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Prediction Intervals of Future Generalized Order Statistics Based on Generalized Extreme Value Distribution

Amany E. Aly

Department of Mathematics, Faculty of Science, Helwan University, Ain Helwan, Cairo, Egypt.
Current address: Department of Mathematics, Faculty of Science, Taibah University, Madinah, Saudi Arabia.

Abstract. In this paper, three prediction intervals of future generalized order statistics (gos) based on generalized extreme value distribution (GEVD) are constructed. For this purpose, three predictive pivotal quantities are considered, and their exact distributions are established. A Monte Carlo simulation study is carried out to explore the efficiency of the proposed method. The obtained results are then applied to real data for illustrative purposes.

1. Introduction

The prediction of future events without any doubt is one of the most important problems in statistics. This problem has been extensively studied by many authors, including Lingappaiah (25), Aitchison and Dunsmore (3), Lawless (26; 27), Kaminsky and Rhodin (20), Kaminsky and Nelson (21), Patel (28), Raqab et al. (30), Barakat et al. (7), El-Adll (14), El-Adll et al. (16), Barakat et al. (8) and AL-Hussaini et al. (4), among others.

It is well known that the ordered random variables play an important role in prediction methods. Since Kamps (23) had introduced the concept of gos as a unification of several models of ascendingly ordered random variables, the use of such concept has been steadily growing through the years. This is due to the fact that such concept includes important well-known models of ordered random variables that have been treated separately in statistical literature. Kamps (23) defined gos first by defining uniform gos and then using the quantile transformation to obtain the joint probability density function (jpdf) of the random variables $Y(1, n, \tilde{m}, k), \cdots, Y(n, n, \tilde{m}, k)$ based on cumulative distribution function (cdf) F with pdf f. The jpdf of $Y(1, n, \tilde{m}, k), \cdots, Y(n, n, \tilde{m}, k)$ is given by

$$f^{Y(1,n,\tilde{m},k),\ldots,Y(n,n,\tilde{m},k)}(y_1, \ldots, y_n) = k \left(\prod_{j=1}^{n-1} \gamma_j \right) \left(\prod_{i=1}^{n-1} (1 - F(y_i))^{m_i} f(y_i) \right) (1 - f(y_n))^{k-1} f(y_n),$$

on the cone $F^{-1}(0) \leqslant y_1 \leqslant \cdots \leqslant y_n \leqslant F^{-1}(1-)$ of $\mathbb{R}^n$. The model parameters are, $n \in \mathbb{N},\ n \geq 2$, $k > 0,\ \tilde{m} = (m_1, \ldots, m_{n-1}) \in \mathbb{R}^{n-1},\ M_r = \sum_{j=r}^{n-1} m_j$, such that $\gamma_r = k + n - r + M_r > 0$ for all $r \in \{1, \ldots, n-1\}$ and $\gamma_n = k$. Particular choices of the parameters $\gamma_1, \cdots, \gamma_n$ lead to different models, e.g., $m-$gos ($\gamma_n = k,\ \gamma_r = k + (n-r)(m+1),\ r = 1, \cdots, n-1$), oos ($\gamma_n = 1,\ \gamma_r = n - r + 1,\ r = 1, \cdots, n-1$, i.e., $k = 1,\ m_i = 0,\ i = 1, \cdots, n-1$), sos ($\gamma_n = \alpha_n,\ \gamma_r = (n-r+1)\alpha_r,\ \ \alpha_r > 0,\ \ r = 1, \cdots, n-1$), pos with censoring scheme $(R_1, \cdots, R_M)$ ($\gamma_n = R_M + 1,\ \gamma_r = n - r + 1 + \sum_{j=r}^{M} R_j$, if $r \leq M-1$ and

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Email address: a_abusabh@yahoo.com (Amany E. Aly)

$\gamma_r = n-r+1+R_M$, if $r \geq M$) and upper records ($\gamma_r = 1$, $1 \leq r \leq n$, i.e., $k = 1$, $m_i = -1, i = 1, \cdots, n-1$). Therefore, all the results obtained for the gos model can be applied to the particular models choosing the respective parameters. For more details in the theory and applications of gos see Kamps (23), Ahsanullah (2), Kamps and Cramer (22), Cramer (13), Barakat et al (7), El-Adll (15), Barakat (6; 9), and Ahmad et al (1).

