Prediction Intervals of Future Generalized Order Statistics Based on Generalized Extreme Value Distribution

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Abstract. In this paper, three prediction intervals of future generalized order statistics (gos) based on generalized extreme value distribution (GEVD) are constructed. For this purpose, three predictive pivotal quantities are considered, and their exact distributions are established. A Monte Carlo simulation study is carried out to explore the efficiency of the proposed method. The obtained results are then applied to real data for illustrative purposes.

1. Introduction

The prediction of future events without any doubt is one of the most important problems in statistics. This problem has been extensively studied by many authors, including Lingappaiah (25), Aitchison and Dunsmore (3), Lawless (26; 27), Kaminsky and Rhodin (20), Kaminsky and Nelson (21), Patel (28), Raqab (4), among others.

It is well known that the ordered random variables play an important role in prediction methods. Since Kamps (23) had introduced the concept of gos as a unification of several models of ascendingly ordered random variables, the use of such concept has been steadily growing through the years. This is due to the fact that such concept includes important well-known models of ordered random variables that have been treated separately in statistical literature. Kamps (23) defined gos first by defining uniform gos and then using the quantile transformation to obtain the joint probability density function (jpdf) of the random variables $Y(1,n,\hat{m},k), \cdots , Y(n,n,\hat{m},k)$ based on cumulative distribution function (cdf) $F$ with pdf $f$. The jpdf of $Y(1,n,\hat{m},k), \cdots , Y(n,n,\hat{m},k)$ is given by

$$f^{Y(1,n,\hat{m},k),\cdots ,Y(n,n,\hat{m},k)}(y_1, \cdots , y_n) = k \left( \prod_{j=1}^{n-1} \gamma_j \right) \left( \prod_{i=1}^{n-1} (1 - F(y_i))^{m_i} f(y_i) \right) (1 - f(y_n))^{k-1} f(y_n),$$

on the cone $F^{-1}(0) \leq y_1 \leq \cdots \leq y_n \leq F^{-1}(1-) \subseteq \mathbb{R}^n$. The model parameters are, $n \in \mathbb{N}$, $n \geq 2$, $k > 0$, $\hat{m} = (m_1, \cdots , m_{n-1}) \in \mathbb{R}^{n-1}$, $M_r = \sum_{j=r}^{n-1} m_j$, such that $\gamma_r = k + n - r + M_r > 0$ for all $r \in \{1, \cdots , n-1\}$ and $\gamma_n = k$. Particular choices of the parameters $\gamma_1, \cdots , \gamma_n$ lead to different models, e.g., $m$-gos ($\gamma_n = k, \gamma_r = k + (n-r)(m+1), r = 1, \cdots , n-1$), oos ($\gamma_n = 1, \gamma_r = n - r + 1, r = 1, \cdots , n-1$), i.e., $k = 1, m_i = 0, i = 1, \cdots , n-1$, sos ($\gamma_n = \alpha_n, \gamma_r = (n-r+1)\alpha_r, \alpha_r > 0, r = 1, \cdots , n-1$), pos with censoring scheme $(R_1, \cdots , R_M)$ ($\gamma_n = R_M + 1, \gamma_r = n - r + 1 + \sum_{j=r}^{M} R_j$, if $r \leq M - 1$ and

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\[\gamma_r = n - r + 1 + R_M, \text{ if } r \geq M\] and upper records \((\gamma_r = 1, 1 \leq r \leq n, \text{i.e., } k = 1, m_i = -1, i = 1, \ldots, n-1)\). Therefore, all the results obtained for the gos model can be applied to the particular models choosing the respective parameters. For more details in the theory and applications of gos see Kamps (23), Ahsanullah (2), Kamps and Cramer (22), Cramer (13), Barakat et al (7), El-Adll (15), Barakat (6; 9), and Ahmad et al (1).

