hypothesis, (when $M = 10000$) define the cut-off points as $\hat{\theta}_L = \hat{\theta}_{0((\alpha/2)M)} = \hat{\theta}_{0(250)}$ and $\hat{\theta}_U = \hat{\theta}_{0((1-\alpha/2)M)} = \hat{\theta}_{0(9750)}$. Reject H_0 if $\hat{\theta} < \hat{\theta}_L$ or $\hat{\theta} > \hat{\theta}_U$.

A simulation study, with 10000 repetition, is performed to compare the sizes and powers test of the two approaches; i) the CAT, ii) the UMP-test, for testing $H_0 : \theta = 3$ vs. $H_1 : \theta \neq 3$. The actual sizes and powers of the tests for different values of n and θ with $\beta = 2$ are given in Table 1. As shown in Table 1, we conclude that the actual sizes and powers for different sample sizes in CAT are as good as the UMP-test.

3.2. β is unknown

Also we apply the CAT for testing the hypothesis (2) when β is unknown. In this case, there is not UMP-test for θ but there is an exact test with test statistic

$$F = -(n-1)\hat{\beta}\left[\log(\hat{\theta}) - \log(\theta_0)\right],$$

and H_0 is rejected in (2) if $F > F_{(\alpha/2,2,2n-2)}$ or $F < F_{(1-\alpha/2,2,2n-2)}$. We compare the CAT with this exact test. The CAT is given based on the following steps:

a. In Step 1, we derive $\hat{\theta} = \max\{X_i\}$ the MLE of θ.

b. In Step 2, (i) assume that $H_0 : \theta = \theta_0$ is true, then $X_i \sim \text{pow}(\theta_0, \beta)$. The MLE of the parameter β which are called the 'restricted MLE' is $\hat{\beta}_R = (\log(\theta_0) - \frac{1}{n}\sum_{i=1}^{n}\log(X_i))^{-1}$.
(ii) Generate artificial sample $Y_1, ..., Y_n$ *i.i.d.* from $\text{pow}(\theta_0, \hat{\beta}_R)$ a large number of times (say, M times). For each of these replicated samples, recalculated the MLE of θ. Let these recalculated MLE values of θ be $\hat{\theta}_{01}, \hat{\theta}_{02}, ..., \hat{\theta}_{0M}$ $(M = 10000)$.
(iii) Let $\hat{\theta}_{0(1)} \leq \hat{\theta}_{0(2)} \leq ... \leq \hat{\theta}_{0(10000)}$ be the ordered values of $\hat{\theta}_{0l}$, $1 \leq l \leq M$.

c. In Step 3, for testing $H_0 : \theta = \theta_0$ against $H_A : \theta \neq \theta_0$ define the cut-off points as $\hat{\theta}_L = \hat{\theta}_{0((\alpha/2)M)}$ and $\hat{\theta}_U = \hat{\theta}_{0((1-\alpha/2)M)}$. Reject H_0 if $\hat{\theta} < \hat{\theta}_L$ or $\hat{\theta} > \hat{\theta}_U$. Alternatively, the p-value is computed as: $p = 2\min(p_1, 1-p_1)$, where $p_1 = \frac{1}{M}\sum_{l=1}^{M} I_{(\hat{\theta}_{0(l)} > \hat{\theta})}$.

A simulation study, with 10000 repetition, is performed to compare the sizes and powers of the two approaches; i) The CAT ii) the exact F-test, for testing $H_0 : \theta = 3$ *vs.* $H_1 : \theta \neq 3$. For fixed n, θ and $\beta = 2$, generate iid observations of size n from $\text{pow}(\theta, \beta)$. The actual sizes and powers of the tests for different values are given in Table 1. We see that the actual sizes of the CAT are always less than the nominal level, however, this can not happen in the other method. Also, according to this Table we see that the power of the test in our method is as good as, the F-test.

3.3. Real example

We consider a set of the given data by Majumdar (1993) on the failure times of a vertical boring machine. The observations are 376 808 1596 1700 1701 1781 1976 2076 2136 2172 2296 2380 2655 2672 2806 2816 2848 2937 3158 3575 3632 3686 3705 3802 3811 4020

We obtain the MLE's as $\hat{\theta} = 4020$ and $\hat{\beta} = 1.8478$. Based on Kolmogorov-Smirnov test, we observe that these data follow a power law distribution. Consider that our interest is to test the hypotheses

$$H_0 : \theta = 4050 \quad vs. \quad H_1 : \theta \neq 4050.$$

We performed the F-test and the CAT using the proposed steps with $M = 100000$ and obtained the p-values for the CAT and the F-test as 0.5781 and 0.5886, respectively. Therefore, these two approaches do not reject the null hypothesis.