Extreme value data usually is explored from heavy right tailed or excess kurtosis. Particularly in environmental data, e.g., spatial and temporal variability of turbulence (Sanford, (32)), daily maximum ozone measurement (Gilleland, (18)), maximum wind speed (Castillo et al. (11)), largest lichen measurements (Cooley et al., (12)), and maximum water level (Bruxer et al., (10)). *GEVD* was originally defined for the first time by (Jenkinson (19)), and the three possible limiting distributions of the maximum/minimum of random variables are embedded within it. This distribution is also known as the von Mises extreme value, von Mises-Jenkinson, and Fisher-Tippet distribution. A historical review of extreme value theory is provided in Kotz and Nadajarah (24). Recently, a comparative review of *GEVD* is reported in Pinheiro et al. (29).

The *GEVD* distribution is one of the most widely applied models for univariate extreme values. Its pdf and cdf are, respectively, given by

$$f(y;\mu,\sigma,\xi) = \frac{1}{\sigma}\left[1+\xi\left(\frac{y-\mu}{\sigma}\right)\right]^{(-1/\xi)-1}\exp\left\{-\left[1+\xi\left(\frac{y-\mu}{\sigma}\right)\right]^{-1/\xi}\right\}, \quad 1+\xi(y-\mu)/\sigma > 0 \tag{1}$$

and

$$F(y;\mu,\sigma,\xi) = \exp\left\{-\left[1+\xi\left(\frac{y-\mu}{\sigma}\right)\right]^{-1/\xi}\right\}, \quad 1+\xi(y-\mu)/\sigma \geqslant 0, \tag{2}$$

where $\mu \in \mathbb{R}$ is the location parameter, $\sigma > 0$ is the scale parameter and $\xi \in \mathbb{R}$ is the shape parameter. The shape parameter ξ governs the tail behavior of the distribution. The Weibull, Gumbel, Frechet sub-families are corresponding, respectively, to $\xi < 0$, $\xi \to 0$, and $\xi > 0$.

The rest of this paper is organized as follows. In Section 2, the predictive pivotal quantities and their exact distributions are obtained. Section 3, include simulation studies. Some applications to real data are presented in Section 4.

2. Pivotal Quantities and Their Distributions

This section concerns with the proposed pivotal quantities and their exact distributions as well as the construction of the predictive confidence intervals PCI's of future observations from *GEVD* based on gos. Suppose that $Y(1,n,\tilde{m},k),\cdots,Y(n,n,\tilde{m},k)$ are gos based on *GEVD* with cdf given by (2). Define the following three pivotal quantities

$$V_1 := V_1(r,s,n,\tilde{m},k) \quad = \quad \frac{X(s,n,\tilde{m},k)-X(r,n,\tilde{m},k)}{X(r,n,\tilde{m},k)-X(1,n,\tilde{m},k)}, \tag{3}$$

$$V_2 := V_2(r,s,n,\tilde{m},k) \quad = \quad \frac{X(s,n,\tilde{m},k)-X(r,n,\tilde{m},k)}{T_{r,n}}, \tag{4}$$

$$V_3 := V_3(r,s,n,\tilde{m},k) \quad = \quad \frac{X(s,n,\tilde{m},k)-X(r,n,\tilde{m},k)}{X(r,n,\tilde{m},k)}, \tag{5}$$

where

$$T_{r,n} = \sum_{i=1}^{r}\gamma_i(X(i,n,\tilde{m},k)-X(i-1,n,\tilde{m},k)), \tag{6}$$

$$X(i,n,\tilde{m},k) = H(Y(i,n,\tilde{m},k)), \quad i=1,2,\ldots,n, \tag{7}$$

and $H(t) = -\log(1 - F(t))$ is the cumulative hazard function.
Remark 2.1 In view of (7), it is not difficult to show that the random variables $X(i, n, \tilde{m}, k),\ i = 1, 2, ..., n$ can be expressed as gos based on the standard exponential distribution ($Exp(1)$) and therefore, the distributions of V_1, V_2, and V_3 are free of the GEVD parameters, μ, σ, and ξ.
The main aim of this section is to derive the exact distributions of V_1, V_2, and V_3 and the results are formulated in the following three theorems.