Extreme value data usually is explored from heavy right tailed or excess kurtosis. Particularly in environmental data, e.g., spatial and temporal variability of turbulence (Sanford, (32)), daily maximum ozone measurement (Gilleland, (18)), maximum wind speed (Castillo et al. (11)), largest lichen measurements (Cooley et al., (12)), and maximum water level (Bruzer et al., (10)). GEVD was originally defined for the first time by Jenkinson (19)), and the three possible limiting distributions of the maximum/minimum of random variables are embedded within it. This distribution is also known as the von Mises extreme value, von Mises-Jenkinson, and Fisher-Tippet distribution. A historical review of extreme value theory is provided in Kotz and Nadajarah (24). Recently, a comparative review of GEVD is reported in Pinheiro et al. (29).

The GEVD distribution is one of the most widely applied models for univariate extreme values. Its pdf and cdf are, respectively, given by

\[
f(y; \mu, \sigma, \xi) = \frac{1}{\sigma} \left[ 1 + \xi \left( \frac{y - \mu}{\sigma} \right) \right] \left( \frac{1}{\xi} - 1 \right)^{\frac{1}{2}} \exp \left\{ - \left[ 1 + \xi \left( \frac{y - \mu}{\sigma} \right) \right]^{-\frac{1}{\xi}} \right\}, \quad 1 + \xi(y - \mu)/\sigma > 0 \tag{1}\]

and

\[
F(y; \mu, \sigma, \xi) = \exp \left\{ - \left[ 1 + \xi \left( \frac{y - \mu}{\sigma} \right) \right]^{-\frac{1}{\xi}} \right\}, \quad 1 + \xi(y - \mu)/\sigma \geq 0, \tag{2}\]

where \(\mu \in \mathbb{R}\) is the location parameter, \(\sigma > 0\) is the scale parameter and \(\xi \in \mathbb{R}\) is the shape parameter. The shape parameter \(\xi\) governs the tail behavior of the distribution. The Weibull, Gumbel, Frechet sub-families are corresponding, respectively, to \(\xi < 0\), \(\xi \to 0\), and \(\xi > 0\).

The rest of this paper is organized as follows. In Section 2, the predictive pivotal quantities and their exact distributions are obtained. Section 3, include simulation studies. Some applications to real data are presented in Section 4.

2. Pivotal Quantities and Their Distributions

This section concerns with the proposed pivotal quantities and their exact distributions as well as the construction of the predictive confidence intervals PCI’s of future observations from GEVD based on gos. Suppose that \(Y(1, n, \tilde{m}, k), \ldots, Y(n, n, \tilde{m}, k)\) are gos based on GEVD with cdf given by (2). Define the following three pivotal quantities

\[
V_1 \ : \ V_1(r, s, n, \tilde{m}, k) = \frac{X(s, n, \tilde{m}, k) - X(r, n, \tilde{m}, k)}{X(r, n, \tilde{m}, k) - X(1, n, \tilde{m}, k)}, \quad \tag{3}
\]

\[
V_2 \ : \ V_2(r, s, n, \tilde{m}, k) = \frac{X(s, n, \tilde{m}, k) - X(r, n, \tilde{m}, k)}{T_{r,n}}, \tag{4}
\]

\[
V_3 \ : \ V_3(r, s, n, \tilde{m}, k) = \frac{X(s, n, \tilde{m}, k) - X(r, n, \tilde{m}, k)}{X(r, n, \tilde{m}, k)}, \tag{5}
\]

where

\[
T_{r,n} = \sum_{i=1}^{r} \gamma_i (X(i, n, \tilde{m}, k) - X(i - 1, n, \tilde{m}, k)), \quad \tag{6}
\]

\[
X(i, n, \tilde{m}, k) = H(Y(i, n, \tilde{m}, k)), \quad i = 1, 2, \ldots, n, \tag{7}
\]
and $H(t) = -\log(1 - F(t))$ is the cumulative hazard function.