4. Testing the equality of two scale parameters

Let $X_{i1}, ..., X_{in_i}$, $i = 1, 2$, be two independent random samples from the power law distribution, i.e. $X_{ij} \sim \text{pow}(\theta_i, \beta_i)$, $i = 1, 2$, $j = 1, ..., n_i$, where the parameters $(\theta_1, \theta_2, \beta_1, \beta_2)$ are unknown. We consider problem of testing the hypothesis

$$H_0 : \theta_1 = \theta_2 \quad vs. \quad H_1 : \theta_1 \neq (< \text{ or } >)\ \theta_2,$$

which is equivalent to the following test hypothesis

$$H_0^* : \delta = 0 \quad vs. \quad H_1^* : \delta \neq (< \text{ or } >)\ 0,$$

Table 1: The actual sizes and powers of the tests when the nominal level is 0.05.

β is known						
				θ		
n	Test	2.8	2.9	3.0	3.1	3.2
5	CAT	0.0678	0.0532	0.0511	0.3127	0.5062
	UMP	0.0677	0.0526	0.0511	0.3125	0.5065
10	CAT	0.0956	0.0553	0.0480	0.5056	0.7336
	UMP	0.0950	0.0552	0.0479	0.5062	0.7334
15	CAT	0.2008	0.0694	0.0495	0.6436	0.8718
	UMP	0.1985	0.0691	0.0493	0.6437	0.8715
20	CAT	0.3949	0.0962	0.0512	0.7442	0.9316
	UMP	0.3943	0.0962	0.0510	0.7441	0.9313
25	CAT	0.7906	0.1334	0.0494	0.8199	0.9643
	UMP	0.7885	0.1330	0.0493	0.8198	0.9640
30	CAT	1	0.1934	0.0496	0.8668	0.9804
	UMP	1	0.1932	0.0485	0.8666	0.9804
35	CAT	-	0.2731	0.0495	0.9059	0.9893
	UMP	-	0.2704	0.0527	0.9053	0.9891
40	CAT	-	0.3770	0.0503	0.9289	0.9937
	UMP	-	0.3768	0.0508	0.9289	0.9932
45	CAT	-	0.5211	0.0509	0.9490	0.9972
	UMP	-	0.5205	0.0510	0.9488	0.9970
50	CAT	-	0.7409	0.0474	0.9613	0.9981
	UMP	-	0.7401	0.0477	0.9611	0.9979
β is unknown						
5	CAT	0.0595	0.0570	0.0256	0.3004	0.4952
	F-Test	0.0596	0.0576	0.0519	0.3198	0.5076
10	CAT	0.0657	0.0597	0.0378	0.4951	0.7384
	F-Test	0.0658	0.0596	0.0499	0.5006	0.7405
15	CAT	0.1505	0.0695	0.0418	0.6382	0.8716
	F-Test	0.1504	0.0691	0.0483	0.6410	0.8724
20	CAT	0.3157	0.0981	0.0437	0.7366	0.9250
	F-Test	0.3156	0.0978	0.0501	0.7384	0.9254
25	CAT	0.5635	0.1390	0.0450	0.8123	0.9609
	F-Test	0.5630	0.1372	0.0491	0.8129	0.9611
30	CAT	0.8235	0.1998	0.0473	0.8670	0.9825
	F-Test	0.8229	0.1901	0.0506	0.8675	0.9825
35	CAT	0.9638	0.2668	0.0401	0.9060	0.9894
	F-Test	0.9555	0.2640	0.0437	0.9064	0.9895
40	CAT	0.9971	0.3769	0.0482	0.9331	0.9942
	F-Test	0.9883	0.3745	0.0518	0.9330	0.9942
45	CAT	1	0.5144	0.0478	0.9489	0.9979
	F-Test	1	0.5072	0.0507	0.9486	0.9978
50	CAT	-	0.6779	0.0478	0.9651	0.9987
	F-Test	-	0.6715	0.0503	0.9649	0.9981

where $\delta = \theta_1 - \theta_2$.

There is no approach to test these hypotheses. In fact, when the shape parameters β_i are not unequal, testing the equality of scale parameters of the two power law distribution is similar to the Behrens-Fisher problem (Chang et al., 2008a).

4.1. CAT for testing the equality of scales

In what follows, we present the specific version of the CAT as applicable for the given power law problem. Note that $\delta = \theta_1 - \theta_2$ is the parameter of interest.

Step 1. Obtain the MLE of the parameters as $\hat{\theta}_i = \max\{X_{i1}, ..., X_{in_i}\}$, $i = 1, 2$, and then $\hat{\delta} = \hat{\theta}_1 - \hat{\theta}_2$.