Theorem 2.1. *Assume that $Y(1, n, \tilde{m}, k), \cdots, Y(r, n, \tilde{m}, k)$ are the first observed gos based on GEVD with pdf (1). Then the exact cdf of the pivotal quantity V_1, $F_{V_1}(v_1)$, is given by*

$$F_{V_1}(v_1) = 1 - \frac{C_{s-1}}{\gamma_1} \sum_{i=r+1}^{s} \sum_{j=2}^{r} a_i^{(r)}(s)\, a_j^{(1)}(r)\ [\gamma_i\,(\gamma_j + \gamma_i\, v_1)]^{-1}, \quad v_1 \geq 0, \tag{8}$$

where,

$$C_{s-1} = \prod_{j=1}^{s} \gamma_j,\ a_i(r) = \prod_{\substack{j=1\\ j\neq i}}^{r} \frac{1}{\gamma_j - \gamma_i},\ 1 \leq i \leq r \leq n, \quad \text{and} \quad a_i^{(r)}(s) = \prod_{\substack{j=r+1\\ j\neq i}}^{s} \frac{1}{\gamma_j - \gamma_i},\ r+1 \leq i \leq s \leq n.$$

Furthermore, an observed $100(1-\delta)\%$ predictive confidence interval (PCI) for $Y(s, n, \tilde{m}, k),\ s > r$ is (ℓ, u_1), where $\ell = y_r$, and u_1 can be computed numerically from the relation

$$u_1 = H^{-1}\left(\log\left(\left(\frac{1 - F(y_1)}{1 - F(y_r)}\right)^{v_{1,\delta}} (1 - F(y_r))^{-1}\right)\right), \tag{9}$$

y_r is an observed value of $Y(r, n, \tilde{m}, k)$ and $v_{1,\delta}$ satisfies the nonlinear equation $F_{V_1}(v_{1,\delta}) = 1 - \delta$.

Proof. The joint pdf of $Y(r, n, \tilde{m}, k)$ and $Y(s, n, \tilde{m}, k)$, $f_{r,s}(y_r, y_s)$, was derived in Kamps and Cramer (22). Namely,

$$f_{r,s}(y_r, y_s) = C_{s-1,n} \sum_{i=r+1}^{s} \sum_{j=1}^{r} a_i^{(r)}(s) a_j(r) \left(\frac{\overline{F}(y_s)}{\overline{F}(y_r)}\right)^{\gamma_i} \left(\overline{F}(y_r)\right)^{\gamma_j} \frac{f(y_r)}{\overline{F}(y_r)} \frac{f(y_s)}{\overline{F}(y_s)}, \quad r < s \leqslant n,\ y_r < y_s. \tag{10}$$

For simplicity, we write Y_i instead of $Y(i, n, \tilde{m}, k)$ and X_i instead of $X(i, n, \tilde{m}, k)$. Therefore, by (7), the distribution of the subrange, $W_{r,s} = X_s - X_r$, can be obtained by Lemma 3 of Kamps and Cramer (22), namely,

$$f_{W_{r,s}(w_{r,s})} = \frac{C_{s-1}}{C_{r-1}} \sum_{i=r+1}^{s} a_i^{(r)}(s) e^{-\gamma_i w_{r,s}}, \quad w_{r,s} > 0. \tag{11}$$

Thus, we have

$$f_{W_{1,r}(w_{1,r})} = \frac{C_{r-1}}{\gamma_1} \sum_{i=2}^{r} a_i^{(1)}(r) e^{-\gamma_i w_{1,r}}, \quad w_{1,r} > 0. \tag{12}$$

Moreover, by Theorem 3.5.5 of (23), the subrange $W_{r,s}$ can be expressed as

$$W_{r,s} = \sum_{i=r+1}^{s} \left(X(i, n, \tilde{m}, k) - X(i-1, n, \tilde{m}, k)\right) = \sum_{i=r+1}^{s} Z_i/\gamma_i,$$

where the normalizing spacings Z_i, $i = 1, 2, ..., n$, are independent and identically distributed according to $Exp(1)$, which immediately implies, the independence between $W_{1,r}$ and $W_{r,s}$. Therefore, the joint pdf, $f_{W_{1,r}, W_{r,s}}(w_{1,r}, w_{r,s})$, of $W_{1,r}$ and $W_{r,s}$ is given by,

$$f_{W_{1,r}, W_{r,s}}(w_{1,r}, w_{r,s}) = \frac{C_{s-1}}{\gamma_1} \sum_{i=r+1}^{s} \sum_{j=2}^{r} a_i^{(r)}(s) a_j^{(1)}(r) e^{-(\gamma_j w_{1,r} + \gamma_i w_{r,s})}, \quad w_{1,r} > 0,\ w_{r,s} > 0.$$