**Remark 2.1** In view of (7), it is not difficult to show that the random variables $X(i, n, \tilde{m}, k)$, $i = 1, 2, ..., n$ can be expressed as gos based on the standard exponential distribution ($\text{Exp}(1)$) and therefore, the distributions of $V_1$, $V_2$, and $V_3$ are free of the GEVD parameters, $\mu$, $\sigma$, and $\xi$.

The main aim of this section is to derive the exact distributions of $V_1$, $V_2$, and $V_3$ and the results are formulated in the following three theorems.

**Theorem 2.1.** Assume that $Y(1, n, \tilde{m}, k), \ldots, Y(r, n, \tilde{m}, k)$ are the first observed gos based on GEVD with pdf (1). Then the exact cdf of the pivotal quantity $V_1$, $F_{V_1}(v_1)$, is given by

$$
F_{V_1}(v_1) = 1 - \frac{C_{s-1}}{\gamma_1} \sum_{i=r+1}^{s} \prod_{j=2}^{r} a_i^{(r)}(s) a_j^{(1)}(r) [\gamma_i (\gamma_j + \gamma_v v_1)]^{-1}, \quad v_1 \geq 0,
$$

where,

$$
C_{s-1} = \prod_{j=1}^{s} \gamma_j, \quad a_i(r) = \prod_{j=1}^{r} \frac{1}{\gamma_j - \gamma_i}, \quad 1 \leq i \leq r \leq n, \quad \text{and} \quad a_i^{(r)}(s) = \prod_{j=1}^{r} \frac{1}{\gamma_j - \gamma_i}, \quad r + 1 \leq i \leq s \leq n.
$$

Furthermore, an observed $100(1 - \delta)$% predictive confidence interval (PCI) for $Y(s, n, \tilde{m}, k)$, $s > r$ is $(\ell, u_\delta)$, where $\ell = y_r$, and $u_\delta$ can be computed numerically from the relation

$$
u_i = H^{-1} \left( \log \left( \left( \frac{1 - F(y_1)}{1 - F(y_r)} \right)^{v_i, \delta} \right) \right),$$

$y_\delta$ is an observed value of $Y(r, n, \tilde{m}, k)$ and $v_{i, \delta}$ satisfies the nonlinear equation $F_{V_1}(v_{i, \delta}) = 1 - \delta$.

**Proof.** The joint pdf of $Y(r, n, \tilde{m}, k)$ and $Y(s, n, \tilde{m}, k)$, $f_{r,s}(y_r, y_s)$, was derived in Kamps and Cramer (22). Namely,

$$
f_{r,s}(y_r, y_s) = C_{s-1,n} \sum_{i=r+1}^{s} \prod_{j=2}^{r} a_i^{(r)}(s) a_j^{(1)}(r) \left( \frac{F(y_s)}{F(y_r)} \right)^{\gamma_i} \left( \frac{1 - F(y_s)}{1 - F(y_r)} \right)^{\gamma_j} f(y_r) f(y_s) F(y_r) F(y_s), \quad r \leq s \leq n, \quad y_r < y_s. \quad (10)
$$

For simplicity, we write $Y_1$ instead of $Y(i, n, \tilde{m}, k)$ and $X_1$ instead of $X(i, n, \tilde{m}, k)$. Therefore, by (7), the distribution of the sub-range, $W_{r,s} = X_s - X_r$, can be obtained by Lemma 3 of Kamps and Cramer (22), namely,

$$
f_{W_{r,s}}(w_{r,s}) = \frac{C_{s-1}}{\gamma_1} \sum_{i=r+1}^{s} a_i^{(r)}(s) e^{-\gamma_i w_{r,s}}, \quad w_{r,s} > 0. \quad (11)
$$

Thus, we have

$$f_{W_{1,r}}(w_{1,r}) = \frac{C_{r-1}}{\gamma_1} \sum_{i=2}^{r} a_i^{(1)}(r) e^{-\gamma_i w_{1,r}}, \quad w_{1,r} > 0. \quad (12)
$$