Step 2. Assume that H_0^* is true, i.e. $\delta = 0$ ($\theta_1 = \theta_2 = \theta$). Under this restricted model, we have $X_{ij} \sim \text{pow}(\theta, \beta_i)$, $i = 1, 2$, $j = 1, ..., n_i$. Get the MLE's of the three parameters $(\theta, \beta_1, \beta_2)$ which are called the restricted MLEs as given

$$\hat{\theta}_R = \frac{\max\{X_{11}, ..., X_{1n_1}\} + \max\{X_{21}, ..., X_{2n_2}\}}{2},$$

and

$$\hat{\beta}_{i(R)} = \frac{1}{\log(\hat{\theta}_R) - \frac{1}{n_i}\sum_{j=1}^{n_i} \log(X_{ij})}, \quad i = 1, 2.$$

Step 3. Generate artificial data X_{ij}, $i = 1, 2$, $j = 1, ..., n_i$ from $\text{pow}(\hat{\theta}_R, \hat{\beta}_{i(R)})$ a large number of times (say, M times). For each of these replicated samples, recalculate the MLE of δ. Thus we will have $\hat{\delta}_{01}, \hat{\delta}_{02}, ..., \hat{\delta}_{0M}$.

Step 4. Let $\hat{\delta}_{0(1)} \leq \hat{\delta}_{0(2)} \leq ... \leq \hat{\delta}_{0(M)}$ be the ordered values of $\hat{\delta}_{0l}$, $l = 1, ..., M$.

Step 5. (i) For testing $H_0^* : \delta = 0$ against $H_1^* : \delta < 0$, define the critical value $\hat{\delta}_L$ as $\hat{\delta}_L = \hat{\delta}_{0(\alpha M)}$. If $\hat{\delta} < \hat{\delta}_L$, then H_0^* is rejected. Alternatively the p-value can be defined as

$$p = \frac{1}{M}\sum_{l=1}^{M} I(\hat{\delta}_{0(l)} < \hat{\delta}).$$

(ii) For testing $H_0^* : \delta = 0$ against $H_1^* : \delta > 0$, define $\hat{\delta}_U = \hat{\delta}_{0((1-\alpha)M)}$. If $\hat{\delta} > \hat{\delta}_U$, then H_0^* is rejected. Alternatively the p-value can be defined as

$$p = \frac{1}{M}\sum_{l=1}^{M} I(\hat{\delta}_{0(l)} > \hat{\delta}).$$

(iii) For testing $H_0^* : \delta = 0$ against $H_1^* : \delta \neq 0$, define $\hat{\delta}_L = \hat{\delta}_{0((\alpha/2)M)}$ and $\hat{\delta}_U = \hat{\delta}_{0((1-\alpha/2)M)}$. If $\hat{\delta}$, is either greater than $\hat{\delta}_U$ or less than $\hat{\delta}_L$, then H_0^* is rejected. Alternatively the p-value can be defined as

$$p = 2\min(p_1, 1 - p_1),$$

where $p_1 = \dfrac{1}{M}\sum_{l=1}^{M} I(\hat{\delta}_{0(l)} < \hat{\delta})$.

A simulation study is performed to evaluate the sizes and powers of the CAT in order to compare the scale parameters of two power law distributions. Two data sets are generated; the first data set with size n_1, is generated from the power law distribution with parameters $\theta_1 = 3$ and $\beta_1 = 1$ and second data set with size n_2, is generated from the power law distribution with different parameters θ_2 and β_2. The p-value for CAT with $M = 10000$ simulations is computed for testing $H_0 : \theta_1 = \theta_2$ vs. $H_1 : \theta_1 \neq \theta_2$. For $N = 10000$ replication, the p-values are computed. The size/power of the CAT is the number of cases that the p-values are smaller than nominal level $\alpha = 0.05$. The results for different value are given in Table 2. We can find that the actual sizes of the CAT are close to the nominal level and the powers of the CAT are satisfactory.

4.2. Example

To illustrate the CAT, we simulated data on $X \sim \text{pow}(2, 1)$ and on $Y \sim \text{pow}(3, 5)$ with $n_1 = n_2 = 10$. Therefore, two data sets are generated from the power law distribution with different scale parameters. The two set data are

Data 1	0.6436	1.0204	1.7635	1.7734	1.1493
	1.3233	0.0103	1.7982	1.5776	0.1547
Data 2	2.3280	2.8040	1.6894	2.4329	2.6147
	2.9358	2.6694	2.9960	2.6147	2.8855

The MLEs are $\hat{\theta}_1 = 1.7982$, $\hat{\beta}_1 = 0.9874$, $\hat{\theta}_2 = 2.9960$, $\hat{\beta}_2 = 6.4718$. Using given steps in this Section, with $M = 100000$, we computed the p-value for testing $H_0^* : \delta = 0$ vs. $H_1^* : \delta \neq 0$ as 0.0106, which suggests that the data provide evidence against H_0.