By a standard method of transformations of random variables, it is not difficult to show that, the joint pdf, $f_{V_1,W_{1,r}}(v_1, w_{1,r})$, of $V_1 = W_{r,s}/W_{1,r}$ and $W_{1,r}$ is,

$$f_{V_1,W_{1,r}}(v_1, w_{1,r}) = \frac{C_{s-1}}{\gamma_1} \sum_{i=r+1}^{s} \sum_{j=2}^{r} a_i^{(r)}(s) a_j^{(1)}(r)\, w_{1,r}\, e^{-(\gamma_j+\gamma_i v_1)\, w_{1,r}}, \quad w_{1,r} > 0,\ v_1 > 0.$$

Thus we have,

$$f_{V_1}(v_1) = \int_0^\infty f_{V_1,W_{1,r}}(v_1, w_{1,r}) dw_{1,r} = \frac{C_{s-1}}{\gamma_1} \sum_{i=r+1}^{s} \sum_{j=2}^{r} a_i^{(r)}(s) a_j^{(1)}(r)(\gamma_j + \gamma_i v_1)^{-2}, v_1 > 0.$$

Hence (8) follows directly by evaluating the integration $\int_0^{v_1} f_{V_1}(u)du$. The limits of a $100(1-\delta)\%$ PCI of Y_s can be obtained by noting that $F_{V_1}(v_{1,\delta}) = Pr(V_1 \leqslant v_{1,\delta}) = 1-\delta$. Which can be rewritten as

$$Pr(Y_r \leqslant Y_s \leqslant H^{-1}(H(Y_r) + v_{1,\delta}(H(Y_r) - H(Y_1))) = 1 - \delta. \tag{13}$$

Clearly, the lower limit of an observed gos sample is $\ell = y_r$ and an approximate upper limit, u_1 can be accomplished by solving the nonlinear equation (9). Hence the theorem. □

Theorem 2.2. *Under the same conditions of Theorem 2.1, the exact cdf of the pivotal quantity* V_2, $F_{V_2}(v_2)$, *can be written as,*

$$F_{V_2}(v_2) = 1 - \frac{C_{s-1}}{C_{r-1}} \sum_{i=r+1}^{s} \frac{a_i^{(r)}(s)}{\gamma_i}(1 + \gamma_i\, v_2)^{-r}, \quad v_2 \geq 0., \tag{14}$$

Moreover, a $100(1-\delta)\%$ *observed predictive confidence interval (PCI) for* $Y(r, n, \tilde{m}, k)$, $r < s$ *is* (ℓ, u_2), *where* $\ell = y_r$, *and* u_2 *can be calculated numerically from the relation*

$$u_2 = H^{-1}\left(-\log(1 - F(y_r)) + t_{r,n} v_{2,\delta}\right),$$

y_r *is an observed value of* $Y(r, n, \tilde{m}, k)$, $t_{r,n}$ *is an observed value of* $T_{r,n}$ *and* $v_{2,\delta}$ *satisfies the nonlinear equation* $F_{V_2}(v_{2,\delta}) = 1 - \delta$.

Theorem 2.3. *Under the same conditions of Theorem 2.1, the exact cdf of the pivotal quantity* V_3, $F_{V_3}(v_3)$, *is given by*

$$F_{V_3}(v_3) = 1 - C_{s-1} \sum_{i=r+1}^{s} \sum_{j=1}^{r} \frac{a_i^{(r)}(s)\, a_j(r)}{\gamma_i}(\gamma_j + \gamma_i\, v_3)^{-1}, \quad v_3 \geq 0, \tag{15}$$

Consequently, an observed $100(1-\delta)\%$ *predictive confidence interval (PCI) for* $Y(s, n, \tilde{m}, k)$, $s > r$ *is* (ℓ, u_3), *where* $\ell = y_r$, *and* u_3 *can be computed numerically from the relation*

$$u_3 = H^{-1}\left((1 + v_{3,\delta})(-\log(1 - F(y_r)))\right), \tag{16}$$

y_r *is an observed value of* $Y(r, n, \tilde{m}, k)$ *and* $v_{3,\delta}$ *satisfies the nonlinear equation* $F_{V_3}(v_{3,\delta}) = 1 - \delta$.