Moreover, by Theorem 3.5.5 of (23), the sub-range $W_{r,s}$ can be expressed as

$$W_{r,s} = \sum_{i=r+1}^{s} (X(i, n, \tilde{m}, k) - X(i-1, n, \tilde{m}, k)) = \sum_{i=r+1}^{s} Z_i/\gamma_i,$$

where the normalizing spacings $Z_i$, $i = 1, 2, ..., n$, are independent and identically distributed according to $\text{Exp}(1)$, which immediately implies, the independence between $W_{1,r}$ and $W_{r,s}$. Therefore, the joint pdf, $f_{W_{1,r}, W_{r,s}}(w_{1,r}, w_{r,s})$, of $W_{1,r}$ and $W_{r,s}$ is given by,

$$f_{W_{1,r}, W_{r,s}}(w_{1,r}, w_{r,s}) = \frac{C_{s-1}}{\gamma_1} \sum_{i=r+1}^{s} \prod_{j=2}^{r} a_i^{(r)}(s) a_j^{(1)}(r) e^{-(\gamma_j w_{1,r} + \gamma_i w_{r,s})}, \quad w_{1,r} > 0, \quad w_{r,s} > 0.
By a standard method of transformations of random variables, it is not difficult to show that, the joint pdf, $f_{V_1,W_1,r}(v_1,w_1,r)$, of $V_1 = W_{r,s}/W_{1,r}$ and $W_{1,r}$ is,

$$f_{V_1,W_1,r}(v_1,w_1,r) = \frac{C_{s-1}}{\gamma_1} \sum_{i=r+1}^{s} \sum_{j=2}^{r} a_{i}^{(r)}(s) a_{j}^{(1)}(r) w_{1,r} e^{-(\gamma_j + \gamma_i) v_1}, \quad w_{1,r} > 0, \quad v_1 > 0.$$

Thus we have,

$$f_{V_1}(v_1) = \int_{0}^{\infty} f_{V_1,W_1,r}(v_1,w_1,r) dw_{1,r} = \frac{C_{s-1}}{\gamma_1} \sum_{i=r+1}^{s} \sum_{j=2}^{r} a_{i}^{(r)}(s) a_{j}^{(1)}(r)(\gamma_j + \gamma_i v_1)^{-2}, \quad v_1 > 0.$$

Hence (8) follows directly by evaluating the integration $\int_{0}^{1} f_{V_1}(u) du$. The limits of a 100(1 − δ)% PCI of $Y$ can be obtained by noting that

$$Pr(Y_r \leq Y_s \leq H^{-1}(H(Y_r) + v_{1,s}(H(Y_r) - H(Y_1))) = 1 - \delta.$$

Clearly, the lower limit of an observed gos sample is $\ell = y_r$ and an approximate upper limit, $u_1$ can be accomplished by solving the nonlinear equation (9). Hence the theorem.

**Theorem 2.2.** Under the same conditions of Theorem 2.1, the exact cdf of the pivotal quantity $V_2$, $F_{V_2}(v_2)$, can be written as,

$$F_{V_2}(v_2) = 1 - \frac{C_{s-1}}{C_{r-1}} \sum_{i=r+1}^{s} \sum_{j=1}^{r} a_{i}^{(r)}(s) a_{j}^{(1)}(r)(1 + \gamma_i v_2)^{-r}, \quad v_2 \geq 0,.$$

Moreover, a 100(1 − δ)% observed predictive confidence interval (PCI) for $Y(r,n,\hat{m},k)$, $r < s$ is $(\ell, u_2)$, where $\ell = y_r$, and $u_2$ can be calculated numerically from the relation

$$u_2 = H^{-1}(-\log(1 - F(y_r)) + t_{r,n}\gamma_{3,s}),$$

$y_r$ is an observed value of $Y(r,n,\hat{m},k)$, $t_{r,n}$ is an observed value of $T_{r,n}$ and $v_{3,s}$ satisfies the nonlinear equation $F_{V_2}(v_{3,s}) = 1 - \delta$.