5. Conclusions and remarks

In this paper, we consider the problem of testing hypothesis for the scale parameter of power law distribution. In addition, we consider to test of the scale parameters of two power law populations when the shape parameters are unknown. We have in fact developed an approach based on the concept of the computational approach test (CAT) which is relatively easy to implement and does not require the explicit knowledge of the sampling distribution. The CAT is presented for testing the scale parameter of power law distribution and is compared with the other methods. Numerical results show that the CAT gives equivalent results to (if not better than) the UMP-test method and is much more satisfactory than F-test method in terms of the actual size and power of the test. In the main section of this paper, we considered testing the equality of scale parameters of two power law distributions when the shape parameters are unknown. In the latter case, as far as we know, there is no exact approach for testing the equality of scale parameters. Simulation studies indicate that the actual sizes and powers of our approach are satisfactory. A numerical example is given for illustrating the application of the proposed CAT. We believe that our proposed approach would be useful to researchers using the power law distribution in their analysis of the data.

Table 2: The actual sizes and powers of the test at 5% significant level when $\theta_1 = 3$ and $\beta_1 = 1$.

		θ_2					
β_2	(n_1, n_2)	3.0	3.5	4.0	4.2	4.5	5.0
0.5	(10, 10)	0.0717	0.2385	0.5006	0.5943	0.7131	0.8303
	(20, 10)	0.0706	0.4154	0.6986	0.7535	0.8306	0.8961
	(10, 20)	0.0695	0.2413	0.5886	0.7149	0.8416	0.9370
	(20, 20)	0.0657	0.5039	0.8636	0.9161	0.9572	0.9845
	(30, 35)	0.0639	0.7982	0.9803	0.9913	0.9977	0.9993
	(50, 50)	0.0598	0.9487	0.9980	0.9991	1.0000	1.0000
	(100, 50)	0.0599	0.9743	0.9994	0.9999	0.9999	1.0000
	(50, 100)	0.0573	0.9948	1.0000	1.0000	1.0000	1.0000
1.0	(10, 10)	0.0698	0.2323	0.5672	0.6953	0.8265	0.9364
	(20, 10)	0.0682	0.4882	0.8665	0.9178	0.9589	0.9855
	(10, 20)	0.0679	0.2413	0.6311	0.7750	0.9007	0.9797
	(20, 20)	0.0669	0.5805	0.9623	0.9843	0.9971	0.9998
	(30, 35)	0.0615	0.9155	0.9993	0.9999	1.0000	1.0000
	(50, 50)	0.0591	0.9952	1.0000	1.0000	1.0000	1.0000
	(100, 50)	0.0565	0.9988	1.0000	1.0000	1.0000	1.0000
	(50, 100)	0.0553	1.0000	1.0000	1.0000	1.0000	1.0000
1.5	(10, 10)	0.0667	0.2293	0.5901	0.7330	0.8677	0.9662
	(20, 10)	0.0655	0.5326	0.9305	0.9695	0.9874	0.9968
	(10, 20)	0.0631	0.2400	0.6422	0.7853	0.9145	0.9863
	(20, 20)	0.0613	0.6079	0.9872	0.9961	0.9999	1.0000
	(30, 35)	0.0607	0.9518	0.9999	1.0000	1.0000	1.0000
	(50, 50)	0.0570	0.9998	1.0000	1.0000	1.0000	1.0000
	(100, 50)	0.0552	0.9999	1.0000	1.0000	1.0000	1.0000
	(50, 100)	0.0547	1.0000	1.0000	1.0000	1.0000	1.0000
2.0	(10, 10)	0.0656	0.2303	0.6023	0.7515	0.8880	0.9776
	(20, 10)	0.0669	0.5553	0.9603	0.9870	0.9960	0.9997
	(10, 20)	0.0621	0.2410	0.6449	0.7890	0.9195	0.9888
	(20, 20)	0.0609	0.6181	0.9945	0.9991	1.0000	1.0000
	(30, 35)	0.0578	0.9665	1.0000	1.0000	1.0000	1.0000
	(50, 50)	0.0550	0.9999	1.0000	1.0000	1.0000	1.0000
	(100, 50)	0.0555	1.0000	1.0000	1.0000	1.0000	1.0000
	(50, 100)	0.0543	1.0000	1.0000	1.0000	1.0000	1.0000

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