The proof of Theorems 2.2 and 2.3 are similar to the proof of Theorem 2.1 with suitable modifications.

3. Simulation

In this section, two special cases from gos model are considered to investigate the efficiency of the theoretical results given in the preceding section, by executing a simulation study. The selected models are,

1. oos with $\gamma_i = n - i + 1$ for $n = 20,\ r = 12, 14, 16$ and $s = r + 1, r + 2, ..., n - 1$,
2. sos with $\gamma_n = \alpha_n,\ \gamma_i = 2(n - i) + 1$ for $n = 20,\ r = 12, 14, 16$ and $s = r + 1, r + 2, ..., n - 1$.

Clearly, the upper limit, U_1, of the PCI of the gos $Y(s, n, \tilde{m}, k)$, based on the pivotal quantity V_1, is a function of $Y(r, n, \tilde{m}, k)$. Therefore, the expected value of the upper limit of a $100(1-\delta)\%$ PCI of $Y(s, n, \tilde{m}, k)$, can be obtained numerically from the relation $E[U_1] = \int_{\mu-\frac{\sigma}{\xi}}^{\infty} u_1 g_{Y(r,n,\tilde{m},k)}(y_r)\, dy$, where u_1 is defined by (9) and

$$g_{Y(r,n,\tilde{m},k)}(y_r) = C_{r-1} \sum_{i=1}^{r} a_i^{(r)} (1 - F(y_r))^{\gamma_i - 1} f(y_r),$$

with $g_{Y(r,n,\tilde{m},k)}(y_r)$ is the pdf of $Y(r, n, \tilde{m}, k)$ (cf. Kamps and Cramer (22)).

The estimated root mean square error (ERMSE) for the upper PCI, based on $V_i,\ i = 1, 2, 3$, are obtained from the relation

$$ERMSE_{V_i} = \left[\frac{1}{M-1} \sum_{j=1}^{M} (U_{V_i}(j) - Y_{s+1}^*(j))^2 \right]^{\frac{1}{2}}, \quad i = 1, 2, 3. \tag{17}$$

where $U_{V_i}(j),\ i = 1, 2, 3$, denote the upper limits for the PCI of the j^{th} sample, and $Y_i(j)$ denote the i^{th} gos for the j^{th} sample, $i = r - 1$ or $s + 1$. The computations are conducted by Mathematica 10 and the results are presented in Tables 1 and 2.

4. Illustrative Examples

In this section, we illustrate the proposed procedure in Section 2 via analysis of two real data sets.

Example 4.1 (Maximum annual temperature).

The following data from Long Beach, California, represents the maximum annual temperature from 1990 to 2012:

86.7	81.7	84.3	86.4	84.9	85.1	89.7	82.3	84.2	85.8	82.4	81.5
84.3	84.1	90.5	89.4	87.5	88.4	90.3	84.1	88.4	83.0	86.6	

An application to Kolmogorov-Smirnov $(K - S)$ test for the complete data reveals that the two-parameter extreme value distribution is adequate model for this data. The observed value of $(K - S)$ test statistic is 0.1147 and the associated $p-$value is 0.8895. The modified least square method (MLSM) for type II right censoring samples (see El-Adll and Aly (17), and Aly (5)), is applied to the previous data, to obtain estimates of the unknown distribution parameters, in the following two situations:
(a) For $n = 23$ and $r = 15$, $\hat{\mu} = 84.4399$ and $\hat{\sigma} = 2.44168$, **(b)** while $n = 26$ and $r = 22$, yield $\hat{\mu} = 84.3723$ and $\hat{\sigma} = 2.81359$. Table 3 summarize the prediction results for the maximum annual temperature data.

Example 4.2 (Sulfur Dioxide (1-Hour Averages)).

The second set of data were obtained through the courtesy of the South Coast Air Pollution Control District (SCAPCD) of the State of California which was analyzed by Roberts (31). The annual maxima of sulfur dioxide $1 - hr$ average concentrations (pphm) are,

47 41 68 32 27 43 20 27 25 18 33 40 51 55 40 55 37 28 34.