**Theorem 3.3.** Under the same conditions of Theorem 2.1, the exact cdf of the pivotal quantity $V_3$, $F_{V_3}(v_3)$, is given by

$$F_{V_3}(v_3) = 1 - C_{s-1} \sum_{i=r+1}^{s} \sum_{j=1}^{r} a_{i}^{(r)}(s) a_{j}^{(1)}(r)(\gamma_j + \gamma_i v_3)^{-1}, \quad v_3 \geq 0,$$

Consequently, an observed 100(1 − δ)% predictive confidence interval (PCI) for $Y(s,n,\hat{m},k)$, $s > r$ is $(\ell, u_3)$, where $\ell = y_r$, and $u_3$ can be computed numerically from the relation

$$u_3 = H^{-1}((1 + v_{3,s})(-\log(1 - F(y_r)))),$$

$y_r$ is an observed value of $Y(r,n,\hat{m},k)$ and $v_{3,s}$ satisfies the nonlinear equation $F_{V_3}(v_{3,s}) = 1 - \delta$.

The proof of Theorems 2.2 and 3.3 are similar to the proof of Theorem 2.1 with suitable modifications.

3. Simulation

In this section, two special cases from gos model are considered to investigate the efficiency of the theoretical results given in the preceding section, by executing a simulation study. The selected models are,
1. oos with $\gamma_i = n - i + 1$ for $n = 20$, $r = 12,14,16$ and $s = r + 1, r + 2, ..., n - 1$ 
2. sos with $\gamma_n = \alpha_n$, $\gamma_i = 2(n - i) + 1$ for $n = 20$, $r = 12,14,16$ and $s = r + 1, r + 2, ..., n - 1$.

Clearly, the upper limit, $U_1$, of the PCI of the gos $Y(s, n, \tilde{n}, k)$, based on the pivotal quantity $V_1$, is a function of $Y(r, n, \tilde{n}, k)$. Therefore, the expected value of the upper limit of a 100$(1 - \delta)$% PCI of $Y(s, n, \tilde{n}, k)$, can be obtained numerically from the relation $E[U_1] = \int_{\xi}^{\infty} u_1 g_{Y(r, n, \tilde{n}, k)}(y_r) \, dy$, where $u_1$ is defined by (9) and $g_{Y(r, n, \tilde{n}, k)}(y_r) = C_r - 1 \sum_{i=1}^{r} a_i^{(r)} (1 - F(y_r))^{-\gamma_r} f(y_r)$, with $g_{Y(r, n, \tilde{n}, k)}(y_r)$ is the pdf of $Y(r, n, \tilde{n}, k)$ (cf. Kamps and Cramer (22)).

The estimated root mean square error (ERMSE) for the upper PCI, based on $V_i$, $i = 1,2,3$, are obtained from the relation

$$\text{ERMSE}_{V_i} = \left[ \frac{1}{M - 1} \sum_{j=1}^{M} (U_V(j) - \hat{Y}_{s+1}(j))^2 \right]^{\frac{1}{2}}, \quad i = 1,2,3. \quad (17)$$

where $U_V(j), i = 1,2,3$, denote the upper limits for the PCI of the $j^{th}$ sample, and $V_i(j)$ denote the $i^{th}$ gos for the $j^{th}$ sample, $i = r - 1$ or $s + 1$. The computations are conducted by Mathematica 10 and the results are presented in Tables 1 and 2.

4. Illustrative Examples

In this section, we illustrate the proposed procedure in Section 2 via analysis of two real data sets.

Example 4.1 (Maximum annual temperature).