(Long Beach, CA from 1956 to 1974, Data Courtesy South Coast Air Pollution Control District)
First of all, it is shown that the two-parameter extreme value distribution fit the data very well (the value of $K - S$ test statistic is 0.1001 and the associated $p-$value is 0.9809). Moreover, an application to MLSM yields,
(a) $\hat{\mu} = 31.6186$ and $\hat{\sigma} = 11.9152$, for $n = 19$ and $r = 13$, **(b)** $\hat{\mu} = 35.8844$ and $\hat{\sigma} = 18.6466$, for $n = 24$ and $r = 17$. The prediction intervals of future oos $Y_{s:n},\ s = r + 1, ..., n$ with $r = 13(17)$ and $n = 19(24)$ are shown in Table 4.

Table 1: 90% coverage probability, average lower limits, $\bar{Y}_s,\ s = r+1,...,18,\ r = 12,14,16$, expected upper limits based on V_1, simulated average upper limits and root mean square errors based on V_1, V_2 and V_3, respectively, for oos model from $GEVD(6.2, 1.2, 0.5)$.

r	s	CP_{V_1}	CP_{V_2}	CP_{V_3}	$L=\bar{Y}_r$	$\bar{Y}_s$	$E[U_{V_1}]$	U_{V_1} ($ERMSE_{V_1}$)	U_{V_2} ($ERMSE_{V_2}$)	U_{V_3} ($ERMSE_{V_3}$)
12	13	89.858	89.885	89.865	7.10071	7.38735	7.85667	7.87626 (0.58544)	7.86093 (0.56742)	7.86784 (0.57623)
	14	89.958	90.074	89.955	7.10071	7.73163	8.60011	8.63904 (1.06382)	8.59907 (1.00444)	8.61676 (1.03301)
	15	89.965	89.887	89.872	7.10071	8.16233	9.54281	9.61310 (1.76750)	9.53254 (1.63481)	9.56785 (1.69879)
	16	90.008	89.979	89.987	7.10071	8.72005	10.86478	10.99009 (2.89677)	10.83800 (2.62623)	10.90425 (2.75659)
	17	90.052	90.023	90.025	7.10071	9.50494	12.92836	13.16206 (4.95794)	12.86921 (4.38913)	12.99629 (4.66439)
	18	90.082	90.095	90.159	7.10071	10.74222	16.70647	17.19924 (9.65554)	16.57234 (8.29131)	16.84385 (8.95637)
14	15	90.003	89.893	89.970	7.73163	8.16233	8.83346	8.89959 (0.93286)	8.87176 (0.90171)	8.88949 (0.92242)
	16	90.071	90.031	90.085	7.73163	8.72005	10.07398	10.19435 (1.81459)	10.11518 (1.69398)	10.16486 (1.77361)
	17	90.134	90.100	90.133	7.73163	9.50494	11.90520	12.12981 (3.41474)	11.94707 (3.10068)	12.06075 (3.30694)
	18	90.170	90.041	90.159	7.73163	10.74222	15.14771	15.61719 (7.06251)	15.18572 (6.25434)	15.45247 (6.78200)
16	17	90.014	89.998	89.978	8.72005	9.50494	10.58802	10.92243 (2.08276)	10.84111 (2.00597)	10.90534 (2.06707)
	18	90.021	90.077	90.010	8.72005	10.74222	13.41889	14.12340 (5.15911)	13.83343 (4.78010)	14.05984 (5.07909)

5. Concluding Remarks

In this work three pivotal quantities have been proposed to construct prediction intervals of the future gos based on GEVD. The exact distributions of the pivotal quantities are obtained. Simulation studies are conducted to compare the pivotal quantities and two real data sets have been analyzed for illustrative purposes. In view of the results obtained in the preceding sections, the following remarks are reported.

1. The upper limits are closed to each other based on the three pivotal quantities,
2. based on the ERMSE, the pivotal quantity V_2 is better than V_1 and V_3,
3. the values of the ERMSE's increase with $s-r$,
4. according to the results of Section 4, good fitting for real data improve the prediction results.

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Table 2: 90% coverage probability, average lower limits, $\bar{Y}_s,\ s = r+1, ..., 18,\ r = 12, 14, 16$, expected upper limits based on V_1, simulated average upper limits and root mean square errors based on V_1, V_2 and V_3, respectively, for sos model from $GEVD(6.2, 1.2, 0.5)$.

r	s	CP_{V_1}	CP_{V_2}	CP_{V_3}	$L = \bar{Y}_r$	$\bar{Y}_s$	$E[U_{V_1}]$	U_{V_1} $(ERMSE_{V_1})$	U_{V_2} $(ERMSE_{V_2})$	U_{V_3} $(ERMSE_{V_3})$
12	13	90.093	90.085	90.092	6.19528	6.33382	6.55341	6.55510 (0.25180)	6.54922 (0.24571)	6.55175 (0.24856)
	14	90.064	90.017	90.037	6.19528	6.49477	6.86905	6.87601 (0.41822)	6.86188 (0.39989)	6.86796 (0.40836)
	15	89.963	89.934	89.893	6.19528	6.68762	7.23782	7.25162 (0.63128)	7.22577 (0.59520)	7.23690 (0.61190)
	16	89.961	89.967	89.966	6.19528	6.92780	7.71211	7.73596 (0.92743)	7.69242 (0.86384)	7.71116 (0.89326)
	17	89.986	90.033	90.001	6.19528	7.24998	8.38590	8.42643 (1.39119)	8.35340 (1.28201)	8.38485 (1.33264)
	18	90.070	90.010	90.022	6.19528	7.73077	9.50071	9.57446 (2.31685)	9.44280 (2.12605)	9.49957 (2.21489)
14	15	89.955	89.892	89.943	6.49477	6.68762	6.98904	6.99129 (0.36213)	6.98198 (0.35284)	6.98776 (0.35882)
	16	89.917	89.939	89.939	6.49477	6.92780	7.47488	7.48274 (0.62967)	7.45863 (0.59846)	7.47355 (0.61848)
	17	89.965	89.977	89.980	6.49477	7.24998	8.12870	8.14593 (1.04512)	8.09670 (0.97806)	8.12711 (1.02112)
	18	89.958	90.008	89.991	6.49477	7.73077	9.18461	9.22079 (1.89024)	9.12115 (1.76522)	9.18265 (1.84536)
16	17	89.953	89.995	89.969	6.92780	7.24998	7.76209	7.77258 (0.70168)	7.74839 (0.68236)	7.76746 (0.69760)
	18	90.019	90.030	90.031	6.92780	7.73077	8.82007	8.84627 (1.54274)	8.77099 (1.47362)	8.83004 (1.52843)

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Table 3: 90% lower and upper limits for future s^{th} oos, $Y_{n:s},\ s = r+1, ..., n,\ r = 15(22)$ for $n = 23(26)$.

r	s	$L = Y_r$	Y_s	Y_{s+1}	U_{V_1}	U_{V_2}	U_{V_3}
15	16	86.6	86.7	87.5	87.58532	87.52367	87.57103
	17	86.6	87.5	88.4	88.37875	88.26495	88.35125
	18	86.6	88.4	88.4	89.21962	89.04956	89.17749
	19	86.6	88.4	89.4	90.18260	89.94724	90.12316
	20	86.6	89.4	89.7	91.35799	91.04197	91.27715
	21	86.6	89.7	90.3	92.91509	92.49105	92.80602
	22	86.6	90.3	90.5	95.28334	94.69436	95.13169
	23	86.6	90.5	-	100.28658	99.35314	100.04686
22	23	90.3	90.5	–	92.49779	92.46942	92.51020
	24	90.3	–	–	94.66616	94.59468	94.68829
	25	90.3	–	–	97.77715	97.63917	97.81248
	26	90.3	–	–	104.22420	103.95804	104.28938

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Table 4: 90% lower and upper limits for future s^{th} oos, $Y_{n:s},\ s = r+1, ..., n,\ r = 13(17)$ for $n = 19(24)$.

r	s	$L = Y_r$	Y_s	Y_{s+1}	U_{V_1}	U_{V_2}	U_{V_3}
13	14	41.0	43.0	47.0	46.63547	46.74297	46.57447
	15	41.0	47.0	51.0	51.41556	51.57707	51.28873
	16	41.0	51.0	55.0	56.86035	57.07243	56.65356
	17	41.0	55.0	55.0	63.83946	64.11394	63.52728
	18	41.0	55.0	68.0	74.23581	74.61280	73.76649
	19	41.0	68.0	-	95.85188	96.50262	95.07168
17	18	55.0	55.0	68.0	62.60153	62.49035	62.79446
	19	55.0	68.0	-	68.88672	68.65183	69.22139
	20	55.0	-	-	75.77472	75.39272	76.25917
	21	55.0	-	-	84.04004	83.47626	84.70160
	22	55.0	-	-	94.89254	94.08745	95.79309
	23	55.0	-	-	111.31402	110.14885	112.57976
	24	55.0	-	-	145.90869	144.04873	147.98882

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