The following data from Long Beach, California, represents the maximum annual temperature from 1990 to 2012:

86.7 81.7 84.3 86.4 84.9 85.1 89.7 82.3 84.2 85.8 82.4 81.5
84.3 84.1 90.5 89.4 87.5 88.4 90.3 84.1 88.4 83.0 86.6

An application to Kolmogorov-Smirnov ($K - S$) test for the complete data reveals that the two-parameter extreme value distribution is adequate model for this data. The observed value of ($K - S$) test statistic is 0.1147 and the associated $p$-value is 0.8895. The modified least square method (MLSM) for type II right censoring samples (see El-Adll and Aly (17), and Aly (5)), is applied to the previous data, to obtain estimates of the unknown distribution parameters, in the following two situations:

(a) For $n = 23$ and $r = 15$, $\hat{\mu} = 84.3499$ and $\hat{\sigma} = 2.44168$, (b) while $n = 26$ and $r = 22$, yield $\hat{\mu} = 84.3723$ and $\hat{\sigma} = 2.81359$. Table 3 summarize the prediction results for the maximum annual temperature data.

Example 4.2 (Sulfur Dioxide (1-Hour Averages)).

The second set of data were obtained through the courtesy of the South Coast Air Pollution Control District (SCAPCD) of the State of California which was analyzed by Roberts (31). The annual maxima of sulfur dioxide $1 - hr$ average concentrations (pphm) are,

47 41 68 32 27 43 20 27 25 18 33 40 51 55 40 55 37 28 34.

(Long Beach, CA from 1956 to 1974, Data Courtesy South Coast Air Pollution Control District)

First of all, it is shown that the two-parameter extreme value distribution fit the data very well (the value of $K - S$ test statistic is 0.1001 and the associated $p$-value is 0.9809 ). Moreover, an application to MLSM yields,

(a) $\hat{\mu} = 31.6186$ and $\hat{\sigma} = 11.9152$, for $n = 19$ and $r = 13$, (b) $\hat{\mu} = 35.8844$ and $\hat{\sigma} = 18.6466$, for $n = 24$ and $r = 17$. The prediction intervals of future oos $Y_{s,n}, s = r + 1, ..., n$ with $r = 13(17)$ and $n = 19(24)$ are shown in Table 4.
Table 1: 90\% coverage probability, average lower limits, $\bar{Y}_s$, $s = r + 1, \ldots, 18$, $r = 12,14,16$, expected upper limits based on $V_1$, simulated average upper limits and root mean square errors based on $V_1$, $V_2$ and $V_3$, respectively, for oos model from GEVD($6.2,1.2,0.5$).

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5. Concluding Remarks

In this work three pivotal quantities have been proposed to construct prediction intervals of the future gos based on GEVD. The exact distributions of the pivotal quantities are obtained. Simulation studies are conducted to compare the pivotal quantities and two real data sets have been analyzed for illustrative purposes. In view of the results obtained in the preceding sections, the following remarks are reported.

1. The upper limits are closed to each other based on the three pivotal quantities,
2. based on the ERMSE, the pivotal quantity $V_2$ is better than $V_1$ and $V_3$,
3. the values of the ERMSE’s increase with $s - r$,
4. according to the results of Section 4, good fitting for real data improve the prediction results.

Acknowledgements.
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Table 2: 90% coverage probability, average lower limits, $\bar{Y}_s$, $s = r + 1, \ldots, 18$, $r = 12, 14, 16$, expected upper limits based on $V_1$, simulated average upper limits and root mean square errors based on $V_1$, $V_2$ and $V_3$, respectively, for sos model from GEVD(6.2, 1.2, 0.5).

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References

Table 3: 90% lower and upper limits for future \( s^{th} \) order, \( Y_{n,s}, s = r + 1, \ldots, n \), \( r = 15(22) \) for \( n = 23(26) \).

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<th>( L = Y_r )</th>
<th>( Y_s )</th>
<th>( Y_{s+1} )</th>
<th>( U_{V_1} )</th>
<th>( U_{V_2} )</th>
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Table 4: 90% lower and upper limits for future $s^{th}$ oos, $Y_{n,s}, s = r + 1, \ldots, n, r = 13(17)$ for $n = 19(24)$.

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